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PROJECT RAND

RESEARCH MEMORANDUM

NOTES ON THE n-PERSON GAME -- II:
THE VALUE OF AN n-PERSON GAME

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RM-670

ASTIA Document Number ATI 210720

21 August 1951

Assigned to ______________________

The RAND Corporation
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Summary: A "value" for the essential, n-person game is deduced from a set of axioms. A simple bargaining procedure is then presented which leads to the same result. Finally a number of simple properties of the value are established.

NOTES ON THE n-PERSON GAME, II: THE VALUE OF AN n-PERSON GAME

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§1. Introduction

At the foundation of the theory of games is the assumption that a player can evaluate in terms of his utility scale every situation that can result from a play of a game. In attempting to apply game theory to (say) economic or military behavior, we of necessity introduce into the class of relevant economic or military situations the prospect of being required to play a game. The possibility of evaluating this prospect is therefore of crucial importance to the successful application of the theory.

If the theory is unable to assign values to the games which occur most commonly in the field of intended application, then only isolated situations — in which games which occur do not depend on the outcomes of previous games — can be eligible for game-theoretic analysis. In the finite game theory of von Neumann and Morgenstern ([1]), difficulty in evaluating games persists for the "essential" games, and for only those.* The typical and important games of economics and politics are essential, and only small-scale applications of game theory have been made in those fields. Purely military problems, on the other hand, frequently lend themselves to two-person-zero-sum, and hence "inessential," models; here game theory has made more significant contributions.

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* Infinite games, even when inessential, do not necessarily possess values in the usual sense — such cases are usually regarded as mathematical curiosities, devoid of practical importance ([2], [3], [4]). For such games a formal value has been defined ([5], see also [4]).

In this note we shall deduce a value for the "essential" case. Our procedure will be to enunciate a set of axioms relating to the values of games and combinations of games; and then to show that the axioms determine the value uniquely. To give insight into the nature of the value formula obtained, we then present a simple bargaining procedure which leads to the same formula. Finally we establish a number of simple properties of the value.

§2. The characteristic function. Composition and direct sum.

We deal only with the so-called characteristic function of the game Γ ; this is a real-valued set-function $v(S)$ on the subsets of the set I of n players, satisfying:

(21) $$v(0) = 0$$

(22) $$v(S) \geq v(S \cap T) + v(S \cap (I - T))$$

for every $S, T \subseteq I$. Intuitively, $v(S)$ is to represent the most that the "coalition" S can surely obtain, without outside assistance. The constant-sum case is obtained by requiring equality in (22) for $S = I$ — in which case (21) can be dispensed with.* The game Γ is inessential if and only if equality always holds in (22).

If equality always holds in (22) for some particular T, other than 0 or I, the game is said to be decomposable; if such a T consists of just one player, he is said to be a dummy.

The direct sum

$$\Gamma = \Gamma' \oplus \Gamma''$$

of two games with sets I', I'' of players is obtained by the rule

(23) $$v(S) = v'(S \cap I') + v''(S \cap I''), \qquad \text{all } S \subseteq I = I' \cup I'' ;$$

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* Indeed, taking $S = T = I$ in (22), we would have $v(I) = v(I) + v(0)$, from which (21) follows.

clearly $v(S)$ is again a characteristic function. In particular, if I' and I'' are disjoint sets, then Γ is the composition of the games Γ', Γ'' :

$$\Gamma = (\Gamma', \Gamma'') .$$

Composition and decomposability are connected in the obvious way ([1] pp. 353 – 358). Again, if I' and I'' are the same set, then the new characteristic function is simply the sum of the original functions. The direct sum of a finite number of games is constant-sum, inessential, or decomposable if and only if all of the constituent games are constant-sum, inessential, or decomposable with respect to a common subset T .

The scalar multiple

$$\Gamma' = \beta\Gamma$$

of a game Γ , defined by

$$v'(S) = \beta v(S), \qquad \text{all } S \subseteq I$$

is again a game if and only if β is non-negative or Γ is inessential. Two games, Γ , Γ' are isomorphic (see [6]) if and only if $I = I'$ and

$$(24) \qquad \Gamma = \beta\Gamma' \oplus \Gamma''$$

holds for some positive β and inessential Γ'' *.

§3. The axioms.

We shall denote the value of the game Γ by a vector

$$\phi = (\phi_i)$$

which associates a real number (a utility) ϕ_i with each player $i \in I$.

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* If $\beta = 1$, they are "strategically equivalent" in the sense of von Neumann and Morgenstern ([1], pp. 245 ff).

AXIOM 1. ϕ <u>depends only on the characteristic function</u> v :

$$\phi = F\left(v(0), \cdots, v(S), \cdots, v(I)\right) = F[\Gamma] \quad .$$

AXIOM 2. ϕ <u>depends symmetrically on the elements of</u> I . <u>Precisely, if</u> Γ' <u>is defined by</u>:

$$v'(S) = v(S^*) \ ,$$

<u>where</u> S^* <u>is like</u> S <u>with</u> i <u>replacing</u> j, j <u>replacing</u> i, <u>then</u>

$$\phi_i' = \phi_j \ , \qquad \phi_j' = \phi_i \quad .$$

AXIOM 3. $\sum_{i \in I} \phi_i = v(I)$.

AXIOM 4. <u>If</u> $\Gamma' = \beta\Gamma$, $\beta > 0$, <u>then</u> $\phi' = \beta\phi$.

AXIOM 5. <u>If</u> $\Gamma = \Gamma' \oplus \Gamma''$ <u>then</u>

$$\phi_i = \begin{cases} \phi_i' + \phi_i'' & \text{for } i \in I' \cap I'' \\ \phi_i' & \text{for } i \in I - I'' \\ \phi_i'' & \text{for } i \in I - I' \ . \end{cases}$$

<u>Comments on the axioms</u>. Axiom 1: Serious doubt has been raised as to the adequacy with which the characteristic function describes the strategic possibilities of a general-sum game.* The difficulty, intuitively, is that the characteristic function does not distinguish between threats that damage just the threatened party and threats that damage both parties. This criticism, however, does not apply with any force to the constant-sum case.

If the objections are allowed to stand, then our present work, so far as general-sum games are concerned, can be construed as applying only to those games which can be reconstructed completely

* See the chapter on von Neumann's theory of general-sum games in [7]; see also [8].

from their characteristic functions, without introducing new strategic elements. (Such games exist for every characteristic function; see [1] pp. 530-532.) For constant-sum games, on the other hand, we can still claim the wider generality, restricted only by the assumption of a linear, transferable utility.

Axiom 2: Here we leave behind the possibility of taking account of differences in bargaining ability among the players, or of any other asymmetry associated with the identities (the names "i") of the players.

Axiom 3: "The value must describe a distribution* of the total proceeds of the game." This axiom excludes, for example, the pessimistic evaluation $\phi_i = v(\{i\})$ in an essential game.

In the general-sum case, this axiom involves an assumption about the "group rationality" of the players — namely that they will behave so as to maximize their common gain. In general it is better to steer clear of such assumptions in the axiom schema since our goal is, after all, to derive (or delimit) an as yet unknown principle of group rationality.** Thus, the generality of our present work is again weakened for the non-constant-sum case.

Axiom 4: "The value is independent of the units in which utility is expressed." This axiom is hardly open to question. It is also virtually unnecessary, since it can be deduced for all positive rational δ from Axiom 5. The companion axiom which one might expect to find here, saying that the value is independent of shifts in the zero points of the individual utility scales, is entirely included in Axiom 5.

Axiom 5: "If two games, played independently, are regarded

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* But not necessarily an "imputation," in the sense of [1]. (See the following footnote.)

** For example, we shall not make the highly "rational" assumption:

$$\phi_i \geq v(\{i\}) \qquad \text{all } i \in I ,$$

although we shall prove it later, as a welcome consequence of our axioms. In [9] we shall describe how critically the von Neumann-Morgenstern solution theory depends upon the way in which this very assumption is introduced.

as sections of a single game, the value to the players who participate in just one section is unchanged, while the value to those who play in both sections is the sum of their sectional values." As we have intimated in § 1, our objective is to apply game theory to situations in which many games take place. Our evaluation scheme must behave properly with respect to ideally independent games, as prescribed in this axiom, or else it will not be effective for analyzing more complicated combinations of games.

In our formal derivation of the value function we shall actually use just two special cases of Axiom 5:*

(31) the case $I' = I'' = I$ (same players for both games);

(32) the case $I' = I - \{i\}$, $I'' = \{i\}$ (adjunction of a dummy to the game Γ').

This economy is not overly surprising, since the most general kind of direct sum can in fact be built up from direct sums of these two types.

§ 4. Determination of the value function.

We now proceed to deduce from our axioms an explicit formula for the value. Applying Axiom 1 we write

$$\phi_i = F_i(v(0), \cdots, v(S), \cdots, v(I)) = F_i[\Gamma] .$$

LEMMA 1. If i is a dummy, then $\phi_i = v(\{i\})$.

Proof. Define Γ' on the set of players $I' = I - \{i\}$ by

$$v'(S) = v(S) \qquad \text{all } S \subseteq I'$$

and Γ'' on the single player i :

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* I owe this economy to a discussion with J. W. Milnor. From a somewhat different set of axioms he had derived, independently, the zero-sum value formula (equation (49) below, with $v(I) = 0$).

$$v''(\{i\}) = v(\{i\}) .$$

Then the direct sum $\Gamma''' = \Gamma' \oplus \Gamma''$ has the characteristic function

$$v'''(S) = \begin{cases} v(S - \{i\}) + v(\{i\}) & \text{if } i \in S , \\ v(S) & \text{if } i \notin S , \end{cases}$$

by (23). But since i is a dummy, v''' is identical to v, and $\Gamma''' = \Gamma$. Now Axiom 5, case (32), gives us

$$\phi_i = \phi_i'' ,$$

while Axiom 3 gives us

$$\phi_i'' = v''(I'') = v(\{i\}) ,$$

and the lemma is proved.

LEMMA 2. F_i _is linear in each_ $v(S)$ (_separately_).

Proof. Take a characteristic function v for which equality never holds in (22).* We still have a characteristic function if we increase or decrease the value of $v(S)$ by a small amount, for any particular $S \neq 0$, leaving $v(T)$ fixed for $T \neq S$. Let v' and v'', defined by

$$v'(T) = v''(T) = v(T) \qquad T \neq S$$

(41)
$$\begin{cases} v'(S) = v(S) - \epsilon_1 & \epsilon_1 > 0 \\ v''(S) = v(S) + \epsilon_2 & \epsilon_2 > 0 \end{cases}$$

(ϵ_1 and ϵ_2 sufficiently small) be two such functions. From (41) we obtain

$$v(S) = \frac{\epsilon_2}{\epsilon_1 + \epsilon_2} v'(S) + \frac{\epsilon_1}{\epsilon_1 + \epsilon_2} v''(S) ,$$

* Thus we are considering a general-sum game. If the constant-sum restriction had been imposed from the start, this proof would have to be modified slightly; the assertion of the lemma is valid in either case.

which holds, trivially, for all other $T \subseteq I$ as well. Hence

$$(42) \quad \Gamma = \frac{\epsilon_2}{\epsilon_1+\epsilon_2}\Gamma' \oplus \frac{\epsilon_1}{\epsilon_1+\epsilon_2}\Gamma'' = \frac{v''(S)-v(S)}{v''(S)-v'(S)}\Gamma' \oplus \frac{v(S)-v'(S)}{v''(S)-v'(S)}\Gamma'' ;$$

the last step following again from (41). Now

$$(43) \qquad \phi_i = F_i[\Gamma] = \frac{v''(S)-v(S)}{v''(S)-v'(S)} F_i[\Gamma'] + \frac{v(S)-v'(S)}{v''(S)-v'(S)} F_i[\Gamma'']$$

by (42) and Axiom 5, case (31). Collecting terms, we have

$$\phi_i = F_i[\Gamma] = \frac{F_i[\Gamma'']-F_i[\Gamma']}{v''(S) - v'(S)} v(S) + \frac{v''(S)F_i[\Gamma']-v'(S)F_i[\Gamma'']}{v''(S) - v'(S)} ,$$

which is a linear function of $v(S)$, the other quantities depending only upon the fixed games Γ' and Γ''.

This linear expression for ϕ_i obviously holds for $v(S)$ anywhere in the interval permitted by the inequalities (22), the quantities $v(T)$, $T \neq S$, being held fixed. Moreover, $F_i[\Gamma]$ must behave continuously at the ends of the interval, for there is nothing to prevent us from taking both Γ' and Γ'' at the end-points, and then, by (43),

$$F_i(\Gamma) \rightarrow \left\{ \begin{matrix} F_i(\Gamma') \\ \\ F_i(\Gamma'') \end{matrix} \right\} \text{ as } \Gamma \rightarrow \left\{ \begin{matrix} \Gamma' \\ \\ \Gamma'' \end{matrix} \right\}.$$

Since S and i were arbitrary, this completes the proof of the lemma.

LEMMA 3. F_i <u>is linear and homogeneous in all the variables</u> $v(S)$ <u>together</u>.

Proof. Lemma 3 will follow from Lemma 2 if we can show that F_i is homogeneous of degree one. By Axiom 4 we already have the requisite homogeneity relation

$$F_i[\beta\Gamma] = \beta F_i[\Gamma] \tag{44}$$

for positive values of β. It suffices to verify (44) for one negative value, say -1:

$$F_i[-\Gamma] = -F_i[\Gamma] \ .$$

But $-\Gamma$ is a game only if Γ is inessential — i.e., only if equality always holds in (22). Hence all players must be dummies, and by Lemma 1 we have

$$F_i[\Gamma] = v(\{i\})$$
$$F_i[-\Gamma] = -v(\{i\})$$

and (44) is confirmed. This completes the proof.

LEMMA 4. $F_i[\Gamma]$ <u>has the form</u>

$$\phi_i = \sum_{S \ni i} \gamma_s v(S) + \sum_{T \not\ni i} \delta_t v(T) \ , \tag{45}$$

<u>the coefficients depending only on the numbers,</u> s <u>and</u> t, <u>of elements in</u> S <u>and</u> T <u>respectively</u>.

(Note: the notation "$S \ni i$" means that the sum is to be taken over all subsets S of I that contain the element i ; similarly "$T \not\ni i$.")

Proof. This is merely the application of the symmetry of Axiom 2 to the result of Lemma 3.

LEMMA 5. $\delta_t = -\gamma_{t+1}$ $\qquad t = 1, \cdots, n-1$.

Proof. Suppose i is a dummy player. Then we can split the first summand of (45):

$$\phi_i = \sum_{S \ni i} \gamma_s [v(\{i\}) + v(S - \{i\})] + \sum_{T \not\ni i} \delta_t v(T)$$

$$= \sum_{S \ni i} \gamma_s v(\{i\}) + \sum_{T \not\ni i} (\gamma_{t+1} + \delta_t) v(T) \quad .$$

By Lemma 1, this expression is identically $v(\{i\})$. But the second sum is independent of $v(\{i\})$, hence

$$\gamma_{t+1} + \delta_t = 0 \qquad \text{for } t = 1, 2, \cdots, n-1 \ ,$$

as was to be shown. (We do not conclude anything about γ_1 and δ_0 because $v(0)$ is not a variable.)

LEMMA 6. $$\gamma_s = \frac{(n-s)!(s-1)!}{n!}$$

Proof. Combining Lemmas 4 and 5 gives us

$$\phi_i = \sum_{S \ni i} \gamma_s v(S) - \sum_{T \not\ni i} \gamma_{t+1} v(T) = \sum_{S \ni i} \gamma_s [v(S) - v(S - \{i\})] \ .$$

Summing on i and making use of Axiom 3, we obtain

$$v(I) = \sum_{i \in I} \sum_{S \ni i} \gamma_s [v(S) - v(S - \{i\})] \ .$$

In this sum, each $v(S)$ will appear with the coefficient γ_s precisely s times (once for each $i \in S$), and with the coefficient $-\gamma_{s+1}$ precisely $n - s$ times (once for each $i \notin S$). Thus:

$$v(I) = \sum_{S \subseteq I} \left(s\gamma_s - (n-s)\gamma_{s+1}\right) v(S) \ .$$

Since $v(I)$ is independent of $v(S)$ for $S \subset I$, we have

$$(46) \qquad s\gamma_s = (n-s)\gamma_{s+1} \qquad \text{for} \quad s = 1, 2, \cdots, n-1 .$$

Moreover, the coefficient of $v(I)$ must be exactly 1:

$$(47) \qquad n\gamma_n = 1 .$$

By means of (46) we may proceed recursively from (47):

$$\gamma_n = \frac{1}{n}, \quad \gamma_{n-1} = \frac{1}{n-1}\gamma_n = \frac{1}{n(n-1)}, \quad \gamma_{n-2} = \frac{2}{n-2}\gamma_{n-1} = \frac{1.2}{n(n-1)(n-2)}, \quad \cdots,$$

to obtain the formula of the lemma. Q.E.D.

We summarize:

THEOREM 1. _Axioms 1 – 5 lead uniquely to the following expression for the value of an n-person game:_

$$(48) \qquad \phi_i = \frac{1}{n!}\sum_{S \ni i}(s-1)!(n-s)!\left[v(S) - v(S-\{i\})\right] .$$

Note 1. The restriction $S \ni i$ is actually irrelevant here, since the summand vanishes for those S which do not contain i.

Note 2. Observe that the entire characteristic function occurs essentially in the formula (48).

Note 3. Strictly speaking, it is necessary still to demonstrate that the axioms are not inconsistent. We may do this by verifying that formula (48) actually satisfies each axiom.

COROLLARY. _In the constant sum case:_

$$v(S) + v(I - S) = v(I)$$

formula (48) _reduces to_

$$(49)\qquad \phi_i = -v(I) + \frac{2}{n!}\sum_{S \ni i} (s-1)!(n-s)!v(S) \quad .$$

Proof. We have, by easy steps,

$$\phi_i = \frac{1}{n!}\sum_{S \ni i}(s-1)!(n-s)!v(S) - \frac{1}{n!}\sum_{T \not\ni i}(t)!(n-t-1)!v(T)$$

$$= \frac{1}{n!}\sum_{S \ni i}(s-1)!(n-s)!v(S) - \frac{1}{n!}\sum_{S \ni i}(n-s)!(s-1)!\left[v(I)-v(S)\right]$$

$$= \frac{2}{n!}\sum_{S \ni i}(s-1)!(n-s)!v(S) - \frac{1}{n!}\sum_{s=1}^{n}\binom{n-1}{s-1}(n-s)!(s-1)!v(I) \; .$$

But the last summand reduces to $(n-1)!v(I)$; hence form (49) results.

§ 5. A bargaining procedure,

The n players of I are assigned, at random, the ordinal numbers* $1,2,\cdots,n$. Player i_1 constitutes himself into the (one-element) set S_1, and receives the amount $v(S_1)$. Player i_2 then joins the set S_1, forming the (two-element) set S_2, and receives the amount $v(S_2) - v(S_1)$ which he contributes to the value of the set by his presence. The process continues: Player i_k joins S_{k-1} to form S_k, and receives the amount $v(S_k) - v(S_{k-1})$. The last player receives $v(I) - v(S_{n-1})$.

We now ask, what is the expectation of each player under this scheme? The randomization assures that the players receive the symmetric treatment required in Axiom 2; the total amount disbursed is clearly $v(I)$, as required by Axiom 3. We could go on to verify the other axioms (only the fifth axiom gives any difficulty), but it is easier and more illuminating to compute the expectations directly. The probability that Player i joins a particular t-element subset $T \not\ni i$ is the product

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* We have so far considered I only as a set of n distinguished objects ("players"), not associated _a priori_ with the natural numbers $1,2,\cdots,$ n in any order.

$$(\text{prob.that } i=i_{t+1})\cdot(\text{prob.that } S_t=T) = \frac{1}{n}\cdot\left[\binom{n-1}{t}\right]^{-1} = \frac{t!(n-t-1)!}{n!} .$$

His over-all expectation is therefore

$$\sum_{T \not\ni i} \frac{t!(n-t-1)!}{n!}\left[v(T\cup\{i\}) - v(T)\right] ,$$

which is at once seen to be his value, ϕ_i, as given by (48).

For constant-sum games, there is a simple variant of this procedure that gives the same outcome. The players "choose up sides" as follows: after the players have been ordered at random, each in turn is assigned randomly to one of two teams, and is awarded the amount his presence adds to the value of the team. The teams then play the game as a two-person, constant-sum game, and it is easy to see that each team wins just enough to pay off all its members. Under this procedure, the expectation of each player is again precisely ϕ_i. (We omit the calculation.)

The two-team procedure will not give the value in a general-sum game; indeed, the sum of the expectations of the players is necessarily less than $v(I)$ if the game is not constant-sum.

§ 6. Elementary properties of the value.

We collect in the next theorem a number of simple properties of the value function (48). Most of them follow immediately from the definitions, axioms, and lemmas of the earlier sections.

THEOREM 2. *The value of an n-person game has the following properties:*

(i) *The value of an inessential game is given by*

$$\phi_i = v(\{i\}) .$$

(ii) *If two games are isomorphic:*

$$v'(S) = \beta v(S) + \sum_{i \in S} \alpha_i \qquad (\beta > 0) ,$$

then their values are related by

$$\phi'_i = \beta\phi_i + \alpha_i \quad .$$

(iii) In a decomposable game, the value to each player may be computed indifferently from the game as a whole or from the component game to which he belongs.

(iv) In particular, the presence of a dummy player does not affect the values to the other players.

(v) If Player i is not a dummy, then

$$\phi_i > v(\{i\}) \ .$$

Proofs. (i): By Lemma 2. (ii): By Axiom 5, $I' = I''$, using (i) and (24). (iii): By Axiom 5, $I' \cap I'' = 0$. (iv): By (iii). (v): By (48), (22) we have

$$\phi_i = \sum_{S \ni i} \gamma_s [v(S)-v(S-\{i\})] \geq \sum_{S \ni i} \gamma_s v(\{i\}) = v(\{i\}) \ ,$$

with equality if and only if i is a dummy. Q.E.D.

For purposes of reference and illustration, we write out the actual formulas for small numbers of players, employing two simple normalizations to select a single representative from each equivalence-class of isomorphic, essential games:

Normalization A: $v(I) = 0$, $v(\{i\}) = -1$, all $i \in I$.

Normalization B: $v(I) = 1$, $v(\{i\}) = 0$, all $i \in I$.

(Compare von Neumann's "reduced form" and "zero-reduced form," [1] pp. 544-5.) For games normalized in these two ways, the formulas are given in Table I. We may remark:

(1) Symmetry alone suffices to determine the value of the three-person, constant-sum game (i.e., Axiom 5 is not required). The same holds for the two-person, general-sum game.

TABLE I

Type of Game	Normalization	Value to Player i
3-person constant-sum	A	0
	B	$\frac{1}{3}$
4-person constant-sum	A	$\frac{1}{6}\sum_{j\neq i} v(\{i,j\})$
	B	$\frac{1}{6}\sum_{j\neq i} v(\{i,j\})$
5-person constant-sum	A	$\frac{1}{10}\sum_{j\neq i} v(\{i,j\}) - \frac{1}{15}\sum_{\{j,k\}\not\ni i} v(\{j,k\})$
	B	$\frac{1}{10}\sum_{j\neq i} v(\{i,j\}) - \frac{1}{15}\sum_{\{j,k\}\not\ni i} v(\{j,k\}) + \frac{1}{5}$
2-person general-sum	A	0
	B	$\frac{1}{2}$
3-person general-sum	A	$\frac{1}{6}\sum_{j\neq i} v(\{i,j\}) - \frac{1}{3} v(I - \{i\})$
	B	$\frac{1}{6}\sum_{j\neq i} v(\{i, j\}) - \frac{1}{3} v(I - \{i\}) + \frac{1}{3}$

(2) The value of a four-person, constant-sum game is an imputation which is related to, but not identical with, the characteristic point ω, defined and discussed in [10]. Under

normalization A, we have

$$\phi = \omega/3 \quad .$$

Thus, geometrically speaking, and regardless of normalization, the value lies on the line connecting the characteristic point with the centroid of the simplex of imputations, 1/3 of the way out from the centroid.

Table II gives the coefficients γ_s for different values of n .

TABLE II

n \ s	1	2	3	4	5	6	7	8
2	$\frac{1}{2}$	$\frac{1}{2}$						
3	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$					
4	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$				
5	$\frac{1}{5}$	$\frac{1}{20}$	$\frac{1}{30}$	$\frac{1}{20}$	$\frac{1}{5}$			
6	$\frac{1}{6}$	$\frac{1}{30}$	$\frac{1}{60}$	$\frac{1}{60}$	$\frac{1}{30}$	$\frac{1}{6}$		
7	$\frac{1}{7}$	$\frac{1}{42}$	$\frac{1}{105}$	$\frac{1}{140}$	$\frac{1}{105}$	$\frac{1}{42}$	$\frac{1}{7}$	
8	$\frac{1}{8}$	$\frac{1}{56}$	$\frac{1}{168}$	$\frac{1}{280}$	$\frac{1}{280}$	$\frac{1}{168}$	$\frac{1}{56}$	$\frac{1}{8}$

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