U. S. AIR FORCE
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RESEARCH MEMORANDUM

AN INFINITE-DIMENSIONAL EXTENSION
OF A SYMETRIC BLOTTO GAME

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RM-718

November 8, 1951

Assigned to ______________________

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Summary: A two-person zero-sum game over a function space and with discontinuous payoff is considered which has an optimal strategy for either player, consisting of randomizing over a certain two-parameter family of functions.

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Let X be the set of all real-valued continuous functions over $[0,1]$ with the properties

$$x(t) \geq 0 \text{ for all } t \in [0,1] \text{ and}$$

$$\int_0^1 x(t)\, dt = 1$$

for any $x \in X$. We set up the normal form of the game thus:

1. A pure strategy for the maximizing player is a point $x \in X$.
2. A pure strategy for the minimizing player is a point $y \in X$.
3. For any selection of pure strategies $\{x,y\}$, the payoff M to the maximizing player is given by

$$M(x,y) = \int_0^1 v(t) \operatorname{sgn} [x(t) - y(t)]\, dt,$$

where v is a prescribed positive-valued element of X.

Remarks

One can envision this game as two lines of infantry of equal strength about to engage in battle in a corridor, the value of a breakthrough to either force in any interval dt along the width of the corridor being $v(t)dt$. A breakthrough is achieved at the interval, if "Blotto's" forces exceed the enemy's there, or conversely. We note that this game is an extension of the finite-dimensional cases considered in [1] and [2].

We propose that the following is an optimal strategy for the maximizing player (hereafter denoted by "Blotto"). It would follow by symmetry that the enemy could do equally as well by employing the same strategy. Hence, to prove we have a solution, we need only verify that the following strategy will ensure Blotto an expectation of zero:

Blotto chooses one of the following 2-parameter family of functions:

(1) $$x_{\alpha,\rho}(t) = v(t)\left[1 + \rho\cos 2\pi\left(\alpha - \int_0^t v(\lambda)d\lambda\right)\right],$$

where the choice is affected by selecting α and ρ independently from the respective distributions

(2) $$f_1^*(\alpha) = \alpha \quad 0 \leq \alpha \leq 1,$$

(3) $$f_2^*(\rho) = 1 - \sqrt{1-\rho^2} \quad 0 \leq \rho \leq 1.$$

Verification

We first show that the foregoing is admissible as a mixed strategy. It should suffice to show that (i) f_1^* and f_2^* are distributions over $[0,1]$, and (ii) for any choice of $\alpha, \rho \in [0,1]$, $x_{\alpha,\rho}$ is an admissible pure strategy. The proof of (i) is trivial. To prove (ii), we observe that $x_{\alpha,\rho}$ is continuous as a function of t and $x_{\alpha,\rho}(t) \geq 0$ for all $t \in [0,1]$. It remains to show that $\int_0^1 x_{\alpha,\rho}(t)dt = 1$. Thus,

$$\begin{aligned}\int_0^1 x_{\alpha,\rho}(t)dt &= \int_0^1 v(t)dt + \rho\int_0^1 v(t)\cos 2\pi\left(\alpha - \int_0^t v(\lambda)d\lambda\right)dt \\ &= 1 + \rho\int_0^1 \cos 2\pi\left(\alpha - \int_0^t v(\lambda)d\lambda\right) d\int_0^t v(\lambda)d\lambda \\ &= 1 + \rho\int_0^1 \cos 2\pi(\alpha - u)\,du \\ &= 1 + 0 = 1, \text{ as required.}\end{aligned}$$

To prove that the strategy given by (1), (2) and (3) is optimal for Blotto, we must show that

(4) $$\int_0^1\int_0^1 M(x_{\alpha,\rho},y)\; df_1^*(\alpha)df_2^*(\rho) \geq 0 \text{ for all } y \in X.$$

On making the required substitution, we obtain the following expression for the left member of (4):

$$(5) \qquad \int_0^1\int_0^1 \left\{\int_0^1 v(t)\ \mathrm{sgn}\left[v(t)\ \left(1+\rho\cos 2\pi\left(\alpha-\int_0^t v(\lambda)d\lambda\right)\right)-y(t)\right]dt\right\}\frac{\rho\, d\rho\, d\alpha}{\sqrt{1-\rho^2}}$$

On making the transformation $\theta = 2\pi\alpha$, and using the Fubini theorem, one obtains for (5):

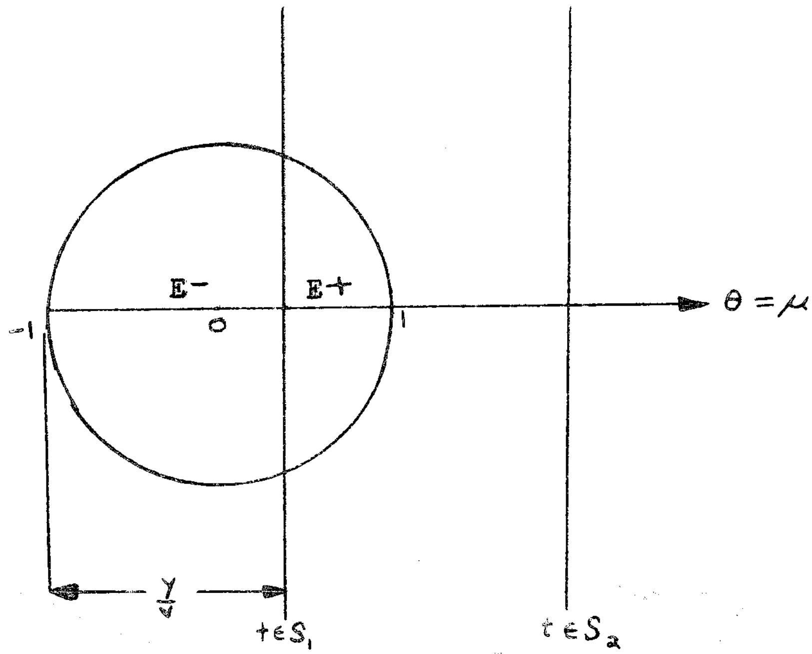
$$(6) \qquad \frac{1}{2\pi}\int_0^1 v\ \left\{\int_0^{2\pi}\int_0^1 \mathrm{sgn}\ \left[v\left(1+\rho\cos\ (\theta-\mu)\right)-y\right]\frac{\rho\, d\rho\, d\theta}{\sqrt{1-\rho^2}}\right\}dt,$$

where, for brevity, we have written v, μ, y, respectively, for $v(t), 2\pi\int_0^t v(\lambda)d\lambda, y(t)$.

Now observe that $\frac{\rho\, d\rho\, d\theta}{\sqrt{1-\rho^2}}$ is an element of surface area of a unit hemisphere erected on the polar coordinate plane and centered at the origin, the point on the surface of the hemisphere having cylindrical coordinates (ρ, θ, z). Moreover, the equation

$$v\ \left(1+\rho\cos\ (\theta-\mu)\right)-y = 0$$

is the equation of a straight line in polar coordinates (i.e., a vertical plane in cylindrical coordinates) at a directed distance $\frac{y}{v}-1$ from the origin, the direction being taken with respect to the ray $\theta=\mu$. (μ can be interpreted as an orientation angle.) Moreover, since $\frac{y}{v}\geqslant 0$ for all t, it follows that $\frac{y}{v}-1\geqslant -1$. Thus, in evaluating the double integral under the braces of (6), we need consider only two cases. Let $S_1 = \left[t\,\middle|\,\frac{y}{v}\leqslant 2\right]$ and $S_2 = \left[t\,\middle|\,\frac{y}{v}>2\right]$. These cases are illustrated in the accompanying figure.



Now, for $t \in S$, the double integral is evaluated as follows:

The signum function is $+1$ to the right of the line (i.e., in E^+ as illustrated) and -1 to the left. Thus, the integral in question is equal to the area of that portion of the hemisphere to the right of the line less the area to the left. But from a well-known property of the surface area of the sphere, this latter area is proportional to $\frac{y}{v}$, i.e., it is equal to $\pi \frac{y}{v}$. Thus, for $t \in S_1$,

$$\left\{\int\int\right\} = \left(2\pi - \pi \frac{y}{v}\right) - \pi \frac{y}{v} = 2\pi\left(1 - \frac{y}{v}\right)$$

Now for $t \in S_2$, the integral becomes simply the negative of the surface area of the hemisphere, i.e., -2π. But for $t \in S_2$, $\frac{y}{v} > 2$, whence $-2\pi > 2\pi(1 - \frac{y}{v})$. Thus, for $t \in S_2$,

$$-2\pi = \left\{\int\int\right\} > 2\pi\left(1 - \frac{y}{v}\right).$$

Hence, in any case, we have

$$(7) \quad \left\{\int\int\right\} \geqslant 2\pi\left(1 - \frac{y}{v}\right).$$

Now, since v is positive, we may multiply the inequality (7) by $\frac{v}{2\pi}$ and integrate:

$$(8) \quad \frac{1}{2\pi}\int_0^1 v\left\{\int\int\right\} dt \geqslant \int_0^1 v\left(1 - \frac{y}{v}\right)dt = \int_0^1 v dt - \int_0^1 y dt$$
$$= 1 - 1$$
$$= 0.$$

Returning to (6) we see that the left member of (8) is Blotto's expectation. This completes the verification.

References

[1] O. Gross and R. Wagner: A Continuous Colonel Blotto Game, RAND RM-408.

[2] O. Gross: The Symmetric Blotto Game, RAND RM-424.