U. S. AIR FORCE

PROJECT RAND

RESEARCH MEMORANDUM

TWO EXAMPLES CONCERNING BEHAVIOR STRATEGIES

F. B. Thompson and R. Wagner

RM-747

18 December 1951

Assigned to ____________________

The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

Summary: Some questions concerning properties of behavior strategies are answered by means of an example.

TWO EXAMPLES CONCERNING BEHAVIOR STRATEGIES

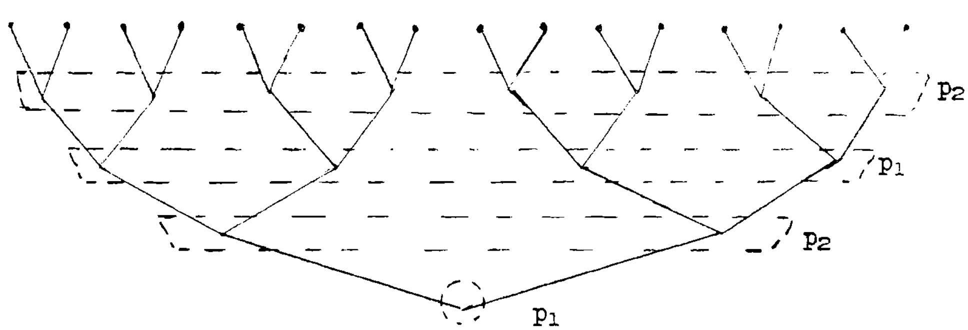
F. B. Thompson and R. Wagner

So far little seems to be known about behavior strategies in finite games. The authors feel that in such a young domain the exhibition of examples is not out of order. There are a number of questions which thus can be quickly answered and the way cleared for deeper inquiry.

The notion of behavior strategy was first formalized by Kuhn in his paper "Extensive Games" [1]. However it is implicitly used by earlier investigators [2]. It will be assumed that the reader is familiar with the concepts involved.

1: Notation

The two examples to be presented differ only in payoff function. Thus we shall exhibit here the game-tree with the player and information partitions shown thereupon:



The i^{th} "player" or team is indicated by p_i. The alternatives at each move are to be thought of as numbered in the clock-wise order. A behavior strategy for p_1 consists of a pair $\ll a_1^1, a_2^1 >, < a_1^2, a_2^2 \gg$ where a_j^i is the probability that p_1 plays the j^{th} alternative at his i^{th} move. Evidently $a_1^1 + a_2^1 = a_1^2 + a_2^2 = 1$. Similarly a behavior strategy for p_2 is $\ll b_1^1, b_2^1 >, < b_1^2, b_2^2 \gg$.

If the players were both to use optimal mixed strategies then the expected payoff to p_1 would be a fixed number v which is usually called the value of the game. It is the maximum p_1 can insure himself and the minimum to which p_2 can restrict p_1. Let "v_1" denote the maximum which p_1 can insure himself when using behavior strategies, and let "v_2" denote the minimum to which p_2 can restrict p_1 when using behavior strategies. It is intuitively evident (and follows formally from a result of Dalkey's) that $v_1 \leq v \leq v_2$. Dresher, Helmer and Wagner [3] have exhibited a game for which $v_1 \neq v_2$. By an optimal behavior strategy for p_1 we shall mean any behavior strategy which insures him v_1. By an optimal behavior strategy for p_2 we shall mean any behavior strategy for p_2 which restricts p_1 to at most v_2.

Pure strategies for player p_1 will be denoted by "$< i,j >$" where $i, j = 1, 2$; $< i,j >$ indicating alternative i at p_1's first move and alternative j at p_1's second move. Similarly for p_2.

2: First example.

Payoff matrix:

	< 1,1 >	< 1,2 >	< 2,1 >	< 2,2 >
< 1,1 >	1	1	-2	0
< 1,2 >	3	2	-1	0
< 2,1 >	0	-1	2	3
< 2,2 >	0	-2	1	1

For this game the following values can be verified:

(i) $v_1 = 0$, $v = \frac{1}{2}$, $v_2 = 1$; thus $v_1 < v < v_2$;

(ii) a behavior strategy for p_1 is optimal if and only if it is of the form: $\ll \frac{1}{2}, \frac{1}{2} >, < h, 1-h \gg$ where $0 \leq h \leq 1$;

(iii) a behavior strategy for p_2 is optimal if and only if it is of the form: $\ll \frac{1}{2}, \frac{1}{2} >, < k, 1-k \gg$ where $0 \leq k \leq 1$;

(iv) if p_1 uses $\ll \frac{1}{2}, \frac{1}{2} >, < h, 1-h \gg$, and p_2 uses $\ll \frac{1}{2}, \frac{1}{2} >, < k, 1-k \gg$, then the expected payoff is: $\frac{1}{2} - (h - \frac{1}{2})(k - \frac{1}{2})$.

It will first be noted that each player has a continuum number of optimal behavior strategies. If both players play optimal behavior strategies, the payoff will be some value between $\frac{1}{4}$ and $\frac{3}{4}$ and any value in the closed interval $\left[\frac{1}{4}, \frac{3}{4}\right]$ is obtainable. Thus if both play optimal behavior strategies the payoff will actually be greater than v_1 and less than v_2.

One is immediately led to the question of "preferred" optimal behavior strategy. This suggests the induced game over the unit square whose payoff is $H(h, k) = \frac{1}{2} - (h - \frac{1}{2})(k - \frac{1}{2})$. This game has a saddle-point: $h = k = \frac{1}{2}$. Thus one might conclude that the "preferred" optimal behavior strategies are respectively $\ll \frac{1}{2}, \frac{1}{2} >, < \frac{1}{2}, \frac{1}{2} \gg$, $\ll \frac{1}{2}, \frac{1}{2} >, < \frac{1}{2}, \frac{1}{2} \gg$.

Now an examination of the above game raises the following comments and questions:

(i) A convex combination of optimal mixed strategies is always an optimal mixed strategy. We note that in our first example the convex combination of optimal behavior strategies is an optimal behavior strategy. Is this always the case?

(ii) We note that the induced game we considered to determine the "preferred" optimal behavior strategies had a saddle-point solution. The corresponding consideration for optimal mixed strategies trivially yields the same result since optimal mixed strategies when played against one-another always give v. Will there always be such a saddle-point?

3: Second example.

Payoff matrix:

	$< 1,1 >$	$< 1,2 >$	$< 2,1 >$	$< 2,2 >$
$< 1,1 >$	0	0	2	4
$< 1,2 >$	2	2	-2	0
$< 2,1 >$	-1	-2	2	2
$< 2,2 >$	4	2	0	0

For this game the following values can be verified:

(i) $v_1 = \frac{2}{3}$, $v = 1$, $v_2 = \frac{4}{3}$;

(ii) p_1 has exactly two optimal behavior strategies:

$\alpha_1 = \ll 0,1 >, < \frac{1}{3}, \frac{2}{3} \gg$ and $\alpha_2 = \ll 1,0 >, < \frac{2}{3}, \frac{1}{3} \gg$;

(iii) p_2 has exactly two optimal behavior strategies:

$\beta_1 = \ll \frac{2}{3}, \frac{1}{3} >, < 0,1 \gg$ and $\beta_2 = \ll \frac{1}{3}, \frac{2}{3} >, < 1,0 \gg$;

(iv) the payoff to p_1 if p_1 plays α_i and p_2 plays β_j is given by the following table:

	β_1	β_2
α_1	$\frac{2}{3}$	$\frac{11}{9}$
α_2	$\frac{4}{3}$	$\frac{2}{3}$

We thus are in a position to answer the questions in Section 2. In fact:

(i) The convex combination of optimal behavior strategies (in whatever meaningful way this may be defined) is not necessarily an optimal behavior strategy.

(ii) The induced game, whose normal form is

	β_1	β_2
α_1	$\frac{2}{3}$	$\frac{11}{9}$
α_2	$\frac{4}{3}$	$\frac{2}{3}$

does not have a saddle-point solution. Thus there appears to be no way for either player to pick out a "preferred" optimal behavior strategy. The existence of a game with this property seems to us to indicate a deficiency either in the usefulness of behavior strategies or in their theory. It seems rational to expect p_2 to play an optimal behavior strategy (unless p_2 is playing under conditions where mixed strategies can be used, a situation which is excluded here). In our first example p_1 could choose a behavior strategy which would not only insure v_1 against all behavior strategies of p_2 but insure v against all optimal behavior strategies of p_2; while p_2 can prevent p_1 from gaining more than v if p_1 plays an optimal behavior strategy. This is not the case in example two where p_1 can insure himself no more than $\frac{2}{3}$ whereas p_2 can only be sure p_1 receives no more than $\frac{11}{9}$.

Thus the whole question of how a team should best play a game in which team members are separated from one another over several plays seems again to be thrown open.

There is an observation concerning the second example which seems interesting. As is well known there is a mixture of pure strategies of p_1 such that if played it insures p_1 at least v. Now pure strategies can be considered as a subclass of the class of behavior strategies. Thus there is a mixture of behavior strategies which if played insures p_1 an expected payoff of at least v. However if one computes the optimal mixture of p_1's optimal behavior strategies one obtains the mixture $\frac{6}{11}\alpha_1$ with $\frac{5}{11}\alpha_2$, which if played insures p_1 no more than $\frac{32}{33}$, thus less than v.

[1] H. W. Kuhn. Extensive Games. Proceedings of the National Academy of Sciences, vol. 36 (1950), pp. 570-576.

[2] For example: J. F. Nash and L. S. Shapley. A simple three-person poker game. Annals of Mathematics Studies, No. 24 (1950), Princeton.

[3] M. Dresher, O. Helmer, and R. Wagner. An analysis of three move games. RM-681.