

U. S. AIR FORCE
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BEHAVIOR STRATEGIES IN FINITE GAMES

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RM-769

22 January 1952

Assigned to _____

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Summary: The relation between behavior strategies and mixed strategies is developed. Those game structures solvable by behavior strategies are characterized.

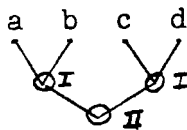
BEHAVIOR STRATEGIES IN FINITE GAMES

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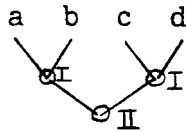
Informal Discussion: In general, this paper is pointed to the development of the relation between behavior strategies and mixed strategies for finite games. The notion of behavior strategy was introduced by Kuhn [1]. A behavior strategy for a player consists, essentially, of a family of distribution functions, one for each information set of the player which assigns the relative frequency with which the various alternatives open to the player at that information set are to be played. Thus a behavior strategy differs from a mixed strategy in that it randomizes the alternatives at each move of a play rather than randomizing the pure strategies for entire plays.

One may advance two reasons for studying behavior strategies. First, one may wish to consider a "player" as consisting of a team rather than just one person. In some cases it may be impossible to decide between plays what common strategy should be played and to communicate adequately during the play. Such a situation leads naturally to the study of behavior strategies. Secondly, and probably more important, it has been shown in a number of cases that it may be considerably easier first to solve a game in behavior strategies and then find a mixed strategy solution directly from the behavior strategies. This, of course, raises the problem as to when this is possible.

We shall again deal with the notion of game structure which was introduced in RM-759 [2]. It will be recalled that two games have the same structure if they differ at most in payoff [3]. For a fixed game and a given choice of mixed (behavior) strategies for the players, we shall denote by MH_p (BH_p) the amount which player p may expect if these strategies are actually played. Dalkey has proved [4] that for a fixed game structure and any choice of behavior strategies $\vec{\beta}$ for the players, there are corresponding mixed strategies $\vec{\alpha}$ such that $MH_p(\vec{\alpha}) = BH_p(\vec{\beta})$ for all players and all payoffs. Kuhn [1] has defined the notion of perfect recall; a game structure has perfect recall if each player recalls at a given move both what he did and what information he had at all of his previous moves. Kuhn then proved that a sufficient condition for the existence of a map carrying any choice of mixed strategies $\vec{\alpha}$ for the players into a choice of behavior strategies $\vec{\beta}$ such that $MH_p(\vec{\alpha}) = BH_p(\vec{\beta})$ for all players and all payoffs is that the game structure have perfect recall. We establish that Kuhn's condition is also necessary. The question arises whether there is such a mapping which is one-one. Although there are game structures for which the answer is "yes," they form a very restricted class. For example, the following game structure, with perfect information, is not such a structure:



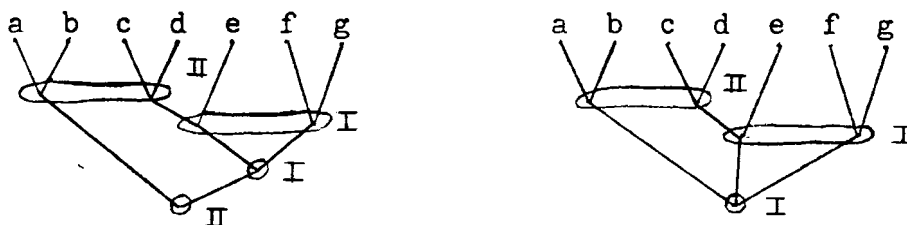
Two mixed (behavior) strategies for a player will be called equivalent if, for all strategies for his opponents and all payoffs, they give the same expectancy. There may be a strategy for a player and an information set U for the player such that the plays resulting from this strategy and any strategies of the opponents will never intersect U . In this case strategies which differ from the given strategy only at U will be equivalent to it. We shall show that this is the only way equivalences among behavior strategies can arise. On the otherhand this is not true of mixed strategies. For example, in the structure:



no two behavior strategies are equivalent, while it is easy to find one dimensional equivalence classes of mixed strategies (one easily shows that equivalence classes of mixed strategies are convex). We establish that a necessary and sufficient condition that there be a one-one correspondence between equivalence classes of mixed strategies and equivalence classes of behavior strategies which preserves expectant payoffs is that the structure have perfect recall.

The notion of characteristic function for a game has been defined by von Neumann [5]. It assigns to each coalition of players the maximum amount which they can insure themselves by mixing their pure strategies against the combined onslaught of their opponents. A similar characteristic function can be defined for behavior strategies. In many cases the values for behavior strategies will be less than those for the mixed strategy

function. We shall say that a game structure is solvable by behavior strategies if, for every payoff, the characteristic function for behavior strategies is the same as the characteristic function (for mixed strategies). By Kuhn's result mentioned above, it follows that a sufficient condition for solvability by behavior strategies is perfect recall. It can be shown that this condition is not necessary. For example, the structures:



are solvable by behavior strategies. In order to characterize those games which are solvable by behavior strategies, we first define a notion of weak equivalence of game structures. If there exist one-one correspondences between sets of players and between end-points of two game structures so that for any pair of payoffs which assign the same number to corresponding end-points the mixed (behavior) strategy characteristic functions assign the same value to corresponding coalitions, then the two structures are weakly equivalent for mixed (behavior) strategies. This notion is contrasted to the notion of strong equivalence where there are one-one correspondences between equivalence classes of strategies which preserves expectancy under all payoffs. It is proved that strong equivalence implies weak equivalence and weak equivalence implies isomorphism of reduced normal forms [6]. The converses are shown to hold for mixed strategies; however neither converse holds for behavior strategies. It is then shown that a game structure is

solvable by behavior strategies if and only if it is weakly equivalent to a structure (in the wider sense) which has perfect recall. The proof is constructive and actually characterizes those games solvable by behavior strategies in a manner independent of the notion of strategy.

Formal Presentation: We shall assume familiarity with notations, definitions and results of the formal part of RM-759. To emphasize this we shall continue the enumeration of definitions and theorems from that paper. There is one correction to a definition given there which is important for our results here. We shall assume that if a is a move, then Aa has at least two elements, i.e. when a player is called upon to move he has a choice of at least two alternatives.

Definition 22: Let $\underline{G} = \langle G, \leq, P, R, I \rangle$ be a game structure. By a play of \underline{G} we shall mean a maximal chain of G under \leq . Let \underline{Q} be the set of all plays of \underline{G} .

Definition 23: Let \underline{G} be a game structure, $p \in \underline{P}^*$. β is a behavior strategy for p if β is such a function on p that (i) for $a \in p, \beta(a) = \beta_a$ is a probability distribution on A_a ; (ii) if $a, b \in p, a I b, a' \in A_a, b' \in A_b$, and $a' R b'$, then $\beta_a(a') = \beta_b(b')$. Let BSp be the set of behavior strategies for p ; let BS be the Cartesian product of all BSp for $p \in \underline{P}^*$.

Definition 24: Let \underline{G} be a game structure $p \in \underline{P}^*$, π is a mixed strategy for p if π is a probability distribution on S_p . Let MSp be the family of all mixed strategies for p ; let MS be the Cartesian product of the MSp for $p \in \underline{P}^*$.

Definition 25: Let $\langle \underline{G}, h \rangle$ be a game, $p \in \underline{P}^*$. Then the behavior strategy payoff matrix for p_1 is the function BHp_1 on BS such that for $\beta \in BS$:

$$BHp_1(\vec{\beta}) = \sum_{Q \in \underline{Q}} \prod_{p \in \underline{P}^*} \prod_{\substack{a \in p \cap Q \\ b \in A_a \cap Q}} \beta_{p,a}(b) \cdot h(e^*(Q), p_1)$$

where $e^*(Q)$ is the unique element of $Q \cap E$.

Definition 26: Let $\langle \underline{G}, h \rangle$ be a game, $p \in \underline{P}^*$. Then the mixed strategy payoff matrix for p_1 is the function MHp_1 on MS such that for $\pi \in MS$:

$$MHp_1(\vec{\pi}) = \sum_{\vec{\alpha} \in S} \prod_{p \in \underline{P}^*} \vec{\pi}_p(\vec{\alpha}_p) \cdot h(e(\vec{\alpha}), p_1).$$

Lemma 27: Let $G' \subseteq G, G' \neq \Lambda$, be such that if $a, b \in G'$ then not $a I b$. Let $\vec{\alpha}_1 \in S; S' = \{ \vec{\alpha} \mid \vec{\alpha} \in S \text{ and if } a \in p, a / I \cap G' = \Lambda \text{ then } \vec{\alpha}_p(a) = \vec{\alpha}_{1,p}(a) \}$. Then for $\beta \in BS, \sum_{\vec{\alpha} \in S'} \prod_{p \in \underline{P}^*} \prod_{a \in G' \cap p} \beta_{p,a}(\vec{\alpha}_p(a)) = 1$

Proof: By induction on the number of information sets which intersect G' .

Theorem 28 (Dalkey): Given a game structure \underline{G} , there are functions ϕ_p , for $p \in \underline{P}^*$, which map BSp into MSp in such a way that whenever $\langle \underline{G}, h \rangle$ is a game and $\beta \in BS$, $BHp(\vec{\beta}) = MHp(\vec{\alpha})$ for $p \in \underline{P}^*$, where $\phi_p(\vec{\beta}_p) = \vec{\alpha}_p$ for all $p \in \underline{P}^*$.

Proof: The theorem follows by direct application of the definitions and simple manipulations using lemma 27.

Definition 29: (Kuhn) Let \underline{G} be a game structure. \underline{G} has perfect recall if, for $p \in \underline{P}^*$ and $a, b \in p$ such that $a < b$, the following condition is satisfied: For $a' \in Aa$, let $Ra = \{d \mid \text{for some } c \in a/I, \text{ and } c' \in Aca'/R, c' < d\}$. Then for some $a' \in Aa$, $b/I Ra$.

Theorem 30: Let $\underline{G} = \langle G, \leq, P, R, I \rangle$ be a game structure such that $I \cap (E \times E)$ is the identity relation on E . Then there are functions θ_p , for $p \in \underline{P}^*$, which map MSp into BSp in such a way that whenever $\langle \underline{G}, h \rangle$ is a game and $\vec{\alpha} \in MS$, $MHp(\vec{\alpha}) = BHp(\vec{\beta})$ for $p \in \underline{P}^*$, where $\theta_p(\vec{\alpha}_p) = \vec{\beta}_p$ for all $p \in \underline{P}^*$, if and only if \underline{G} has perfect recall.

Proof: The sufficiency has been proved by Kuhn. We establish the necessity.

Suppose there are functions θ_p as described and that \underline{G} does not have perfect recall. Thus for some $p_1 \in \underline{P}^*$, $a, b_1, b_2 \in p_1$, $a < b_1, b_1 I b_2$ and if $c_1 \in Aa$, $c_1 \leq b_1$, $c_2 \in Ad$, $c_2 \leq b_2$ and $c_1 R c_2$, then not $a I d$. Let $c_1, c_3 \in Ab_1$, $c_2 \in Ab_2$ be such that $c_1 \neq c_3$ and $c_3 R c_2$. Let $\alpha^{(1)} \in S$ be such that $c_1 \leq e(\alpha^{(1)})$ and, if d is such that not $d I a$ and, for $j = 1$ or 2 and all $d' I d$, $d' \neq e(\vec{\alpha}^{(j)})$, then $\vec{\alpha}^{(1)} d/p(d) = \vec{\alpha}^{(2)} d/p(d)$. By our hypothesis we may also choose $\vec{\alpha}^{(2)}$ so that $\vec{\alpha}^{(2)} p_1(a) \neq b_1$. For $j' = 1, 2, 3$, let $h_{j'}$ be such that $\langle \underline{G}, h_{j'} \rangle$ is a game and $h_{j'}(e, p)$ is 1 if $c_1 \leq e$ and 0 otherwise.

Let $\vec{\pi} \in MS$ be such that for $p \in \underline{P}^*$, $i = 1$ or 2 , $\vec{\pi}_p(\vec{\alpha}_p^{(i)}) = 1/2$ if $\vec{\alpha}_p^{(1)} \neq \vec{\alpha}_p^{(2)}$, and $\vec{\pi}_p(\vec{\alpha}_p^{(i)}) = 1$ otherwise.

$$\begin{aligned} MHP_1^1(\vec{\pi}) &= \sum_{\vec{\alpha} \in S} \prod_{p \in \underline{P}^*} \vec{\pi}_p(\vec{\alpha}_p) h_i(e(\vec{\alpha}), p_1) \\ &= \sum_{\vec{\alpha} \in S, c_i \leq e(\vec{\alpha})} \prod_{p \in \underline{P}^*} \vec{\pi}_p(\vec{\alpha}_p). \end{aligned}$$

$$\therefore MHP_1^1(\vec{\pi}) \geq \prod_{p \in \underline{P}^*} \vec{\pi}_p(\vec{\alpha}_p^{(1)}) > 0. \text{ Similarly } MHP_1^2(\vec{\pi}) > 0.$$

Suppose $c_3 \leq e(\vec{\alpha})$ for some $\vec{\alpha} \in S$. Now $\vec{\alpha}_{p_1}^{(1)}(b_1) = c_1 \neq c_3$, thus $\vec{\alpha}_{p_1}^{(1)} \neq \vec{\alpha}_{p_1}^{(2)}$; $\vec{\alpha}_{p_1}^{(1)}(a) \neq b_1 \leq c_3$ and therefore $\vec{\alpha}_{p_1}^{(1)} \neq \vec{\alpha}_{p_1}^{(2)}$. Consequently $\pi_{p_1}(\vec{\alpha}_{p_1}) = 0$, and

$$MHP_1^3(\vec{\pi}) \leq \sum_{\vec{\alpha} \in S, c \leq e(\vec{\alpha})} \vec{\pi}_{p_1}(\vec{\alpha}_{p_1}) = 0$$

Let $\vec{\beta} \in BS$ be such that $\theta_p(\vec{\pi}_p) = \vec{\beta}_p$ for $p \in \underline{P}^*$.

$$\begin{aligned} BHP_1^1(\vec{\beta}) &= \sum_{Q \in \underline{Q}} \prod_{p \in \underline{P}^*} \prod_{\substack{c \in p \cap Q \\ d \in Ac \cap Q}} \vec{\beta}_{p,c(d)} h_i(e^*(Q), p_1) \\ &= \sum_{Q \in \underline{Q}, c_i \in Q} \prod_{p \in \underline{P}^*} \prod_{\substack{c \in p \cap Q \\ d \in Ac \cap Q}} \vec{\beta}_{p_1,c(d)} \\ &= \prod_{p \in \underline{P}^*} \prod_{\substack{c \in p, d \leq c_i \\ d \in Ac}} \vec{\beta}_{p,c(d)} \text{ (by lemma 27)} \\ &= MHP_1^1(\vec{\pi}) \end{aligned}$$

$$\therefore \prod_{p \in \underline{P}^*} \prod_{\substack{c \in p, \\ d \in Ac}} \vec{\beta}_{p_1,c(d)} > 0 \text{ for } i = 1, 2,$$

$$= 0 \text{ for } i = 3.$$

Thus for some $p \in \underline{P}^*$, $c \in p$ and $d \in A_c$ such that $d \leq c_3$, $\vec{\beta}_{p,c}(d) = 0$.
 If $c \neq b_1$, then $d \leq c_1$ as well and $BHp_1^1(\vec{\beta}) = 0$ which is a contradiction,
 Thus $c = b_1, d = c_3$. But $\vec{\beta}_{p,c}(d) = \vec{\beta}_{p,b_1}(c_3) = \vec{\beta}_{p,b_2}(c_2)$ since
 $b_1 I b_2$ and $c_2 R c_3$. This implies $BHp_1^2(\vec{\beta}) = 0$ which is again a contra-
 diction.

Definition 31: Let \underline{G} be a game structure, $p_1 \in \underline{P}^*$. Then, for
 $\sigma_1, \sigma_2 \in \left\{ \begin{matrix} MSP_1 \\ BSP_1 \end{matrix} \right\}$, σ_1 is equivalent to σ_2 if, for $\vec{\alpha}, \vec{\beta} \in \left\{ \begin{matrix} MS \\ BS \end{matrix} \right\}$ such
 that $\vec{\alpha}_{p_1} = \sigma_1, \vec{\beta}_{p_1} = \sigma_2$ and $\vec{\alpha}_p = \vec{\beta}_p$ for $p \in \underline{P}^* - \{p_1\}$, then for all
 h for which $\langle \underline{G}, h \rangle$ is a game, $\left\{ \begin{matrix} Mhp(\vec{\alpha}) = Mhp(\vec{\beta}) \\ Bhp(\vec{\alpha}) = Bhp(\vec{\beta}) \end{matrix} \right\}$ for all $p \in \underline{P}^*$.
 Let $\left\{ \begin{matrix} MSP^* \\ BSP^* \end{matrix} \right\}$ be the family of all equivalence classes of $\left\{ \begin{matrix} \text{mixed} \\ \text{behavior} \end{matrix} \right\}$
 strategies; let $\left\{ \begin{matrix} MS^* \\ BS^* \end{matrix} \right\}$ be the Cartesian product of the $\left\{ \begin{matrix} MSP^* \text{'s} \\ BSP^* \text{'s} \end{matrix} \right\}$.

Theorem 32: Let \underline{G} be a game structure such that $I \cap (EXE)$ is
 the identity relation on E . Let $\sigma_1, \sigma_2 \in BSP_1$ with σ_1 equivalent
 to σ_2 . Suppose $a \in G$ is such that $\sigma_1, a \neq \sigma_2, a$. Then, for $i = 1, 2$,
 there are $c_i \leq a, c_i \in Ab_i$ such that $\sigma_i, b_i(c_i) = 0$.

Proof: Let σ_1, σ_2, a be such that the hypothesis is true,
 but the conclusion false. Without loss of generality we can assume
 that for all $b \in p, b < a, \sigma_1, b = \sigma_2, b$. Let $d \in Aa$ be such that

$\sigma_1, a(d) \neq \sigma_2, a(d)$; let h be such that $\langle \underline{G}, h \rangle$ is a game and
 $h(e, p_1) = 1$ if $d < e, h(e, p) = 0$ otherwise. Let $\vec{\alpha}^i \in BS$ be such
 that $\vec{\alpha}_{p_1}^i = \sigma_i, \vec{\alpha}_p^1 = \vec{\alpha}_p^2$ for $p \in \underline{P}^* - \{p_1\}$, and, if $c \leq a, c \in Ab$ and
 $b \in p \in \underline{P}^* - \{p_1\}$, then $\vec{\alpha}_{p,a}^i(b) = 1$. Thus

$$\begin{aligned}
 BHp_1(\vec{\alpha}^i) &= \sum_{Q \in Q} \prod_{p \in \underline{P}^*} \prod_{\substack{b \in p \cap Q \\ c \in Aa \cap Q}} \vec{\alpha}_{p,b}^i(c) \cdot h(e^*(Q), p_1) \\
 &= \sigma_i, a(d) \quad (\text{by lemma 27 and above assumption})
 \end{aligned}$$

$\therefore \text{BHp}_1(\vec{\alpha}^1) \neq \text{BHp}_1(\vec{\alpha}^2)$ which is a contradiction.

Theorem 33: Let \underline{G} be a game structure such that $I \cap (E \times E)$ is the identity relation on E . Then there are one-one correspondences between BSp^* and MSp^* such that if $\vec{\beta}_p$ and $\vec{\alpha}_p$ correspond for all p , $\text{BHp}(\vec{\beta}) = \text{MHp}(\vec{\alpha})$ for all p and all payoffs if and only if \underline{G} has perfect recall.

Proof: Follows from theorems 28, 30 and definition 31.

Definition 34: Let $\underline{G}_1, \underline{G}_2$ be two game structures. \underline{G}_1 is strongly equivalent to \underline{G}_2 for $\left\{ \begin{matrix} \text{mixed} \\ \text{behavior} \end{matrix} \right\}$ strategies if there are biunique functions r, s and t such that:

- i) r is on \underline{P}_1^* onto \underline{P}_2^* ;
- ii) s is on \underline{P}_1^* such that for $p \in \underline{P}_1^*$, $s(p) = s_p$ is a biunique function on Sp^* onto $\text{Sr}(p)^*$;
- iii) t is on E_1/I_1 onto E_2/I_2 ;
- iv) if h is a function such that $\langle \underline{G}_1, h \rangle$ is a game, and h' is defined on E_2 so that $h'(e_2) = h(e_1)$ whenever $e_1 \in E_1$ and $e_2 \in t(e_1/I_1)$, then $\left\{ \begin{matrix} \text{MHp}(\vec{\alpha}) = \text{Mh}'a(p)(\vec{\beta}) \\ \text{BHp}(\vec{\alpha}) = \text{Bh}'r(p)(\vec{\beta}) \end{matrix} \right\}$ for $\vec{\alpha} \in \vec{\alpha}_* \in S_1^*, \vec{\beta} \in \vec{\beta}_* \in S_2^*$ and $s_p(\vec{\alpha}_p^*) = \vec{\beta}_*^* r(p)$.

Definition 35: Let $\langle \underline{G}, h \rangle$ be a game. U^m is the characteristic function of $\langle \underline{G}, h \rangle$ for mixed strategies if U^m is such a function on the family of all subsets of \underline{P}^* that for $\underline{p}_1 \subseteq \underline{P}^*$ and $\text{MSp}_1 = \{ \pi \mid \pi \text{ is a function on } \prod_{p \in \underline{P}_1} \text{Sp}; \text{ for } \alpha \in \prod_{p \in \underline{P}_1} \text{Sp}, 0 \leq \pi(\alpha) \leq 1; \text{ and}$

$$\left. \sum_{\substack{\alpha \in \prod_{p \in \underline{P}_1} \text{Sp} \\ \pi(\alpha) = 1 \}} \right\}, U^m(\underline{P}_1) = \text{Max}_{\pi \in \text{MSp}_1} \text{Min}_{\sigma \in \text{MSp}^* - \underline{p}_1}$$

$$\sum_{\vec{\alpha} \in S} \left[\pi(\vec{\alpha} \cap (P_1 \times V)) - \sigma(\vec{\alpha} \cap ((P^* - P_1) \times V)) \cdot \sum_{p \in P_1} h(e(\vec{\alpha}), p) \right]$$

Definition 36: Let $\langle G, h \rangle$ be a game. U^B is the characteristic function of $\langle G, h \rangle$ for behavior strategies if U^B is such a function on the family of all subsets of P^* that for $P_1 \subseteq P^*$,

$$U^B(P_1) = \text{Max}_{p \in P_1, \vec{\beta} p \in BSp} \text{Min}_{p \in P^* - P_1, \vec{\beta} p \in BSp} \sum_{p \in P_1} BHp(\vec{\beta}).$$

Definition 37: Let G_1, G_2 be two game structures. G_1 is weakly equivalent to G_2 for $\left\{ \begin{matrix} \text{mixed} \\ \text{behavior} \end{matrix} \right\}$ strategies if there are biunique functions r, t such that:

- i) r is on P_1^* onto P_2^* ;
- ii) t is on E_1/I_1 onto E_2/I_2 ;

iii) if h is a function such that $\langle G_1, h \rangle$ is a game and h' is defined on E_2 so that $h'(e_2) = h(e_1)$ whenever $e_1 \in E_1$ and $e_2 \in t(e_1/I_1)$, then $U_{1,P}^M = U_{2,r^*(P)}^M$ for $P \in P_1^*$, where $r^*(P) = \{r(p) \mid p \in P\}$.

Theorem 38: If G_1, G_2 are strongly equivalent, then they are weakly equivalent.

Proof: Immediate.

Theorem 39: If G_1, G_2 are such that $I_1 \cap (E_1 \times E_1)$ is the identity relation on E_1 , and G_1, G_2 are weakly equivalent, then they have isomorphic reduced normal forms.

Proof: Although not strictly true, we shall write as if

$E_1 = E_2 = E, P_1^* = P_2^* = P^*$. For $\alpha \in Sp$, let $E_{\alpha}^{(i)} = \{e \mid \text{for some } \vec{\alpha} \in S_i, \vec{\alpha} p = \alpha \text{ and } e_i(\vec{\alpha}) = e\}$.

(1) For $\alpha_1, \alpha_2 \in Sp_1$, if $E_{\alpha_1} \subseteq E_{\alpha_2}$, then $E_{\alpha_1} = E_{\alpha_2}$.

Suppose not. Let $E\alpha_1 \subset E\alpha_2$, $e_2 \in E\alpha_2 - E\alpha_1$.

Let $\vec{\beta}_1, \vec{\beta}_2 \in S$ such that $\vec{\beta}_{1,p_1} = \alpha_1$ and $\vec{\beta}_{1,p} = \vec{\beta}_{2,p}$ for $p \in \underline{P}^* - \{p_1\}$, and $e(\vec{\beta}_2) = e_2$. Let $e(\vec{\beta}_1) = e_1$. Let a be the largest move such that $a < e_1, a < e_2$. Since both $e_1, e_2 \in E\alpha_2$, we see that $a \notin p_1$. But by the definition of $\vec{\beta}_1$ and $\vec{\beta}_2$, $a \notin p$ for $p \in \underline{P}^* - \{p_1\}$.

(2) Let $p_1 \in \underline{P}^*$, $\alpha_1 \in Sp_1^{(1)}$. Then there is a $\alpha_2 \in Sp_1^{(2)}$ such that $E\alpha_1^{(1)} = E\alpha_2^{(2)}$.

Suppose not. We consider two cases.

(i) For every $\alpha \in Sp_1^{(2)}$, there is an $e \in E\alpha^{(2)} - E\alpha_1^{(1)}$. Let h be such a function on $E \times \underline{P}^*$ that $\langle G_1, h \rangle$ are games and $h(e, p_1) = 1$ if $e \in E\alpha_1^{(1)}$, $h(e, p_1) = 0$ otherwise. Let each $p \in \underline{P}^* - \{p_1\}$ play a mixed strategy which assigns the same frequency to each of his pure strategies. Since for every $\alpha \in Sp_1^{(2)}$, there is an $\alpha \in S^{(2)}$ such that $\alpha p_1 = \alpha^{(2)}$ and $h(e(\alpha), p_1) = 0$, we see that no matter what mixed strategy $\alpha \in Sp_1^{(2)}$ might play his expectancy would be less than 1. However, if he plays α_1 then his expectancy would be 1. However, in G_1 , if he plays α_1 , then he would receive 1 against any strategies of his opponents. Thus we have a contradiction of weak equivalence for mixed strategies. Clearly, using theorem 28, and the fact that α_1 is a pure strategy the same result holds for behavior strategies.

(ii) There is an $\alpha_2 \in Sp_1^{(2)}$ such that $E\alpha_2^{(2)} \subset E\alpha_1^{(1)}$. Now for every $\alpha \in Sp_1^{(1)}$, there is an $e \in E\alpha^{(1)} - E\alpha_2^{(2)}$, for if not there would be an $\alpha \in Sp_1^{(1)}$ such that $E\alpha \subseteq E\alpha_2^{(2)} \subset E\alpha_1^{(1)}$, contradicting (1). Now we simply interchange the roles of G_1 and G_2 in (i).

(3) For $\alpha_1, \alpha_2 \in Sp_1$, α_1 is equivalent to α_2 if and only if $E\alpha_1 = E\alpha_2$. Clearly if α_1 and α_2 are equivalent, then $E\alpha_1 = E\alpha_2$. If $E\alpha_1 = E\alpha_2$, we prove α_1 and α_2 to be equivalent by proceeding in a manner similar to the proof of (1).

(4) If $\vec{z} \in S$, then $\bigcap_{p \in P^*} E_{\vec{z}p}$ contains exactly one element. In fact $e(\vec{z})$.

Putting (2), (3) and (4) together, the theorem is now immediate.

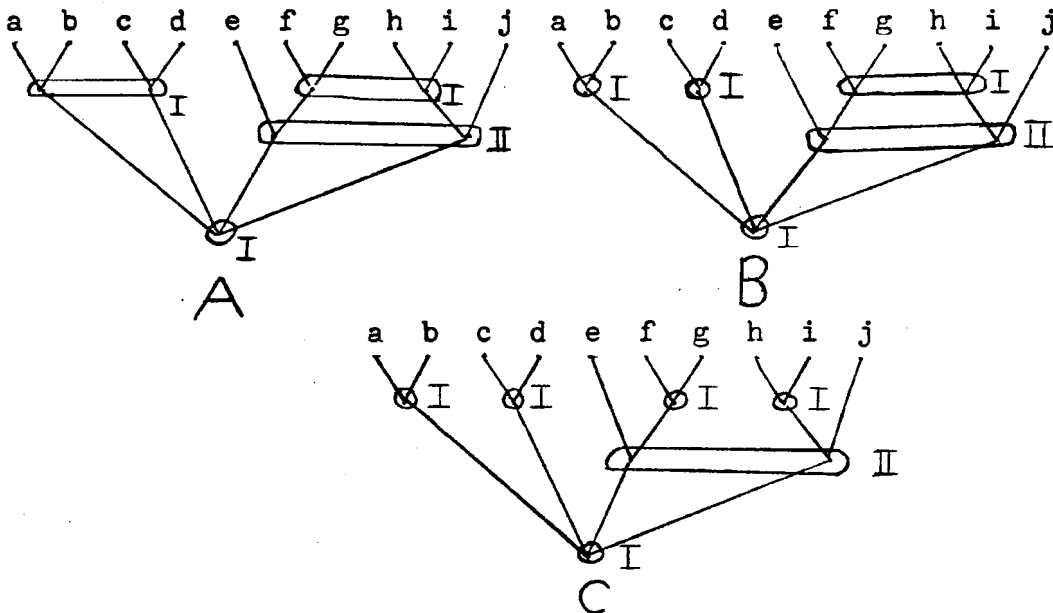
Theorem 40: Let G_1, G_2 be such that $I_1 \cap (E_1 \times E_1)$ is the identity relation on E_1 . Then the following statements are equivalent:

- i) G_1 and G_2 are strongly equivalent for mixed strategies;
- ii) G_1 and G_2 are weakly equivalent for behavior strategies;
- iii) G_1 and G_2 have isomorphic reduced normal forms.

Proof: After theorems 38 and 39, it is sufficient to prove that iii) implies i). This is perfectly straight forward.

Theorem 41: Of the three statements for behavior strategies analogous to those of theorem 40, no two are equivalent.

Proof: It is sufficient to exhibit counter-examples.



A and B are weakly but not strongly equivalent; B and C are not weakly equivalent but have isomorphic reduced normal forms.

Definition 42: A game structure \underline{G} is solvable by behavior strategies if, for every game $\langle \underline{G}, h \rangle$, $U_p^B = U_p^M$ for all $p \in \underline{P}^*$.

Theorem 43: Let \underline{G} be a game structure such that $I \cap (E \times E)$ is the identity relation on E . Then \underline{G} is solvable by behavior strategies if and only if it is weakly equivalent to a structure with perfect recall.

Proof: Suppose \underline{G} is weakly equivalent for behavior strategies to a structure \underline{G}' with perfect recall. Then \underline{G} , \underline{G}' have isomorphic reduced normal forms, and thus are weakly equivalent in mixed strategies. The mappings involved can easily be seen to correspond. The solvability of \underline{G} now follows from Kuhn's result and the definitions of weak equivalence.

Suppose \underline{G} is solvable by behavior strategies. We first check that if $\underline{G}' \stackrel{j}{\sim} \underline{G}''$ for $i = 2, 3, 4$ (where $\stackrel{j}{\sim}$ is defined in definition 16 of RM-759) then \underline{G}' and \underline{G}'' are weakly equivalent. Therefore, by lemma 18, there are game structures $\underline{G}_1, \dots, \underline{G}_t$ such that $\underline{G} \cong \underline{G}_1$, $\underline{G}_i \stackrel{j}{\sim} \underline{G}_{i+1}$ for $1 \leq i < t$ and $j = 2, 3, 4$, such that:

- i) if a is a move of \underline{G}_t , then Aa has exactly two elements;
- ii) if $a_i < b_i$ for $i = 1, 2$, then not $a_1 I b_2$;
- iii) for some ordering of \underline{P}^* , say $\underline{P}^* = \{p_1, \dots, p_n\}$, if $a \in p_i$, $b \in p_j$ and $i < j$, then $a < b$.

We have assumed that in the structure \underline{G} , I was the identity relation on E . Now this property is not preserved by $\stackrel{j}{\sim}$. However, we consider the following condition:

- iv) if for $\vec{\alpha}_1, \vec{\alpha}_2 \in S$, $e(\vec{\alpha}_1) I e(\vec{\alpha}_2)$, then for any $\vec{\beta} \in S$ such that $\vec{\beta}_p = \vec{\alpha}_{i,p}$, $i = 1$ or 2 , for each $p \in \underline{P}^*$, then $e(\vec{\beta}) I e(\vec{\alpha}_1)$.

It is straight forward to check that the fact that I is the identity relation on E implies iv) and that iv) is preserved under $\underline{2}$, $\underline{3}$ and $\underline{4}$. Thus without loss of generality we may assume that \underline{G} has properties i) through iv).

Now the question is; how far can we inflate? We wish to show that if there is an information set which can be inflated under the relation $\underline{1}$, but where this inflation would not preserve weak equivalence, then \underline{G} is not solvable by behavior strategies. If we establish this, then our proof is complete, for then the solvability of \underline{G} would imply that \underline{G} was in fact weakly equivalent for behavior strategies to its normal form.

Although the argument would not be brief, it can be ascertained that we may limit our attention to that $p \in \underline{P}^*$ for which $a \in \underline{P}^* - \{p_n\}$ implies $a < b$ for some $b \in p_n$. This is based on the possibility of reordering the players by a modification of definition 16 iv).

Suppose there is an information set $\{a_1, \dots, a_k, b_1, \dots, b_k\}$, for p_n such that $a_i R a_j$, $b_i R b_j$ and, for some c_i , $A c_i = \{a_i, b_i\}$. Suppose that if we break this information set into two: $\{a_1, \dots, a_k\}$, $\{b_1, \dots, b_k\}$, then the resulting structure $\underline{G}' \underline{1} \underline{G}$ is not weakly equivalent for behavior strategies to the original \underline{G} . Thus, for some h , $\langle \underline{G}, h \rangle$ and $\langle \underline{G}', h \rangle$ are games and $U^B(\underline{P}) \neq U^{B'}(\underline{P})$ for some $\underline{P} \subseteq \underline{P}^*$.

Now $U^B(\underline{P}) \leq U^{B'}(\underline{P})$ for all $\underline{P} \subseteq \underline{P}^*$. To show this it is sufficient to show that $U^B(\{p\}) \leq U^{B'}(\{p\})$ for $p \in \underline{P}^*$, by the nature of U^B . A straight forward calculation using the definition of U^B gives the desired result. It is immediate that $U^{B'}(\underline{P}) \leq U^{M'}(\underline{P}) = U^M(\underline{P})$ for all $\underline{P} \subseteq \underline{P}^*$. Thus for some $\underline{P} \subseteq \underline{P}^*$ $U^B(\underline{P}) < U^M(\underline{P})$ and therefore \underline{G} is not solvable by behavior strategies.

A careful examination of the full proof of the last few steps yields enough information to characterize in terms of the game tree those situations of the above kinds where inflation preserves weak equivalence for behavior strategies.

FOOTNOTES AND REFERENCES

- 1 H. W. Kuhn, Extensive Games. Proceedings of the National Academy of Sciences, vol. 36 (1950), pp. 570-576.
- 2 F. B. Thompson, Equivalence of Games in Extensive Form. RM-759.
- 3 A more restrictive notion was introduced in RM-759 and will be used in the formal part of this paper. However, this less restrictive notion is intended here.
- 4 With the permission of Dr. Dalkey, his theorem and proof are being published here for the first time.
- 5 J. von Neumann and O. Morgenstern, Theory of Games and Economic Behavior. Princeton, 1947.
- 6 This result implies the statement made in the last sentence of the first paragraph on page 2 of RM-759, which was given there without proof.

