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NOTES ON THE n-PERSON GAME, III: SOME VARIANTS OF THE VON NEUMANN-MORGENSTERN DEFINITION OF SOLUTION

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NOTES ON THE n-PERSON GAME, III: Some Variants of the von Neumann-Morgenstern Definition of Solution

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Summary.

A von Neumann-Morgenstern solution to an n-person game can be characterized as a set which is "A-stable", where A is the set of outcomes of the game which maximize the total profit and which are compatible with individual rationality on the part of the players. We introduce an alternative notion of "G-stability", G being the set of all possible outcomes. From this concept we obtain several new definitions of "solution", differing in the extent to which they take account of individual rationality. The new solutions are similar in appearance to those of von Neumann-Morgenstern, but none of the new definitions is exactly equivalent to the old. Some theorems are proved and a number of examples are given.

1. Preliminaries. Stability defined.

We follow in general the notation and terminology of the earlier notes of this series (RM-656, RM-670), unless otherwise indicated. The first departure is a slight relaxation in the definition of characteristic function - we continue to require superadditivity:

$$v(S \cap T) + v(S - T) \leq v(S) \qquad (\text{all } S, T \subseteq I); \tag{1}$$

but we discard the condition $v(\emptyset) = 0$. The quantity $v(\emptyset)$, which is non-positive by (1), is to be thought of as the possible spread in the total profit of the game; in terms of the payoff functions $H_i(x_1, \ldots, x_n) = H_i(\vec{x})$ we would have

$$v(\emptyset) = \min_{\vec{x}} \sum_I H_i(\vec{x}) - \max_{\vec{x}} \sum_I H_i(\vec{x}) . \tag{2}$$

In a constant-sum game, $v(\emptyset) = 0$.

DEFINITION 1. Let G be the set of points α which satisfy

$$(3) \qquad v(\emptyset) + v(I) \leq \sum_I \alpha_i \leq v(I) .$$

Thus, G is the set of all possible outcomes of the game.

DEFINITION 2. Let E be the set of points α which satisfy

$$(4) \qquad \sum_I \alpha_i = v(I) .$$

Thus, E is the set of outcomes which maximize the total profit.

DEFINITION 3. Let A be the set of points α which satisfy (4) and

$$(5) \qquad \alpha_i \geq v(\{i\}) \qquad (\text{all } i \in I) .$$

A comprises the "imputations" of the von Neumann-Morgenstern theory.

DEFINITION 4. A set K is <u>C-stable</u>, for any $C \subseteq G$, if

$$(6) \qquad K = C - \text{dom } K .$$

Thus, a C-stable set is contained in C , and dominates its complement in C .

LEMMA 1. If K is C-stable, and if $K \subseteq B \subseteq C$, then K is B-stable.

<u>Proof</u>. Immediate from the definition.

<u>2. The definitions of "solution".</u>

DEFINITION 5. Let $\mathcal{S}_1$ denote the class of A-stable sets:

$$K \in \mathcal{S}_1 \Longleftrightarrow K = A - \text{dom } K .$$

These are the "solutions" of von Neumann and Morgenstern.

von Neumann and Morgenstern impose no direct limitation on the amount of side payments. They nevertheless quietly assume that only points in A are of importance. In justification they invoke the principle of individual rationality, saying (TGEB 29.2.1) that the players "will certainly block" any distribution not in A. The exact nature of this blocking process is not made clear*, but it is evidently quite different from the "domination" process, whereby one distribution is rejected in favor of another. In the von Neumann-Morgenstern theory, "blocking" takes precedence over "domination", in that it actually prevents the players from considering, let alone accepting, distributions which are outside of A.

The propriety of this restriction to A may be challenged on several grounds. In the first place, it is not at all obvious that the notion of group rationality, as exemplified by the solution of an n-person game, must necessarily be a refinement of the principle of individual rationality, as embodied in the inequalities (5). In the second place, it would seem methodologically more correct to study the consequences of the domination process separately from those of the blocking process. One might even hope that the former, apparently the more powerful, might make the latter superfluous. (In that case, the restriction to A would be only a technical convenience, and would not prejudice the conceptual substructure of the theory.) Failing this, the restriction to A might better be applied (if it is desired to exclude "irrational" solutions) after stability under domination has been secured.

These considerations suggest the following definitions of "solution" as competitors to Definition 5.

DEFINITION 6. Let $\mathcal{S}_2$ denote the class of G-stable sets:

$$K \in \mathcal{S}_2 \Longleftrightarrow K = G - \text{dom } K .$$

DEFINITION 7. Let $\mathcal{S}_3$ denote the class of intersections of $\mathcal{S}_2$-sets with A:

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* We intend to devote a later note of this series to an investigation of this process.

$$K \in \mathcal{S}_3 \Longleftrightarrow (\text{some } L \in \mathcal{S}_2)\ K = L \cap A .$$

DEFINITION 8. Let $\mathcal{S}_4$ denote the class of $\mathcal{S}_2$-sets which are contained in A :

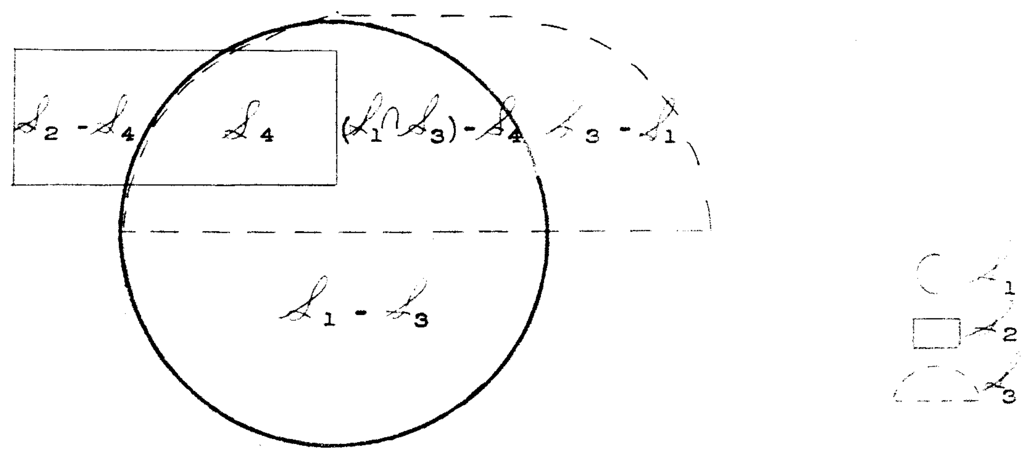
$$K \in \mathcal{S}_4 \Longleftrightarrow K \in \mathcal{S}_2 \text{ and } K \subseteq A .$$

Definition 6 ignores individual rationality. Definitions 7 and 8 comply with individual rationality, but in different ways - the former by excluding <u>points</u> which are outside of A, the latter by excluding <u>sets</u> which are not included in A. $\mathcal{S}_3$ will turn out to be less satisfactory than $\mathcal{S}_4$ (see Example 3 below).

THEOREM 1. $\mathcal{S}_4 = \mathcal{S}_1 \cap \mathcal{S}_2 = \mathcal{S}_2 \cap \mathcal{S}_3$.

<u>Proof</u>. Immediate from the definitions and Lemma 1.

The set-theoretic disposition of $\mathcal{S}_1, \ldots, \mathcal{S}_4$ is indicated in the following diagram.



The areas which remain in the diagram are all non-empty. Thus, Example 2 at the end of this paper provides instances of sets in $\mathcal{S}_2 - \mathcal{S}_4$ and in $(\mathcal{S}_1 \cap \mathcal{S}_3) - \mathcal{S}_4$; and Example 4 gives games with sets in $\mathcal{S}_1 - \mathcal{S}_3$ and in $\mathcal{S}_3 - \mathcal{S}_1$.

3. Some theorems.

THEOREM 2. K is G-stable if and only if K is E-stable.

<u>Proof</u>. As preparation we note that if α is in $G - E$ and if α' is the

projection of α in E :

$$\alpha'_j = \alpha_j + \frac{1}{n}\left[v(I) - \sum_I \alpha_i\right] \quad \text{(all } j \in I) ,$$

then α' I-dominates α, and any point which dominates α' dominates α as well. Now suppose that K is E-stable. Then every α in G - E is dominated by K, for α' is either in K or in dom K . Hence E - dom K = G - dom K , and K is G-stable. Conversely, suppose that K is G-stable. Again, every α in G - E must be in dom K . Consequently $K \subseteq E$. By Lemma 1, K is E-stable. Q.E.D.

Lemma 1 tells us that a G-stable set is A-stable if and only if it is contained in A . The next theorem gives a simple criterion for the converse relation.

THEOREM 3. If K is in $\mathcal{S}_1$, then K is in $\mathcal{S}_2$ * if and only if, for each $i \in I$, there is an α in K with $\alpha_i = v(\{i\})$.**

DEFINITION 9. Let $\gamma^{(i)}$ be given by

$$(7) \qquad \gamma_j^{(i)} = \begin{cases} v(\{j\}) - (n-2)e(v) & \text{if } j = i \\ v(\{j\}) + e(v) & \text{if } j \neq i \end{cases}$$

where $e(v)$:

$$e(v) = v(I) - \sum_I v(\{i\}) ,$$

is the amount available in the game v to be divided up by cooperative action.

LEMMA 2. If α S-dominates $\gamma^{(i)}$, then i is in S and

$$(8) \qquad \gamma_i^{(i)} < \alpha_i \leq v(\{i\}) .$$

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* And hence in $\mathcal{S}_3$ and $\mathcal{S}_4$, by Theorem 1.

** Geometrically, K must meet every (n-2)-dimensional face of A . The solutions of RM-656, and, more generally, the quota solutions of AMS-28, all have this property, and hence belong to $\mathcal{S}_2$, $\mathcal{S}_3$, and $\mathcal{S}_4$ as well as $\mathcal{S}_1$.

Proof. Suppose that $i \notin S$. Then

$$v(S) > \sum_{j\in S} \gamma_j^{(i)} = \sum_{j\in S} v(\{j\}) + se(v)$$

$$= v(I) - \sum_{j\notin S} v(\{j\}) + (s-1)e(v) .$$

Since $s \geq 1$ and $e(v) \geq 0$, the strict inequality above violates the superadditivity (1) of v . Hence i is in S , and the first inequality of (8) follows. To complete the proof we assume that $a_i > v(\{i\})$. Then

$$\text{(10)} \qquad v(S) \geq \sum \alpha_j > v(\{i\}) + \sum_{\substack{j\in S \\ j\neq i}} \gamma_j^{(i)} = \sum_{j\in S} v(\{j\}) + (s-1)e(v)$$

$$= v(I) - \sum_{j\notin S} v(\{j\}) + (s-2)e(v) .$$

By superadditivity we have $S = 1$, and hence $S = \{i\}$. Then the first line of (10) gives the absurdity

$$v(\{i\}) > v(\{i\}) .$$

Hence $\alpha_i \leq v(\{i\})$, as was to be shown.

Proof of Theorem 3. Necessity: Suppose that for some $i \in I$, $\alpha_i > v(\{i\})$ for all $\alpha \in K$. Then, by the lemma, $\gamma^{(i)}$ is not in dom K . Since $\gamma^{(i)}$ cannot be in K either, K is not G-stable. Sufficiency: Every β in $E - A$ has $\beta_i < v(\{i\})$ for some $i \in I$, by (5). But such β is $\{i\}$-dominated by any α with $\alpha_i = v(\{i\})$. The condition of the theorem therefore implies that

$$E - \text{dom } K = A - \text{dom } K = K .$$

K is therefore E-stable, and hence, by Theorem 2, G-stable. Q.E.D.

We next show that the $\mathcal{A}_2$-sets all lie in a bounded region of the hyperplane E .

DEFINITION 10. Let F be the set of points α which satisfy (4) and

$$\alpha_i \geq v(\{i\}) - (n-2)e(v) \qquad (\text{all } i \in I) \ . \tag{9}$$

Thus, F is the simplex in E, concentric with A, which just includes the points $\gamma^{(i)}$. The ratio between the edge lengths of A and F is $1:(n-1)^2$.

THEOREM 4. K is G-stable if and only if K is F-stable.

<u>Proof</u>. Suppose that K is <u>either</u> E-stable <u>or</u> F-stable. The points $\gamma^{(i)}$ are then in either K or dom K. In any case, by the lemma, there will be a point $\alpha^{(i)}$ in K satisfying

$$\gamma_i^{(i)} \leq \alpha_i^{(i)} \leq v(\{i\}) \ ,$$

for each $i \in I$. These points clearly dominate all of E - F. Hence

$$F - \text{dom } K = E - \text{dom } K \ ,$$

and K is <u>both</u> E-stable <u>and</u> F-stable. The result now follows by Theorem 2.

<u>4. Some examples</u>.

EXAMPLE 1. For two-person games, the unique solution under all four definitions is the set A. For inessential n-person games, also, the unique solution is A - i.e., the set consisting of the single point $(v(\{1\}), \ldots, v(\{n\}))$.

EXAMPLE 2. For the three-person constant-sum game in reduced form we list all* the G-stable sets:

(a) the triple $(\frac{1}{2}, -1, -1), (-1, \frac{1}{2}, -1), (-1, -1, \frac{1}{2})$;

(b) the set consisting of the point $(-2+2c, 1-c, 1-c)$ and the line segment $(c, t, -c-t), -1 \leq t \leq 1-c$, where c is an arbitrary constant satisfying $-1 < c < \frac{1}{2}$;

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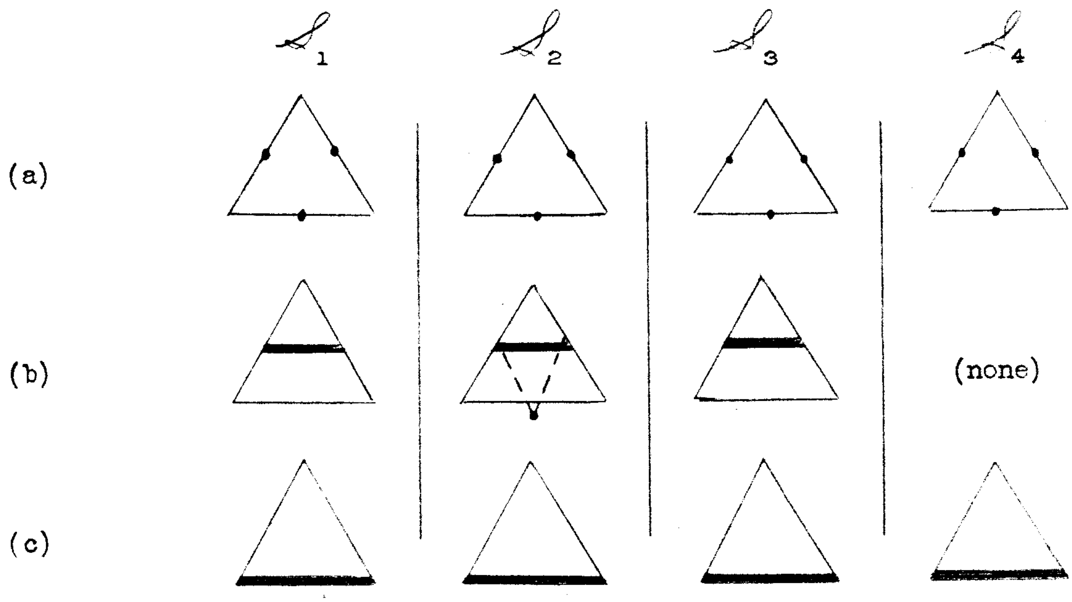
* The proof that our list is complete resembles the proof for A-stable sets in TGEB 32.

(c) the line segment $(-1, t, 1-t)$, $-1 \leq t \leq 2$;

(d) the sets resulting from those of (b) and (c) by permutation of the players.

We illustrate the different types of solutions which arise:



We observe that in this case, $\mathcal{S}_1 = \mathcal{S}_3$. The several definitions diverge only in the "incomplete exploitation" case, (b); the von Neumann-Morgenstern solutions of this type are not G-stable.

EXAMPLE 3. In a four-person constant-sum game with weak player (see RM-656) the characteristic point ω lies outside A . Hence no A-stable sets contain ω . However, it is easily shown that the set K :

$$K = \bigcup_{i \neq k} L_{i j_i} \qquad (k \text{ weak}, \quad j_i \neq i, k)$$

(in the notation of RM-656 page 7), is G-stable, even though K lies completely outside of A .* Hence, for these games, the empty set is a solution in the sense of $\mathcal{S}_3$.

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* A similar result holds for any "quota" game with weak player; see AMS-28 on this subject.

EXAMPLE 4. Consider the symmetric five-person "simple majority" game:

$$v(S) = \begin{cases} 0 & \text{if } s = 0, 1, 2, \\ 1 & \text{if } s = 3, 4, 5. \end{cases}$$

The 2-face of the 4-dimensional simplex A , consisting of points in A of the form

$$(0, 0, \alpha_3, \alpha_4, \alpha_5) ,$$

is both A-stable and G-stable. This solution is analogous to case (c) in Example 2. The triangular region K_c , of points in A of the form

$$(c, c, \alpha_3, \alpha_4, \alpha_5) ,$$

with $c > 0$, behaves somewhat like the line segment in case (b) of Example 2; thus:

(i) If $0 < c < 1/6$, then K_c is A-stable.

By Theorem 3 we see that K_c is not G-stable. However, if the two points (not in A)

$$\eta_c' = \left(\frac{4c-1}{2}, c, \frac{1-2c}{2}, \frac{1-2c}{2}, \frac{1-2c}{2} \right) ,$$

$$\eta_c'' = \left(c, \frac{4c-1}{2}, \frac{1-2c}{2}, \frac{1-2c}{2}, \frac{1-2c}{2} \right)$$

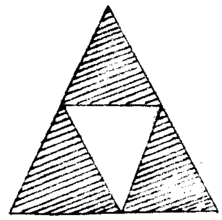
are added, the resulting set is G-stable. Hence K_c is in $\mathcal{S}_1$ and $\mathcal{S}_3$, and $K_c \cup \{\eta_c', \eta_c''\}$ is in $\mathcal{S}_2$.

(ii) If $1/6 \leq c < 1/4$, then K_c is still A-stable, and still not G-stable. The points η_c' and η_c'' now dominate part of K_c , so that $K_c \cup \{\eta_c', \eta_c''\}$ is no longer G-stable, nor is there any other extension of K_c which is G-stable. Hence K_c is in $\mathcal{S}_1 - \mathcal{S}_3$.

Define $L_c = K_c - \text{dom}\, \{\eta_c', \eta_c''\}$. For $c \geq 1/6$, L_c is that subset of K_c whose points satisfy

$$\max\, (\alpha_3, \alpha_4, \alpha_5) \geq (1-2c)/2 ;$$

geometrically L_c is the triangle K_c with the central quarter removed (see illustration).



It can be shown that $L_c \cup \{\eta_c', \eta_c''\}$ is G-stable. But L_c, being a subset of K_c, is certainly not A-stable. Hence L_c is in $\mathcal{S}_3 - \mathcal{S}_1$.

(iii) If $c = 1/4$ the points η_c', η_c'' are in the boundary of A, so the situation is analogous to case (a) of Example 2. The set $L_c \cup \{\eta_c', \eta_c''\}$ is both A-stable and G-stable, and is therefore a solution under all four definitions. This is an example of a solution which is not simply connected.

The solutions given here can be multiplied tenfold by considerations of symmetry. There are of course many other solutions. The use of this game to illustrate the difference between $\mathcal{S}_1$ and $\mathcal{S}_3$ was suggested by D. Gillies, who, with R. Bott, investigates games of this type in articles in AMS-28.

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