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RESEARCH MEMORANDUM

NOTES ON THE n-PERSON GAME, IV:
A THEOREM ON C-STABLE SETS

L. S. Shapley

RM-881

11 July 1952

Assigned to ____________________

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Summary: Solutions are obtained to the following set-equation, relevant to n-person game theory:

$$X = C - (X+R),$$

where C and R are respectively a convex set and a cone in an n-dimensional vector space, satisfying certain conditions.

NOTES ON THE n-PERSON GAME, IV: A THEOREM ON C-STABLE SETS

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The theory of stable-set solutions to n-person games* focuses attention on the somewhat more general question of determining conditions on C and R to ensure the existence of solutions to the equation

$$X = C - (X+R) \tag{1}$$

where X, C, and R are point sets in an n-dimensional vector space E^n; $0 \notin R$; and (X+R) is the set of points $x + r$, $x \in X$, $r \in R$.

Intuitively, C may be regarded as a space of alternatives, R as an intransitive preference rule, whereby y is preferred to x if and only if $x - y$ is in R, and X as a

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* See the third note of this series: "Some variants of the von Neumann-Morgenstern definition of solution", RM-817.

set of conventional choices. When X satisfies (1) we say that X is C-stable with respect to R. If R is a convex cone, then we get a partial ordering of the alternatives, and the only C-stable set is the set of maximal elements in C.

In this note we obtain solutions for (1) under the conditions that C be convex (in a strong sense described below), and that R be a cone, large enough so that every point will be comparable with points near any other point, but still permitting a hyperplane of mutually incomparable points.** The solutions obtained resemble the "discriminatory" solutions of von Neumann and Morgenstern.

If H is any hyperplane through the origin of E^n, we arbitrarily label the two associated open half-spaces H^+ and H^-. The translated hyperplane of points h + x, where x is a fixed vector perpendicular to H, and h ranges over H, we denote by H_c, where $|c|$ is the length of x and $c > 0$ if x is in H^+, $c < 0$ if x is in H^-. H_c^+ and H_c^- denote the corresponding translations of the half-spaces H^+ and H^-. The closure of a set A is denoted $\overline{A}$, its boundary ∂A.

** For example, the domination cone of the constant-sum three-person game has this property.

A closed set is said to be *strictly* convex if the open segment connecting any two of its points is in the interior of the set.

Theorem: *Let* C *be bounded and convex; let* $\bar{C}$ *and its dual be strictly convex;*[***] *let* H *be a hyperplane through the origin, and let* R *be a cone which intersects* H^{+} *and* H^{-}*, but not* H*, and whose closure contains at least one of* x *and* $-x$ *for every* x *in* E^{n}*. Then* (1) *has solutions of the form* $X = H_c \cap C$.

Proof: Define

$$X_c = H_c^{-} \cap (C - \partial C).$$

Let a be the greatest lower bound of those c for which X_c is not empty. The set (X_c+R) is open. For any point z in (X_c+R) can be written $z = x + r$, for some $x \in X_c$, $r \in R$. Then the points of the form $x' + \alpha r$, for $x' \in X_c$, $\alpha > 0$ comprise an open set containing z and contained in (X_c+R).

Lemma A: For all $c > a$ sufficiently close to a,

(2) $$(X_c+R) \supseteq H_c^{-} \cap \bar{C}.$$

*** Equivalently, at each boundary point there is a unique supporting hyperplane, and in each supporting hyperplane there is a unique boundary point. It follows that C is n-dimensional. C need not be closed.

<u>Proof of Lemma A</u>: Select $r \in R \cap H^-$ and let $\theta \leq \pi/2$ be the angle which r makes with the (negative) normal to H. Consider the set of all supporting hyperplanes to C whose normals (pointing out from C) make angles of less than $\pi/2 - \theta$ with the negative normal to H; this set includes of course the supporting hyperplane H_a. The contact points of these hyperplanes with ∂C form a set U which is open in ∂C, since the supporting hyperplane is a continuous function of its contact point. It is possible to find a number $c' > a$ such that

$$(3) \qquad \bar{X}_c \cap \partial C \subseteq U \qquad \text{all } c, \quad a < c < c'.$$

For otherwise we would have a sequence of points x_i, not in U:

$$x_i \in \bar{X}_{c_i} \cap \partial C, \qquad \{c_i\} \to a,$$

which clearly must converge to a point in $H_a \cap \bar{C} \subset U$. Granting (3), let y be any point in $H_c^- \cap \bar{C}$, $a < c < c'$. Project y parallel to r into a point $y' \in H_c$ (see the figure).

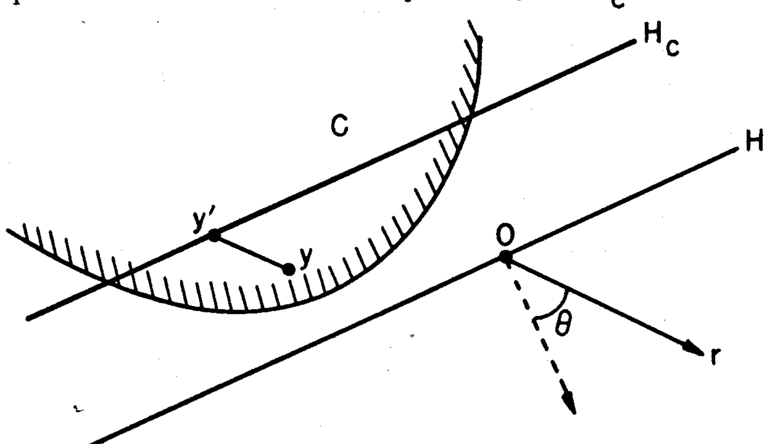

Fig. 1

It is clear that $y \in (\{y'\} + R)$; we must therefore only show that $y' \in X_c$, that is, that y' is interior to C. But if y' is not interior, then some point of the segment joining y and y' must be in ∂C. The supporting hyperplane at this point will then make an angle of more than $\pi/2 - \theta$ with H, giving us a contradiction. This completes the proof of the lemma.

Let c_o be the least upper bound of those c' such that (2) holds for all $c < c'$. There must be at least one point z_o in $H^-_{c_o} \cap C$ not contained in $(X_{c_o} + R)$. This is true because the complement of $(X_{c_o} + R)$ is a closed set which varies continuously with c.

Lemma B: For c_o as just defined,

$$(X_{c_o} + R) \supseteq H^+_{c_o} \cap \overline{C}. \tag{4}$$

Proof of Lemma B: Take $y \in H^+_{c_o} \cap \overline{C}$. The segment joining y and z_o meets H_{c_o} in a point x_o. By the strict convexity of $\overline{C}$, x_o is interior to C, hence $x_o \in X_{c_o}$. By the definition of z_o, the vector $s = z_o - x_o$ is not in R. For the same reason, the open set S of vectors of the form $\alpha(z_o - x)$, $x \in X_c$, $\alpha > 0$, is also outside R, and S contains s. By our assumption on R, we must therefore have a sequence of vectors $\{r_i\} \longrightarrow -s$ in R. We can use these to determine a sequence $\{x_i\} \longrightarrow x_o$ as follows:

$$x_i = y - \alpha_i r_i, \qquad x_i \in H_{c_o}, \quad \alpha_i > 0.$$

Since X_{c_o} is open in H_{c_o}, the x_i will eventually lie in X_{c_o}. It follows that $y \in (X_{c_o}+R)$, as was to be shown.

Using again the continuity of the complement of (X_c+R) as a function of c, we see that (4) must hold over at least a small interval of values of c less than c_o, say, for $c_1 < c < c_o$. But within this interval, Lemma A applies. Therefore, for such a value of c, every point in C lies either in (X_c+R) or in $H_c \cap C$. This remains true if we replace X_c by its closure in C, that is, by $H_c \cap C$. Therefore,

$$H_c \cap C \supseteq C - ([H_c \cap C]+R) \qquad (c_1 < c < c_o).$$

The reverse statement:

$$H_c \cap C \subseteq C - ([H_c \cap C]+R)$$

follows immediately from the fact that R and H are disjoint. This completes the proof.