MORE ON GAMES OF SURVIVAL

M. P. Peisakoff

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SUMMARY. Let $\mathcal{N}$ be the game of survival which is the repetition (until the bankruptcy of one of the players) of a normalized finite zero-sum two-person game, $\Gamma = (\Gamma_{ij})$, where each $\Gamma_{ij}$ is a non-zero integer. It is shown that $\mathcal{N}$ is inessential and has some easily described optimal strategies. It is also shown that if $\max_{i,j} |\Gamma_{ij}|$ is small enough compared to the combined fortunes, then playing at the $n$-th play a $\delta^n$-optimal strategy for $\Gamma$ is an $\varepsilon$-optimal strategy for $\mathcal{N}$, if $\delta$ is sufficiently small.

MORE ON GAMES OF SURVIVAL

M. P. Feistakoff

We are interested in the games $\{\mathcal{N}(f_1, f_2)\}$ in which two players with finite fortunes, $f_1$ and $f_2$, respectively, in chips repeat a normalized finite zero-sum two-person game, $\Gamma = (\Gamma_{ij}|(1, 1) \leq (1, j) \leq (i_0, j_0))$. At least one of $f_1$ and $f_2$ is positive and play is continued until the fortune of one of the players is non-positive, or ad infinitum if this never occurs. The payoff in money is $(1, 0)$ if player 2 ends with a non-positive fortune and $(0, 1)$ if player 1 ends with a non-positive fortune. If the game goes on indefinitely, then the payoff is $(\alpha(C), \beta(C))$, which can depend on the course of the game, $C$, but which satisfies $(\alpha(C), \beta(C)) \leq (1, 1)$ and $\alpha(C) + \beta(C) \leq 1$. We shall show that if all the $\Gamma_{ij}$'s are non-zero integers, then $\mathcal{N}$ is inessential and has some easily described optimal strategies. (In an inessential game, an optimal strategy for a player is one which secures for the player the maximum amount he can insure for himself. An $\varepsilon$-optimal strategy secures for

*An inessential game is one in which the players together can secure only the sum of the (minorant) amounts insurable without cooperation.
him at least that amount less $\varepsilon$.) We shall also show that if $\max_{i,j} |r_{ij}|$ is small enough compared to the combined fortunes, then playing at the n-th play a $\delta^n$-optimal strategy for $\Gamma$ is an $\varepsilon$-optimal strategy for $\Omega$, if $\delta$ is sufficiently small. ($\delta^n$ is the n-th power of $\delta$.)

We assume that every column of $\Gamma$ has a positive entry, and every row has a negative entry. Otherwise, there would be a negative column or a positive row. In the first case, player 2 can always force player 1's fortune to become non-positive, by playing the negative column repeatedly. In the second case, player 1 can force player 2's fortune to become non-positive by playing the positive row repeatedly.

Let $\bar{R}^{(n)}(f_1, f_2)$ be the game in which the two players repeat $\Gamma$ n times, or until one of the players has a non-positive fortune if this occurs first. The payoff in money is (0, 1) if player 1 ends with a non-positive fortune, and (1, 0) otherwise. $\bar{R}^{(n)}(f_1, f_2)$ is a constant-sum two-person game with value, say, $(\nu^{(n)}(f_1, f_2), 1 - \nu^{(n)}(f_1, f_2))$. We observe

(1) Player 2 can always win as much money in $\bar{R}^{(n+1)}$ as in $\bar{R}^{(n)}$ by playing a $\bar{R}^{(n)}$-optimal strategy during the first n moves of $\bar{R}^{(n+1)}$ and arbitrarily on the $(n + 1)$th move. Hence

$$\nu^{(n)}(f_1, f_2) \geq \nu^{(n+1)}(f_1, f_2).$$

(2) Since each column has a positive entry, by repeatedly playing the strategy which assigns each pure strategy probability $1/i$, player 1 insures that no matter what player 2 does, player 2's fortune will decrease each time with probability
at least $1/i_0$. Player 1 thereby insures that with probability
at least $i_0^{-[f_2]-1}$, player 2 will be bankrupted in at most
$[f_2] + 1$ trials. ($[f_2]$ is the largest integer not larger than
$f_2$.) Hence if $n \geq [f_2] + 1$ and $(f_1, f_2) > (0, 0)$, then

$$\nu{(n)}(f_1, f_2) \in [\delta, 1]$$

where $\delta = i_0^{-[f_2]-1}$. By definition we have also

$$\nu{(n)}(f_1, f_2) = 0 \text{ if } f_1 \leq 0$$

and

$$\nu{(n)}(f_1, f_2) = 1 \text{ if } f_2 \leq 0.$$  

(3) Let $G(\Delta)$ be the game value of $\Delta$, for each game $\Delta$. If $(f_1, f_2) > 0$, after one move of $\bar{F}{(n+1)}(f_1, f_2)$, the players are playing $\bar{F}{(n)}(f_1 + \delta_{ij}, f_2 - \delta_{ij})$. Hence

$$\nu{(n+1)}(f_1, f_2) = G\left(\nu{(n)}(f_1 + \delta_{ij}, f_2 - \delta_{ij})\right).$$

(4) Let $\epsilon > 0$. Player 1 can always win as much in $\bar{F}{(n)}(f_1 + \epsilon, f_2 - \epsilon)$ as in $\bar{F}{(n)}(f_1, f_2)$. Hence

$$\nu{(n)}(f_1 + \epsilon, f_2 - \epsilon) \geq \nu{(n)}(f_1, f_2).$$

We can now conclude:

(A) From (1) and (2),
\( \tau^{(n)}(f_1, f_2) \rightarrow \bar{\nu}(f_1, f_2) \in [0, 1] \) if \((f_1, f_2) > (0, 0)\)

\[ = \begin{cases} 0 & \text{if } f_1 \leq 0 \\ 1 & \text{if } f_2 \leq 0 \end{cases} \]

(B) From (3), if \((f_1, f_2) > (0, 0),\)

\[ \bar{\nu}(f_1, f_2) = c(\bar{\nu}(f_1 + \Gamma_{i+j}, f_2 - \Gamma_{i+j})). \]

(C) From (4), for \(\xi \geq 0,\)

\[ \bar{\nu}(f_1 + \xi, f_2 - \xi) \geq \bar{\nu}(f_1, f_2). \]

Definition: A strategy for player 1 is called conditionally optimal if the conditional distribution of his strategy at any play of \(\Gamma_i\), given the course of the game up to that play, is an optimal strategy for the game \((\nu(\phi_1 + \Gamma_{i+j}, \phi_2 - \Gamma_{i+j}))\) where \((\phi_1, \phi_2)\) is the fortune distribution immediately before the play in question.

Lemma 1: If player 1's strategy is conditionally optimal, and if with probability one the fortune of one of the players (not necessarily always the same one) eventually becomes non-positive, then player 1 can expect at least \(\bar{\nu}(f_1, f_2)\) in payoff.

Proof: It is sufficient to show that the probability that player 2's fortune becomes non-positive is at least \(\bar{\nu}(f_1, f_2)\). Let \(((F_1^n, F_2^n) | n \geq 1)\) be the random variable of fortunes at play \(n\), where if the game ends at play \(N, (F_1^{N+j}, F_2^{N+j}) \equiv (F_1^N, F_2^N)\) for \(j \geq 1\). Then since player 1's
strategy is conditionally optimal, if \((p_1^n, p_2^n) > (0, 0)\),

\[
E\tilde{V}(p_1^{n+1}, p_2^{n+1}) \geq E\tilde{V}(p_1^n + \Gamma_{1j}^n, p_2^n - \Gamma_{1j}^n)
\]

\[
= E\tilde{V}(p_1^n, p_2^n),
\]

while

\[
E\tilde{V}(p_1^{n+1}, p_2^{n+1}) = E\tilde{V}(p_1^n, p_2^n)
\]

otherwise. Hence, by induction,

\[
E\tilde{V}(p_1^n, p_2^n) \geq \tilde{V}(f_1, f_2).
\]

Let \(\{(p_1^n, p_2^n) \mid n \geq 1\}\) be the random variable which is

- \((0, 0)\) if neither player's fortune is non-positive by the end of the \(n\)-th play,
- \((0, 1)\) if the first player's fortune is non-positive by the end of the \(n\)-th play,
- \((1, 0)\) if the second player's fortune is non-positive by the end of the \(n\)-th play.

Then

\[
E\tilde{V}(p_1^n, p_2^n) \leq E\tilde{p}_1^n + E(1 - p_2^n - p_1^n).
\]

But by assumption the second term on the right tends to zero. Hence,

where \(\xi_n \to 0\).
\[
\mathbb{E} \mu_1 + \varepsilon_n \geq \bar{\nu}(f_1, f_2)
\]

which is the desired result.

Lemma 2: There is a conditionally optimal strategy for the first player which insures that the probability that the game ends by the \(n\)-th play tends to one as \(n\) tends to \(\infty\), uniformly in the opponent's strategy.

Proof: First we show that for each \((\phi_1, \phi_2) > (0, 0)\), there is an optimal strategy \(I\) for the first player for the game \((\bar{\nu}(\phi_1 + \Gamma_{ij}, \phi_2 - \Gamma_{ij})\)

such that for all \(J\), \(\Pr\{\Gamma_{ij} > 0\} > 0\). Suppose, to the contrary, that for some \((\phi_1, \phi_2) > (0, 0)\), for all optimal \(I\), there is a \(J\) such that \(\Pr\{\Gamma_{ij} > 0\} = 0\), or what is the same since \(\Gamma_{ij} > 0\), \(\Pr\{\Gamma_{ij} < 0\} = 1\). Then since player 1 is playing optimally,

\[
\bar{\nu}(\phi_1, \phi_2) \leq \mathbb{E} \bar{\nu}(\phi_1 + \Gamma_{ij}, \phi_2 - \Gamma_{ij})
\]

From the monotonicity of \(\bar{\nu}(\phi_1 + \varepsilon, \phi_2 - \varepsilon)\),

\[
\bar{\nu}(\phi_1, \phi_2) \geq \bar{\nu}(\phi_1 + \Gamma_{ij}, \phi_2 - \Gamma_{ij})
\]

Combining,

\[
\bar{\nu}(\phi_1, \phi_2) = \bar{\nu}(\phi_1 + \Gamma_{ij}, \phi_2 - \Gamma_{ij})
\]

or weaker, from monotonicity again,

\[
\bar{\nu}(\phi_1, \phi_2) = \bar{\nu}(\phi_1 - 1, \phi_2 + 1)
\]

If \((\phi_1 - 1, \phi_2 + 1) > (0, 0)\), this implies that an optimal strategy \(I\) for
the first player for the game \((\bar{v}(d_1 + G_{1j} - 1, d_2 - G_{1j} + 1))\) is an optimal strategy for \((\bar{v}(d_1 + G_{1j}, d_2 - G_{1j}))\), since by using it against any \(J\), the first player insures himself

\[
\bar{v}(d_1 + G_{1j}, d_2) \geq \bar{v}(d_1 + G_{1j} - 1, d_2 - G_{1j} + 1)
\]

\[
\geq \bar{v}(d_1 - 1, d_2 + 1)
\]

\[
= \bar{v}(d_1, d_2).
\]

Thus for a fortune division \((d_1 - 1, d_2 + 1) > (0, 0)\), and by induction for a fortune division, \((d_1 - n, d_2 + n) > (0, 0)\), for all optimal strategies, \(I\), there is a \(J\) such that \(G_{IJ} < 0\). But eventually, perhaps for \(n = 0\),

\[
(d_1 - n, d_2 + n) > (0, 0)
\]

while \(d_1 \leq n + 1\). Therefore, for an optimal \(I\) and some \(J\),

\[
0 < G \leq v(d_1 - n, d_2 + n) \leq \bar{v}(d_1 - n + G_{IJ}, d_2 + n - G_{IJ})
\]

\[
\leq \bar{v}(d_1 - n - 1, d_2 + n + 1)
\]

\[
= 0,
\]

which is the contradiction we were looking for.

We have now proved that for \((d_1, d_2) > 0\), there is an optimal \(I\) such that for all \(J\), \(Pr \{G_{IJ} > 0\} > 0\). For each \((d_1, d_2) > (0, 0)\), fix such an \(I\). Call it \(I(d_1, d_2)\). From the compactness of the second player's set of strategies and the fact that \(Pr \{G_{IJ} > 0\}\) is a continuous
function of his strategy, \( \Pr \{ f_{ij} > 0 \} \geq \rho(\phi_1, \phi_2) > 0 \). Define \( c(\phi_1, \phi_2) = \min \rho(\phi_1 + k, \phi_2 - k) > 0 \), where \( k \) is an arbitrary positive, zero, or negative integer such that \((\phi_1 + k, \phi_2 - k) > (0, 0)\).

Let now player 1 use the conditionally optimal strategy which consists of playing \( I(\phi_1, \phi_2) \) when the fortune distribution is \((\phi_1, \phi_2)\). Let \( Q^{(n)} \) be the probability that one or the other player's fortune is exhausted on or before the \( n \)-th play. Then, where \( \sigma = c(f_1, f_2) \),

\[
Q^{(n+1)} \geq Q^{(n)} + (1 - Q^{(n)}) c[f_1 + f_2 + 1].
\]

By induction,

\[
Q^{(N+1)} \geq 1 - (1 - c[f_1 + f_2 + 1])^{N-1}.
\]

Hence \( Q^{(N)} \rightarrow 1 \) as \( N \rightarrow \infty \), which is the lemma.

Let \( \Omega^{(n)}(f_1, f_2) \) be the game in which the two players repeat \( \rho \) \( n \)-times, or until one of the players has a non-positive fortune if this occurs first, and the money payoff is \((1, 0)\) if player 2 ends with a non-positive fortune, \((0, 1)\) otherwise. \( \Omega^{(n)}(f_1, f_2) \) is a constant-sum two-person game with value \((v^{(n)}(f_1, f_2), 1 - v^{(n)}(f_1, f_2))\). Obviously

\[
v^{(n)}(f_1, f_2) \leq v^{(n)}(f_1, f_2),
\]

since any strategy for player 1 in \( \Omega^{(n)}(f_1, f_2) \), will insure him as much money in \( \overline{\Omega}^{(n)}(f_1, f_2) \). We therefore conclude, by the same reasoning as
earlier,

\[(A) \quad \nu^{(n)}(f_1, f_2) \rightarrow \nu(f_1, f_2) \in [0, 1 - \delta'] \quad \text{if} \quad (f_1, f_2) > (0, 0) \]

\[= 0 \quad \text{if} \quad f_1 \leq 0 \]

\[= 1 \quad \text{if} \quad f_2 \leq 0 \]

where \(\delta' > 0\).

\[(B) \quad \text{If} \quad (f_1, f_2) > (0, 0), \]

\[\nu(f_1, f_2) = G(\nu(f_1 + \Gamma_{ij}, f_2 - \Gamma_{ij})) . \]

\[(C) \quad \text{If} \quad \epsilon \geq 0, \]

\[\nu(f_1 + \epsilon, f_2 - \epsilon) \geq \nu(f_1, f_2) . \]

In addition

\[(D) \quad \nu(f_1, f_2) \leq \overline{\nu}(f_1, f_2) . \]

**Definition:** A strategy for player 2 is called conditionally optimal if the conditional distribution of his strategy at any play of \(\Gamma_i\), given the course of the game up to that play, is an optimal strategy for the game \((\nu(\phi_1 + \Gamma_{ij}, \phi_2 - \Gamma_{ij}))\) where \((\phi_1, \phi_2)\) is the fortune distribution immediately before the play in question.

From Lemmas 1 and 2, and the analogous Lemmas 1 and 2 which we do not write down, we conclude that each player has a conditionally optimal strategy which insures that play ends by the n-th play with probability
tending to 1 as \( n \) tends to \( \infty \), uniformly in the opponent's strategy.

The first player's strategy insures him \( \bar{v}(f_1, f_2) \) on the average and the second player's strategy insures him \( 1 - \bar{v}(f_1, f_2) \geq 1 - \bar{v}(f_1, f_2) \), on the average. Since together the players can win no more than 1, we get

\[
1 \geq \bar{v}(f_1, f_2) + (1 - \bar{v}(f_1, f_2)) \\
\geq \bar{v}(f_1, f_2) + (1 - \bar{v}(f_1, f_2)) \\
= 1.
\]

This means \( v(f_1, f_2) = \bar{v}(f_1, f_2) \) (say) \( v(f_1, f_2) \), and \( \cap(\varphi_1, \varphi_2) \) is inessential with the solution \( \{ (v(f_1, f_2), 1 - v(f_1, f_2)) \} \).

\( v \) can be characterized as the unique solution of

\[
0 \leq v(\varphi_1, \varphi_2) = G(v(\varphi_1 + f_1, \varphi_2 - f_1)) \leq 1 \\
\text{if } (f_1, f_2) > (0, 0) \\
= 0 \text{ if } f_1 \leq 0 \\
= 1 \text{ if } f_2 \leq 0.
\]

For if \( v^* \) is a solution,

\[
v^{(0)}(\varphi_1, \varphi_2) \leq v^*(\varphi_1, \varphi_2) \leq v^{(0)}(\varphi_1, \varphi_2)
\]

by definition, and so by induction, using \((A)\), \((B)\), \((A)\), and \((B)\),

\[
v^{(n)}(\varphi_1, \varphi_2) \leq v^*(\varphi_1, \varphi_2) \leq v^{(n)}(\varphi_1, \varphi_2).
\]

Hence

\[
v(\varphi_1, \varphi_2) = v(\varphi_1, \varphi_2) \leq v^*(\varphi_1, \varphi_2) \leq v(\varphi_1, \varphi_2) = v(\varphi_1, \varphi_2),
\]
giving

\[ v(\phi_1, \phi_2) = v^*(\phi_1, \phi_2), \]

as was to be proved.

We thus have

**Theorem 1:** \( \Lambda(f_1, f_2) \) is inessential with the solution

\[ \{ (v(f_1, f_2), 1 - v(f_1, f_2)) \} \]

where \( v \) is the unique solution

in \( \{ (\phi_1, \phi_2) | \phi_1 > 0 \text{ or } \phi_2 > 0 \} \) of

\[ 0 \leq v(\phi_1, \phi_2) = G(v(\phi_1 + r_{ij}, \phi_2 - r_{ij})) \leq 1 \]

if \((\phi_1, \phi_2) > (0, 0)\)

\[ = 0 \] if \( \phi_1 \leq 0 \)

\[ = 1 \] if \( \phi_2 \leq 0 \).

Each player has a conditionally optimal strategy which is optimal and which insures that play ends by the \( n \)-th play with probability tending to one uniformly in the opponent's strategies.

Let us turn now to the problem of effectively computing an \( \varepsilon \)-optimal strategy for \( \Lambda(f_1, f_2) \). This is easy if we are not interested in efficiency. Namely, we need only find an \( n \) such that

\[ \bar{v}(n)(f_1, f_2) - v(n)(f_1, f_2) \leq \varepsilon - \delta \]

where \( \delta > 0 \). Then a \( \delta \)-optimal strategy for the first player for
\( \mathcal{U}^{(n)}(f_1, f_2) \) provides an \( \epsilon \)-optimal strategy for him for \( \mathcal{U}(f_1, f_2) \). Namely, he can use the strategy on the first \( n \) moves of \( \mathcal{U}(f_1, f_2) \) and act arbitrarily thereafter. Similarly, a \( \delta \)-optimal strategy for the second player for \( \mathcal{U}^{(n)}(f_1, f_2) \) provides an \( \epsilon \)-optimal strategy for him for \( \mathcal{U}(f_1, f_2) \).

If \( \max_{i,j} |\gamma_{ij}| \) is small enough compared to \( f_1 \) and \( f_2 \), there is another class of interesting \( \epsilon \)-optimal strategies. Repeatedly playing an optimal strategy for \( \Gamma \) is an \( \epsilon \)-optimal strategy for \( \mathcal{U} \). More precisely, let us remove the restriction that each \( \gamma_{ij} \) be a non-zero integer. Let us require instead, say, that \( G(\Gamma) \geq 0 \) and that for some optimal strategy \( I \), \( \Pr \{ \gamma_{ij} > 0 \} > 0 \) for all \( j \). If \( G(\Gamma) = 0 \), we require in addition that for some optimal \( J \), \( \Pr \{ \gamma_{ij} < 0 \} > 0 \) for all \( i \). Define

\[ \alpha = G(\Gamma), \quad \beta = \min_j \Pr \{ \gamma_{ij} > 0 \}, \quad \gamma = \max_{i,j} |\gamma_{ij}|. \]

We assume that both \( f_1 \) and \( f_2 \) are positive and define \( f = f_1 + f_2 \). Define for \( \alpha = 0 \)

\[
\Pr(\phi_1) = \begin{cases} 
\frac{1}{f + \gamma} \phi_1 & \text{if } 0 < \phi_1 < f \\
0 & \text{if } \phi_1 \leq 0 \\
1 & \text{if } \phi_1 \geq f
\end{cases}
\]

and for \( \alpha > 0 \)
\[ p_{\alpha}(\phi_1) = \frac{1 - \exp \left\{ -\frac{\alpha}{\gamma^2} \phi_1 \right\}}{1 - \exp \left\{ -\frac{\alpha}{\gamma^2} (f + \gamma) \right\}} \quad \text{if } 0 < \phi_1 < f \]

\[ = 0 \quad \text{if } \phi_1 \leq 0 \]

\[ = 1 \quad \text{if } \phi_1 \geq f . \]

Lemma 3: If player 1 plays \textbf{I} repeatedly, then he can expect at least \( p_{\alpha}(f_1) \) in payoff. (\( I \) is any optimal strategy for \( \Gamma \) satisfying \( \Pr (\Gamma_{I} > 0) > 0 \).)

Proof: Since \( \beta > 0 \), by the method of proof of Lemma 2, it follows that if player 1 plays \textbf{I} repeatedly, the probability that the game ends by the \( n \)-th play tends to one as \( n \) tends to \( \infty \). Hence, in order to prove Lemma 3, it is sufficient to show that for all \( N \),

\[ \mathbb{E}p_{\alpha}(F_1^N) \geq p_{\alpha}(f_1) . \]

By induction, this would follow from

\[ \mathbb{E}\left\{ p_{\alpha}(F_1^{N+1}) \mid F_1^N \right\} \geq p_{\alpha}(F_1^N) . \]

We prove the latter.

Suppose first that \( \alpha = 0 \). If \( 0 < F_1^N < f \), then for all \((i, j)\), since \( \gamma_{ij} \leq \gamma \),

\[ p_{0}(F_1^N + \gamma_{ij}) \geq \frac{1}{f + \gamma_{ij}} (F_1^N + \gamma_{ij}) . \]
Hence, if $0 < F_i^N < f$,
\[
E \left\{ p_0 (F_i^{N+1}) | F_i^N \right\} \geq \min_j E_p (F_i^N + r_{ij})
\]
\[
\geq \frac{1}{f + \gamma} \min_j E (F_i^N + r_{ij})
\]
\[
\geq \frac{1}{f + \gamma} F_i^N
\]
\[
= p_0 (F_i^N).
\]

Since if $F_i^N \leq 0$ or $F_i^N \geq f$ our proposition is trivial, we have disposed of the case $\alpha = 0$.

Suppose now that $\alpha > 0$. Again we need only consider $0 < F_i^N < f$. Then
\[
p_0 (F_i^N + r_{ij}) \geq \frac{1 - \exp \left\{ - \frac{\alpha}{\gamma^2} (F_i^N + r_{ij}) \right\}}{1 - \exp \left\{ - \frac{\alpha}{\gamma^2} (f + \gamma) \right\}}.
\]

Hence,
\[
E \left\{ p_0 (F_i^{N+1}) | F_i^N \right\} \geq \min_j E_p (F_i^N + r_{ij})
\]
\[
\geq \min_j \frac{1 - E \exp \left\{ - \frac{\alpha}{\gamma^2} (F_i^N + r_{ij}) \right\}}{1 - \exp \left\{ - \frac{\alpha}{\gamma^2} (f + \gamma) \right\}}
\]
\[
\geq \frac{1 - M \exp \left\{ - \frac{\alpha}{\gamma^2} F_i^N \right\}}{1 - \exp \left\{ - \frac{\alpha}{\gamma^2} (f + \gamma) \right\}}
\]
where

\[ M = \max_j E \exp \left\{ -\frac{\alpha}{\gamma^2} \Gamma_{ij} \right\} \]

\[ \leq \max_j \left\{ 1 - \frac{\alpha}{\gamma^2} E \Gamma_{ij} + (e - 2) \left( \frac{\alpha}{\gamma} \right)^2 \right\} \]

\[ \leq \left\{ 1 - \frac{\alpha}{\gamma^2} + (e - 2) \frac{\alpha^2}{\gamma^2} \right\} \]

\[ < 1. \]

Hence

\[ E \left\{ p_\alpha(F^{N+1}) | F^N \right\} \geq \frac{1 - \exp \left\{ -\frac{\alpha}{\gamma^2} F^N \right\}}{1 - \exp \left\{ -\frac{\alpha}{\gamma^2} (f + \gamma) \right\}} \]

\[ = p_\alpha(F^N), \]

as was to be proved.

By symmetry, if \( \alpha = 0 \) we conclude

Lemma 3. If \( \alpha = 0 \), and if player 2 plays \( J \) repeatedly, then he can expect at least \( p_0(f_2) \) in payoff. (\( J \) is any optimal strategy for \( \Gamma \) satisfying \( \Pr(\Gamma_{ij} < 0) > 0 \).)

If \( \alpha = 0 \), Lemmas 3 and 2 give us, whenever \( \Lambda(f_1, f_2) \) is inessential with solution \( \{(v(f_1, f_2), 1 - v(f_1, f_2))\} \),

\[ p_0(f_1) \leq v(f_1, f_2) \leq 1 - p_0(f_2) = p_0(f_1) + \frac{\gamma}{f + \gamma}. \]
Thus repeating I is \( \left(\frac{\gamma}{1 + \gamma}\right) \)-optimal for player 1, and repeating J is \( \left(\frac{\gamma}{1 + \gamma}\right) \)-optimal for player 2. If \( \alpha > 0 \), Lemma \( \delta \) gives us whenever \( \Omega(f_1, f_2) \) is inessential with solution \( \{(v(f_1, f_2), 1 - v(f_1, f_2))\} \),

\[
1 - \exp\left\{-\frac{\alpha}{\gamma^2} f_1 \right\} \leq p_\alpha(f_1) \leq v(f_1, f_2) \leq 1 .
\]

Thus repeating I is \( \exp\left\{-\frac{\alpha}{\gamma^2} f_1 \right\} \)-optimal for player 1 and any strategy is \( \exp\left\{-\frac{\alpha}{\gamma^2} f_1 \right\} \)-optimal for player 2.

What if, instead of repeating I, player 1 repeated a \( \delta \)-optimal I \( \delta \), where \( \delta \) is the smallest number for which I \( \delta \) is \( \delta \)-optimal? If \( \alpha > \delta \) no great harm is done, since it can be verified by precisely the proof given above that this is an \( \exp\left\{-\frac{\alpha - \delta}{\gamma^2} f_1 \right\} \)-optimal strategy for player 1. If, however, \( \alpha < \delta \), player 2 could expect at least \( 1 - \exp\left\{-\frac{\delta - \alpha}{\gamma^2} f_2 \right\} \) in payoff. When \( \frac{\delta - \alpha}{\gamma^2} f_2 \) is large, this payoff is close to one, so I \( \delta \) is not a good strategy. Thus if \( \alpha = 0 \), no matter how small \( \gamma \) is, it is not enough to repeat a \( \delta \)-optimal strategy for sufficiently small \( \delta \). On the other hand, suppose that \( (I^n) \) is a sequence of strategies for player 1 whose \( n \)-th member is \( \delta^n \)-optimal for \( \gamma \) and satisfies

\[
\min_j \Pr \{I^n_j > 0\} \geq \beta' > 0 ,
\]

where \( \beta' \) does not depend on \( n \). Then
Lemma 4: If $\alpha = 0$ and player 1 plays $I_n$ at the $n$-th stage, then he can expect at least $p_\alpha(f_1) - \frac{\delta}{(1 - \delta)(f + \gamma)}$ in payoff.

Proof: The proof is almost identical with that of Lemma 3 where instead of proving

$$\text{Ep}_\alpha(F_1^N) \geq p_\alpha(f_1)$$

one proves

$$\text{Ep}_\alpha(F_1^N) \geq p_\alpha(f_1) - \frac{\delta + \ldots + \delta^{N-1}}{f + \gamma}.$$ 

It is an easy step (left to the reader) now to

Theorem 2. If $G(\Gamma) = \alpha > \delta$ and $\mathcal{N}(f_1, f_2)$ is inessential, repeating a strategy which is $\delta$-optimal for $\Gamma$ is $\exp \left\{ \frac{-\alpha - \delta f_2}{\gamma^2} \right\}$-optimal for $\mathcal{N}(f_1, f_2)$. Let $G(\Gamma) = 0$, and let $(I_n)$ be a sequence of strategies for player 1 whose $n$-th member is $\delta^n$-optimal for $\Gamma$ and satisfies

$$\min_j \Pr \{ \Gamma_{I_nj} > 0 \} \geq \beta' > 0,$$

where $\beta'$ does not depend on $n$. Then playing $I_n$ at the $n$-th stage is a \((\frac{\gamma}{f + \gamma} + \frac{2\delta}{(1 - \delta)(f + \gamma)})\)-optimal strategy for player 2.
The reader will observe that when each $\Gamma_{ij} \neq 0$, say
$|\Gamma_{ij}| \geq \delta$, we automatically have for a $\delta^n$-optimal $I_n$, when $\delta$
is sufficiently small,

$$\min_j \Pr \left\{ \Gamma_{I_nj} > 0 \right\} \geq \frac{\delta^n}{\delta + \delta} \geq \frac{\delta}{\delta + \delta} > 0.$$ 

In closing, we wish to point out that the method of
proof leading to Theorem 1 is trivially sufficient to handle
the following generalized game of survival in which the result
of a play is a random state instead of a definite number.
However, the method is apparently insufficient to handle more
than a finite number of possible states, or the possibility
of "zeros." A finite set $\Sigma$ with two distinguished points $\sigma_1$
and $\sigma_2$ is given. $\Sigma$ is partially ordered by $<$, which satisfies
for some fixed $n$ and all $\{x_i | 1 \leq i \leq n\},$

$$x_1 < x_2 < \cdots < x_{n-1} < x_n \rightarrow x_1 = \sigma_2, x_n = \sigma_1.$$ 

For each $x \in \Sigma$, there is a set of random variables on $\Sigma$,
$\{Y_{ij}(x) | 1 \leq i \leq i_0, 1 \leq j \leq j_0\}$ such that for all $i$ and $j$
$Y_{ij}(\sigma_1) = \sigma_1, Y_{ij}(\sigma_2) = \sigma_2$, and for $x \neq \sigma_1, \sigma_2$

$$\Pr \{Y_{ij}(x) < x\} = 0 \rightarrow \Pr \{x < Y_{ij}(x)\} = 1$$

$$\Pr \{x < Y_{ij}(x)\} = 0 \rightarrow \Pr \{Y_{ij}(x) < x\} = 1.$$
In addition, for \( x \neq \sigma_1, \sigma_2 \), for each \( i \), there is a \( j \) such that

\[
\Pr \left\{ Y_{ij}(x) < x \right\} > 0 ,
\]

and for each \( j \), there is an \( i \) such that

\[
\Pr \left\{ x < Y_{ij}(x) \right\} > 0 .
\]

Define \( \prod_{n=1}^{N} Y_{i_n j_n}^{(n)}(x) \) by induction by

\[
\prod_{n=1}^{M+1} Y_{i_{M+1} j_{M+1}}^{(n)}(x) = \prod_{n=1}^{M} Y_{i_{M+1} j_{M+1}}^{(n)} \prod_{n=1}^{M} Y_{i_{M+1} j_{M+1}}^{(n)}(x) ,
\]

where \( \left\{ (Y_{ij}^{(n)}(x)|1 \leq i \leq i_0, 1 \leq j \leq j_0, x \in \Sigma) \right\} \) is a set of independent random variables, each distributed like \( (Y_{ij}(x)) \).

Then we finally require that \( x < x' \) implies that for all \( N \)

\[
\Pr \left\{ \prod_{n=1}^{N} Y_{i_n j_n}^{(n)}(x') = \sigma_1 \right\} \geq \Pr \left\{ \prod_{n=1}^{N} Y_{i_n j_n}^{(n)}(x) = \sigma_1 \right\} ,
\]

\[
\Pr \left\{ \prod_{n=1}^{N} Y_{i_n j_n}^{(n)}(x') = \sigma_2 \right\} \leq \Pr \left\{ \prod_{n=1}^{N} Y_{i_n j_n}^{(n)}(x) = \sigma_2 \right\} .
\]

All that we have said about \( \{ \Lambda(f_1, f_2) \} \) up to Theorem 1, trivially modified, applies to the games \( \{ \Lambda(x) \} \) in which two players repeatedly and simultaneously choose integers \( i_n \) and \( j_n \).
at each time \( n \), until \( \prod_{n=1}^{N} Y_{n}^{(n)}(x) = \sigma_1 \) or \( \sigma_2 \), or ad infinitum if this never occurs. The payoff is (1, 0) if the game ends in the state \( \sigma_1 \), and (0, 1) if the game ends in the state \( \sigma_2 \).

If the game goes on indefinitely, then the payoff is \((\alpha(C), \beta(C))\) where \((\alpha(C), \beta(C)) \leq (1, 1)\) and \(\alpha(C) + \beta(C) \leq 1\), where \((\alpha(C), \beta(C))\) can depend on the course of the game, \( C \).

Similarly, Theorem 2 can be generalized by the use of expected values to the situation where \( \sum \) is a set of reals and for \( \sigma_1 < x < \sigma_2 \)

\[
Y_{ij}(x) = x + a_{ij} \quad \text{if} \quad \sigma_2 < x + a_{ij} < \sigma_1
\]

\[
= \sigma_1 \quad \text{if} \quad \sigma_1 \leq x + a_{ij}
\]

\[
= \sigma_2 \quad \text{if} \quad \sigma_2 \geq x + a_{ij},
\]

where \( a_{ij} \) is a real-valued random variable whose distribution depends on \((i, j)\).