SOLUTION SETS FOR GAMES ON THE SQUARE

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Summary: Some necessary and sufficient conditions that a pair of non-void weak closed convex sets of strategies form the solution set of a game with continuous payoff on the square are given.

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Let $K$ denote the set of all optimal strategies for one player, $L$ the corresponding set for his opponent in a game. We shall refer to $K \times L$, the set of all pairs $(f,g)$, $f \in K$, $g \in L$, as the solution set of the game. Any non-void weak* closed convex set $K$ is the set of all optimal strategies for one player in some game with continuous payoff, as was shown in [1], but of course not all pairs $K,L$ of such sets will yield solution sets. By means of constructions similar to those used in [1] we shall determine which pairs do occur in terms of the spectra, $\sigma_K, \sigma_L$ of these sets and the number of independent containing hyperplanes.

1. Preliminaries. As was shown in [1], any non-void weak* $(w^*)$ closed convex set $K$ of strategies is the intersection

\[ \sigma K = \bigcup_{f \in K} \sigma(f), \]
which is easily seen to be a closed set.
of a sequence of half spaces, which we may express by

\[(1) \quad K = \{ f \mid (\varphi_n, f) = \int \varphi_n(x) df(x) \geq 0, \ n = 1, \ldots \} \]

where \( \{\varphi_n\} \) is a sequence of continuous functions and we may assume, for each \( n \), \( (\varphi_n, f) = 0 \) for some \( f \) in \( K \). Certain of these \( \varphi \)'s will yield \( (\varphi, f) = 0 \) for all \( f \) in \( K \), and these we shall denote by \( p \)'s. Thus we shall write

\[ K = S(\varphi_n; p_n)^2 \]

to express the fact that \( K = \{ f \mid (\varphi_n, f) = 0 = (p_n, f) \} \) as well as the fact that \( (\varphi_n, K) \) is a non-degenerate interval. The functions \( p_m \) thus define hyperplanes containing \( K \) while the \( \varphi_m \) do not. If the set \( K \) is the intersection of a set of hyperplanes, one may show exactly as in the proof of \( (1) \) that it is the intersection of a sequence of these and one may write \( K = S(p_m) \).

What we shall be concerned with in large part in the following constructions will be the hyperplanes containing \( K \). It is immediately evident that if we select from the functions \( \{p_n\} \) a maximal subsequence \( \{p'_n\} \) which is linearly independent on \( \varphi \) \( K \) then the relations \( (p_n, f) = 0 \) are consequences of the relations \( (p'_n, f) = 0 \) for \( f \) for which

\[ 2) \quad \text{For the opponent we shall write } L = S(\varphi_m; q_n) \text{ where we take } (\varphi_m, g) \neq 0. \]
σ(ψ) ⊂ ψK. Consequently if we set \( p^*(x) = \text{dist}(x, \psi K) \),
then \( (p^*, f) = 0 \), \( (p'_n, f) = 0 \) all \( n \), \( (p_n, f) = 0 \) all \( n \);
thus in most of what follows we shall assume the \( \{p_n\} \) to be
linearly independent\(^{3)}\) and actually orthonormal:

Suppose we define a measure on \( \psi K \) in the following
way: select a sequence \( \{x_n\} \) dense in \( \psi K \) and place weight
\( 2^{-n} \) at \( x_n \). Then clearly we may apply the Gram–Schmidt process
to the \( \{p_n\} \) to obtain an orthonormal sequence \( \{p'_n\} \) of
the same length (we take \( \{x_n\} \) dense to insure that only the
function 0 has the integral of its square zero). Just as
clear is the fact that \( (p'_n, f) = 0 \) for all \( n \) is equivalent
to \( (p_n, f) = 0 \) for all \( n \).

2. Constructions. We shall now construct payoffs which
will have three types of solution sets. That these are the
only types which occur will be shown later.

Case I: Suppose \( \psi K = [0,1] = \psi L \) and \( K \) and \( L \) are
the intersections of the same number of independent hyper-
planes. The orthonormal sequences \( \{p_n\} \) and \( \{q_n\} \) defining \( K \)
and \( L \) are thus of the same length, and if we set
\[
M(x, y) = \sum a_n p_n(x) q_n(y),
\]

\(^{3)}\) We shall say that the hyperplanes \( H_n \) defined by
\( H_n = \{f | (p_n, f) = 0\} \) are independent hyperplanes containing \( K \)
if the \( p_n \) are linearly independent on \( \psi K \), \( K \subset H_n \).
where the $a_n$ are chosen to insure uniform convergence of the series, then for $f$ in $K$ and $g$ in $L$,

$$
\int Mdf = \sum a_n(p_n,f)q_n(y) = 0 = \sum a_n p_n(x)(q_n,g) = \int Mdg;
$$
on the other hand, if $f$ is optimal

$$
\int Mdf = \sum a_n(p_n,f)q_n(y) = 0,
$$
and in view of the orthogonality of the $q_n$; $f$ is in $K$.

Similarly every optimal $g$ is in $L$, and $K \times L$ is the solution set.

Case II: Suppose $\sigma K = [0,1] \not\subset \sigma L$, and $K = S(\Phi_m; p_n)$.

$L = S(q_n)$ where there are at least as many independent hyperplanes containing $K$ as there are containing $L$ (thus we may assume that a maximal linearly independent set of $p_n$'s is at least as long as the set of $q_n$'s linearly independent on $\sigma L$). Since $\sigma L$ is not the full unit interval we may select an open interval $I$ which has one end point $y_o$ in $\sigma L$.

Select a disjoint sequence $\{I_n\}$ of open subintervals of $I$ for which $\text{dist} (y_o, I_n) \to 0$, and an open subinterval $I_n^*$ of each $I_n$ whose closure lies entirely in $I_n$. Let $k_n$ be a continuous non-negative function which vanishes outside $I_n$ but is non-zero inside $I_n$, and which assumes the value 1 at a point $y_n$ of $I_n^*$. Define a continuous function $m_n$ which vanishes at $y_n$ and outside $I_n^*$, but takes on the values $\pm 1$. 
If we then set \( q(y) = \text{dist}(y, \sigma L \cup \bigcup I^*_n) \), then for every \( y \) not in \( \sigma L \) one of the non-negative functions \( q, k_n \) is non-zero at \( y \).

We now define our payoff as follows: we divide the sequence \( \{p_n\} \) into \( \{p'_n\} \), orthonormal and of the same length as the \( \{q_n\} \), and \( \{p^*_n\} \). If either of the sequences \( \{p'_n\} \) or \( \{\varphi_n\} \) are finite we use repetitions to form a sequence, and if there are no \( \psi_n \)'s say, we take \( \varphi_n = 1 \) for all \( n \).

We set (for \( b_n > 0 \), chosen to insure uniform convergence)

\[
M(x,y) = \sum a_n p_n(x)q_n(y) + \sum b_n [k_n(y)\varphi_n(x) + n m_n(y)p_n^*(x)] + q(y)
\]

where \( \{N_n\} \) is an enumeration of the integers in which each integer occurs infinitely often. For \( f \) in \( K \) and \( g \) in \( L \)

\[
\int Mdf = \sum b_n k_n(y)(\varphi_n,f) + q(y) \geq 0 = \int Mdg,
\]

so that both are optimal.

Suppose \( f \) is optimal; then for \( y \) in \( \sigma L \),

\[
\sum a_n(p_n,f)q_n(y) = 0
\]

whence \( (p_n,f) = 0 \), and thus

\[
0 \leq \sum b_n[k_n(y)(\varphi_n,f) + n m_n(y)(p_n^*,f)] + q(y),
\]

and at setting \( y = y_n \), \( b_n(\varphi_n,f) = 0 \), so that \( (\varphi_n,f) = 0 \) for all \( n \). For \( y \) in \( I^*_n \) we have

\[
0 \leq b_n[k_n(y)(\varphi_n,f) + n m_n(y)(p_n^*,f)]
\]
whence $0 \leq (\varphi_{n_{m_{n}}}(f) + nm_{n}(y)(p_{n_{m_{n}}}(f)$, and since $m_{n}$ assumes the values $\pm 1$,

$$(\varphi_{n_{m_{n}}}(f) \geq \pm n(p_{n_{m_{n}}}(f),$$

hence

$$(\varphi_{n_{m_{n}}}(f) \geq n|p_{n_{m_{n}}}(f)|.$$ 

Since $N_{n}$ takes on the value $n_{o}$ infinitely often, $$(\varphi_{n_{o}}(f) \geq n|p_{n_{o}}(f)|$$ for arbitrarily large $n$, and $(p_{n_{o}}(f) = 0$ for each $n_{o}$. Thus $f$ is in $K$.

If $g$ is optimal, then for any $f$ in $K$,

$$Q = \int \int Mdfdg = \sum b_{n}(k_{n},g)((\varphi_{n_{m_{n}}}(f) + (q_{n},g).$$

But each term of this sum is non-negative $((k_{n},g) \geq 0$ since $k_{n} \geq 0)$ so that surely $(q_{n},g) = 0$. If $(k_{n},g) > 0$ for some $n$ then since there is an $f$ in $K$ for which $(\varphi_{n}(f) > 0$, we would have a contradiction. Thus

$$(q_{n},g) = 0, (k_{n},g) = 0, and (m_{n},g) = 0$$

since $(k_{n},g) = 0$ implies $g$ places no weight on $I_{n}$. Thus $g \in \sigma L$, and since we now may write

$$0 = \sum a_{n}(q_{n},g), x \in \sigma K,$$

and $(q_{n},g) = 0$, $g$ is in $L$.  


Case III: \( \sigma K \neq [0,1] \neq \sigma L \). Here we may take any \( K \) and \( L \) without further restriction, so that \( K = S(\psi_m; p_n) \) and \( L = S(\psi_m; q_n) \) (if \( \psi_m \neq 0 \) here, however, in our definitions). We construct functions \( h_n \) similar to the \( k_n \) of case II, and \( l_n \) similar to the \( m_n \), on an interval abutting \( \sigma K \). We set

\[
M(x,y) = \sum a_n [h_n(x) \psi_n(y) + n l_n(x) q_n(y) + k_n(y) \psi_n(x) + m_n(y) p_n(x)].
\]

Arguments entirely similar to those used in case II show \( K \times L \) to be the solution set.

3. Generality. In case I (\( \sigma K = [0,1] = \sigma L \)) we restricted our attention to the case in which \( K \) and \( L \) were intersections of the same number of independent hyperplanes. Suppose now that a game with payoff \( M \) has as its solution set \( K \times L \) where \( \sigma K \) and \( \sigma L \) are the full intervals. \( K \) is determined as the set of all \( f \) for which

\[
\int M(x,y) df(x) = 0
\]

(for convenience we take the value to be zero), and thus is the intersection of hyperplanes given by the functions \( \{M(\cdot,y)\} \), and similarly \( L \) is the intersection of the hyperplanes determined by the functions \( \{M(x,\cdot)\} \).

If a maximal linearly independent set \( \{M(x_i,\cdot)\} \) of the first set, say, is finite, \( i = 1, \ldots, n \), then the same is true of the second, indeed there are just as many. For, as is
well known, $n$ functions $F_1, \ldots, F_n$ are linearly independent on a set $X$ if and only if there exist $x_1, \ldots, x_n$ in $X$ for which

$$\det (F_i(x_j)) \neq 0;$$

consequently we have $y_1, \ldots, y_n$ for which

$$(2) \quad \det (M(x_i, y_j)) \neq 0,$$

so that the functions $\left\{ M(\cdot, y_j) \right\}_{j=1, \ldots, n}$ are linearly independent. Of course if $\left\{ M(\cdot, y_j) \right\}_{j=1, \ldots, n+1}$ were linearly independent by the same argument we should have an $x_{n+1}$ for which $\left\{ M(x_i, \cdot) \right\}_{i=1, \ldots, n+1}$ were, which contradicts our assumption, and there are exactly $n$. Thus the type of solution sets considered in case I are the only type which can occur. (One might note that here finite set of independent containing hyperplanes can only occur in a polynomial-like game, since for every $x$ we have coefficients $a_1(x)$ for which

$$M(x, y) = \sum a_1(x)M(x_1, y),$$

and (2) shows the functions $a_1$ to be continuous.)

In case II, $(\sigma K = [0,1] \neq \sigma L)$ we considered only those $K$ and $L$ for which we had as many independent hyperplanes containing $K$ as there are containing $L$. But if $M$ is the payoff of a game with solution set $K \times L$, $\sigma K = [0,1] \neq \sigma L$, then as before since $L$ is determined by

$$\int M(x, y)dg(y) = 0, \quad \text{all } x,$$
L is just the intersection of hyperplanes. If there are only \( n \) independent containing hyperplanes, then, as we shall see in a moment, these must be given by the functions 
\[
\{ M(x_i, \cdot) \}_{i=1, \ldots, n}
\]
linearly independent on \( \sigma L \), for some set \( x_1, \ldots, x_n \); consequently there exist \( y_1, \ldots, y_n \) in \( \sigma L \) for which (2) holds, and 
\[
\{ M(\cdot, y_j) \}_{j=1, \ldots, n}
\]
are linearly independent.

Since \( \int M(x,y)df(x) = 0 \) for \( y \) in \( \sigma L \), these functions define \( n \) independent hyperplanes containing \( K \).

To see that the \( n \) independent hyperplanes containing \( L \) arise from functions \( M(x_i, \cdot) \) we note that for each \( x_i \), \( M(x_i, \cdot) \) defines a containing hyperplane since \( x_i \) is in \( \sigma K = [0,1] \).

Consequently there can be only \( m \) points, \( m \neq n \), \( x_1, \ldots, x_m \) for which \( \{ M(x_i, \cdot) \} \) are linearly independent, so that clearly
\[
L = \{ g | (M(x_i, \cdot), g) = 0, i = 1, \ldots, m \}.
\]

If \( m < n \), we can find a function \( q_0 \) for which, denoting \( M(x_i, \cdot) \) by \( q_i \), the set \( q_0, \ldots, q_m \) is linearly independent on \( \sigma L \) and \( (q_0, g) = 0 \) for all \( g \) in \( L \). But then the mapping
\[
T: g \mapsto ((q_0, g), \ldots, (q_m, g)),
\]
of the set \( S \) of all strategies into \( m + 1 \) space, takes \( S \) into a convex subset containing \( (0, \ldots, 0) \) (since \( L \) is non-void).

But \( T(S) \) intersects the line \( (t, 0, \ldots, 0) \) in only one point (since \( (q_0, g) = 0 \) for \( g \) in \( L \))—thus \( (0, 0, \ldots, 0) \) is a boundary point and we have a supporting hyperplane at this point given by constants (not all zero) \( a_0, \ldots, a_m \). Thus
\[
\sum_{i=0}^{\infty} a_i(q_i, g) \geq 0
\]
for all $g$ in $S$, hence $\sum a_i q_i(y) \leq 0$ for $y$ in $\sigma$-$L$. If
inequality holds for any $y$ it holds in some neighborhood, and
this is, of course, of positive measure with respect to some
$g$ in $L$ (from the definition of $\sigma$-$L$), whence $\sum a_i(q_i, g) > 0$
for some $g$ in $L$ — a contradiction. Thus $\sum a_i q_i = 0$ on
$\sigma$-$L$, which contradicts the linear independence on $\sigma$-$L$, and
we must have $m = n$.

Thus the theme of things is as follows: The necessary
and sufficient condition that $K \times L$ be the solution set
for a game with continuous payoff on the square (where
$K$ and $L$ are non-void $\omega^*$ closed convex sets of
strategies) is that one of the following hold:

(a) $\sigma K = [0, 1] = \sigma$-$L$ and $K$ and $L$ are the inter-
section of the same number (finite if and only if
the game is polynomial-like) of independent
containing hyperplanes

(b) $\sigma K = [0, 1] \neq \sigma$-$L$, $L$ is the intersection of
hyperplanes and $K$ has as many independent contain-
ing hyperplanes as $L$

(c) $\sigma K \neq [0, 1] \neq \sigma$-$L$.

The constructions we have used can be duplicated in
in the case of a game with continuous payoff played on a pair
of infinite compact metric spaces; the character of solution
sets, however, involves slightly different conditions:
\( \sigma K = [0,1] = \sigma L \) must be replaced by \( \sigma K, \sigma L \) open,
\( \sigma K = [0,1] \not\subset \sigma L \) by \( \sigma K \) open, \( \sigma L \) not open,
\( \sigma K \not\subset [0,1] \not\subset \sigma L \) by \( \sigma K \) and \( \sigma L \) not open. In the case of
a unique optimal strategy forming \( K \) and another forming \( L \)
we are thus guaranteed a game having \( K \times L \) as the solution
set, which generalizes the result of [2].

As a final remark, we note that solution sets for
symmetric games on the square (where \( M(x,y) = -M(y,x) \)) can be
easily described. For such games the value is always zero
and any optimal strategy for one player is optimal for his
opponent, so that a solution set is of the form \( K \times K \). The
necessary and sufficient condition that \( K \times K \) be the solution
set of a symmetric game is that either

(a) \( \sigma K = [0,1] \) and \( K \) is the intersection of an even
   (we take \( \infty \) as even) number of independent hyper-
   planes, or

(b) \( \sigma K \not\subset [0,1] \).

For if \( \sigma K = [0,1] \) and \( K \) is the intersection of an
even number of independent hyperplanes given by functions
\( \{ p_n \} \) (which we may take orthonormal), then, dividing these
into two sets \( \{ p_n \}, \{ p'_n \} \) of equal cardinality, we may set

\[
M(x,y) = \sum a_n [p_n(x)p'_n(y) - p'_n(x) p_n(y)],
\]

which is easily seen to have \( K \times K \) as its solution, and is
symmetric. On the other hand, if \( K \times K \) is the solution set
of a game with payoff $M$ and $\sigma K = [0,1]$, then $K$ is, of course, the intersection of a set of hyperplanes. If only a finite number of these are independent, then, as before, $M$ is polynomial-like, that is,

$$M(x,y) = \sum_{n=1}^{k} \psi_n(x) \psi_n(y),$$

where $\{\psi_n\}$ and $\{\psi_n\}$ are linearly independent sets of functions. Since $M$ is symmetric

$$M(x,y) = -M(y,x) = -\sum_{n=1}^{k} \psi_n(y) \psi_n(x),$$

so

$$M(x,y) = \frac{1}{2} \sum_{n=1}^{k} [\psi_n(x) \psi_n(y) - \psi_n(y) \psi_n(x)].$$

If the functions $\{\psi_n, \psi_n\}$ are not a linearly independent set, replacement of a dependent $\psi$ or $\psi$ again yields a sum of the same type, and we finally obtain a similar expression for $M$ in which the set $\{\psi_n, \psi_n\}$ is linearly independent; however, there are an even number of terms in the resulting sums, and thus there must be an even number of independent hyperplanes determining $K$.

In case (b), $K = S(\psi_n; p_n)$, and we may set

$$M(x,y) = \sum a_n \left[ k_n(y) \psi_n(x) + n m_n(y) p_n(x) - k_n(x) \psi_n(y) - n m_n(x) p_n(y) \right]$$

to obtain a symmetric game in which $K \times K$ is the solution set.
REFERENCES

1. I. Glicksberg and O. Gross, Optimal Sets for Games over the Square, RM-889.

2. I. Glicksberg and O. Gross, Continuous Games with Given Unique Solutions, RM-620.