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RESEARCH MEMORANDUM

OPTIMAL SEQUENTIAL TESTING

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SUMMARY

A problem of importance to the Air Force is that of trouble-shooting to find a malfunctioning part of a complex piece of electronic equipment. The process of identifying and replacing a faulty part is becoming more costly and time consuming as the equipment becomes more complex. At the same time, skilled mechanics with extensive training are becoming more difficult to find and retain. This points out the value of being able to prescribe to the mechanic a series of checks for him to make, preferably in terms of a training manual. To aid in this purpose, some mathematical versions of the general problem and their solutions are presented in this paper.

A complicated machine may break down because of the failure of some of its components. In what sequence should its components be tested and repaired in order to minimize the expected delay time? The cost in time for each step is assumed to be known, together with the probability that each part is working. Simple rules give the optimal sequence. In addition, for each component there is an analogous optimal sequence for testing its parts.

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OPTIMAL SEQUENTIAL TESTING *

1. INTRODUCTION

A complicated machine, e.g., a radar set, which consists of many components each having many parts, is not working because of failure of some of its components. In what sequence should its components be tested and repaired in order to minimize the expected delay time before it is working again?

There are various technical interpretations which can be assumed for this problem; the ones made here will be apparent from the analysis.

We assume a time study has been made so that the time for each step in the procedure is known. Also the probabilities that the various parts are in working order are given. These are assumed to be independent. The machine has several disjoint components each containing one or more parts which may fail. There are tests available for the components and for their parts.

2. DEFINITIONS

Let

n = the number of components in the machine.

W = the time to start the machine and test whether it works.

R_i = the time to remove the i-th component from the machine in order to test it.

W_i = the time to test the i-th component to see if it works.

* This problem was suggested by Dr. Ward Edwards of the Air Force Personnel and Training Research Center, Lowry Air Force Base.

R_i' = the time to return the i-th component or a working replacement back into position in the machine.

$T_i = R_i + W_i + R_i'$.

q_i = probability that the i-th component is working.

$p_i = 1 - q_i$ = probability that the i-th component is not working.

$Q_i = \prod_{j=i}^{n} q_j$ = probability that all components numbered from i to n, inclusive, are working.

E_i = minimum expected delay time to get the i-th component in working order after it has been tested. The determination of E_i is a subproblem of the same type as the original problem. (See Sec. 5.) The parts making up this component are assumed to be tested in an optimal order.

S = the component test sequence (1, 2, ..., n).

E(S) = expected delay time after the machine has been tested initially and before it works if the components are tested in the sequence S.

3. ASSUMPTIONS

1. The component tests used are efficient enough to save expected time.

2. After each faulty part is repaired the component is then tested; after each faulty component is repaired the machine itself is tested.

3. The component is tested before its parts are tested (except that the final component is not tested initially).

4. The parts of the final component are tested after all other components are repaired.

4. SEVERAL FAULTY PARTS

From the assumptions, we obtain

$$(1)\quad E(S) = (1 - Q_1)T_1 + E_1 + p_1W + (1 - Q_2)T_2 + E_2 + p_2W + \cdots + (1 - Q_n)T_n + E_n + p_nW - p_nW_n$$

$$= \sum_{i=1}^{n} (T_i + E_i + p_iW) - \sum_{i=1}^{n} Q_iT_i - p_nW_n \; .$$

Note that if we get to the final component we do not test it before testing its parts, for we know it is defective since all other components are working while the machine is not.

Let S' be the sequence S after interchanging components (i) and (i + 1), where $i + 1 < n$. Then we have

$$(2)\qquad E(S') - E(S) = - Q_{i+2}(- q_iT_ip_{i+1} + q_{i+1}T_{i+1}p_i)$$

which is $\begin{Bmatrix}\text{positive}\\ 0\\ \text{negative}\end{Bmatrix}$ according as $\frac{q_iT_i}{p_i} \begin{Bmatrix}>\\ =\\ <\end{Bmatrix} \frac{q_{i+1}T_{i+1}}{p_{i+1}}$, a transitive relation.

If we fix the final component, and successively interchange consecutive components, we are thus led to the optimal sequence (for this fixed final component) given by the rule that

(3) i precedes j if $\frac{q_i T_i}{p_i} < \frac{q_j T_j}{p_j}$.

Thus $n!$ arrangements have been reduced to n related arrangements. The special consideration for the final component is due to the nonsymmetric term $-p_n W_n$ in the expression for $E(S)$. However, the final component can be determined as follows.

Let the n components be numbered according to (3). Let $S_n = (1, 2, 3, \ldots, n)$ and $S_k = (1, 2, \ldots, k-1, k+1, \ldots, n, k)$; i.e., $S_k = S_n$ with the k-th component transferred to the end.

Let $G_k = E(S_k) - E(S_n)$. We wish to determine j such that $G_j = \min_k G_k$, $k = 1, 2, \ldots, n$.

Now

$$G_k = - q_k Q_{k+1} T_{k+1} - q_k Q_{k+2} T_{k+2} \cdots - q_k Q_n T_n - q_k T_k - p_k W_k$$
$$+ Q_{k+1} T_{k+1} + Q_{k+2} T_{k+2} + \cdots + Q_n T_n + Q_k T_k + p_n W_n \ ,$$

or

$$(4) \quad G_k = p_k \sum_{i=k+1}^{n} Q_i T_i - q_k(1 - Q_{k+1}) T_k - p_k W_k + p_n W_n \ .$$

By generating $\sum_{i=k+1}^{n} Q_i T_i = M_{k+1}$, say, and G_k in some convenient table, we can easily calculate $G_j = \min_k G_k$.

Theorem 1. Number the n components according to increasing value of $q_i T_i / p_i$. Then this is the optimum order except that the j-th component is transferred to the final position, where j is determined by

$$G_j = \min_k G_k \ , \quad k = 1, 2, \ldots, n ,$$

and G_k *is given by* (4).

5. THE PARTS OF A FAULTY COMPONENT

The problem of finding the minimum expected delay time E_i for the i-th component is just a subproblem of the main problem already treated.

We define all the terms with double subscripts for the j-th part of the i-th component in an analogous way to those defined for the machine and its components, and define

$$E_{ij} = p_{ij}L_{ij} \ ,$$

where L_{ij} is the time taken to get a new part to replace the faulty one. Then if S is the test sequence $(1, 2, \ldots, n_i)$ and if the assumptions of Sec. 3 are followed, we have

$$(5) \quad E_i(S) = (1-Q_{i1})T_{i1} + E_{i1} + p_{i1}W_1 + (1-Q_{i2})T_{i2} + E_{i2} + p_{i2}W_i + \cdots + (1-Q_{in_i})T_{in_i} + E_{in_i} - p_{in_i}W_{in_i} - p_{in_i}W_i \ ,$$

since the last part would not be tested and the component would not be tested after the last part was repaired. Thus we have

$$(6) \quad E_i(S) = \sum_{j=1}^{n_i} (T_{ij} + E_{ij} + p_{ij}W_i) - \sum_{j=1}^{n_i} Q_{ij}T_{ij} - p_{in_i}(W_{in_i} + W_i) \ ,$$

which is analogous to (1). The proof of Theorem 1, applied to (6), now yields the following result.

Theorem 2. Number the n_i parts of the i-th component according to increasing value of $q_{im}T_{im}/p_{im}$. Then this is the optimal order of testing the parts except that the j-th part is transferred to the final position, where j is given by

$$G_{ij} = \min_k G_{ik} , \qquad k = 1, 2, \ldots, n_i ,$$

and

$$G_{ik} = p_{ik} \sum_{h=k+1}^{n_i} Q_{ih}T_{ih} - q_{ik}(1 - Q_{i,k+1})T_{ik}$$

$$- p_{ik}(W_{ik} + W_i) + p_{in_i}(W_{in_i} + W_i) .$$

The testing procedures outlined in Theorems 1 and 2 are optimal over the set of permissible procedures of Sec. 3. The exact conditions for the most general problem require much more complicated analysis, which may be carried out in a future study. Some progress along this line has been made already.

If we assume the machine breaks down due to a failure of just one of its parts, the analysis is somewhat easier, as will be shown in the next section. There we still make assumptions 1, 3, and 4 of Sec. 3.

6. ONE FAULTY PART

Assume it is known that the machine has failed because just one part (e.g., a radar tube) has broken down. We still have all the terms defined in Secs. 2 and 5; but now the probabilities of failure are normalized to sum to unity, so that

$$p_i = \sum_{j=1}^{n_i} p_{ij}\ , \quad \sum_{i=1}^{n} p_i = 1\ .$$

Define, further,

H_i = expected delay time to fix the i-th component, knowing it has exactly one faulty part. (It is understood that we test its parts in the optimal manner outlined below in Theorem 4.)

S = (1, 2, 3, ..., n) = a component test sequence.

$H(S)$ = expected delay time to fix the machine, knowing just <u>one</u> of its parts is faulty.

$$V_i = \sum_{j=1}^{n} p_j\ .$$

Then we have

$$H(S) = \sum_{i=1}^{n} p_i T_1 + p_1 H_1 + \sum_{i=2}^{n} p_i T_2 + p_2 H_2 \tag{7}$$

$$+ \cdots + p_n T_n + p_n H_n - p_n W_n\ .$$

Note that the last component would not be tested; rather we would go directly to its parts.

From (7) we obtain

$$H(S) = \sum_{i=1}^{n} V_i T_i + \sum_{i=1}^{n} p_i H_i - p_n W_n\ . \tag{8}$$

As before, we define S' = S with the i-th and (i + 1)-th components interchanged. Then if $i + 1 < n$, we have

$$(9) \quad H(S') - H(S) = (p_{i+1} + p_i)T_{i+1} + p_i T_i - (p_i + p_{i+1})T_i - p_{i+1}T_{i+1}$$

$$= p_i T_{i+1} - p_{i+1} T_i \ ,$$

which is positive whenever $\frac{T_{i+1}}{p_{i+1}} > \frac{T_i}{p_i}$. Thus if we rank the components in order of increasing value of T_i/p_i, we would have the optimal order for each fixed final component.

Next we find the optimal final component.

First we number the n components according to increasing value of T_i/p_i .

Let

S_k = S with k transferred to the end.

$$U_j = \sum_{i=j}^{n} T_i \ .$$

$$G_k = H(S_k) - H(S_n)$$

$$= - T_k V_k + p_k(T_{k+1} + \cdots + T_n) + p_n W_n + p_k T_k - p_k W_k$$

$$= - T_k V_k + p_k(U_k - W_k) + p_n W_n \ .$$

$$(10) \qquad F_k = G_k - p_n W_n = - T_k V_k + p_k(U_k - W_k) \ .$$

The optimal sequence is obtained by minimizing F_k .

Theorem 3. When there is just one faulty part, the machine is put into working operation in the minimum expected time when the components are numbered and tested in order of increasing value of T_i/p_i, except that the j-th component is transferred to the end of the sequence, where j is given by the minimum F_k in (10):

$$F_j = \min_k F_k \ , \quad k = 1, 2, \ldots, n \ .$$

We can easily compute the minimum F_k by generating U_k and V_k in a convenient table. Since (10) is linear in the p_i's, we can use frequence of failure for probability of failure.

The optimal sequence for testing parts of a component known to have one faulty part is given by the same rules as for the component testing since this is a subproblem of the same type as the original one.

Define all the parts terms with double subscripts analogously to those defined for the components. Then we have the following result.

Theorem 4. For a component known to have one faulty part, the parts are numbered and tested in order of increasing value of T_{im}/p_{im}, with the exception that the j-th part is transferred to the end, where j is given by

$$F_{ij} = \min_k F_{ik} \ , \quad k = 1, 2, \ldots, n_i \ ,$$

and

(11) $$F_{ik} = -T_{ik}V_{ik} + p_{ik}(U_{ik} - W_{ik}) .$$

7. EXAMPLE

Consider a machine with five components whose failure matrix is

i	1	2	3	4	5
T_i	3	4	4	5	6
p_i	6	11	5	13	20
W_i	1	2	2	3	4

Renumber and order the components according to increasing value of T_i/p_i, thus

k	1	2	3	4	5
T_k	6	4	5	3	4
p_k	20	11	13	6	5
W_k	4	2	3	1	2
V_k	55	35	24	11	5
U_k	23	17	13	7	4
U_k-p_k	19	15	10	6	2
F_k	50	15	-14	3	-10

F_3 is minimum, so the optimal sequence for testing the components is (1, 2, 4, 5, 3). The parts of the faulty component would be tested according to Theorem 4.