U. S. AIR FORCE

PROJECT RAND

RESEARCH MEMORANDUM

A MATHEMATICAL MODEL
OF AN AIR TRANSPORTATION SYSTEM

T. W. Anderson, K. J. Arrow, J. E. Walsh

RM-224

26 August 1949

Assigned to ______________________

The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

A MATHEMATICAL MODEL OF AN AIR TRANSPORTATION SYSTEM

One air transportation problem may be defined loosely as follows: There are B air bases given. It is desired to transport a given amount of goods q_β^α (α, $\beta = 1, \ldots, B$; $\alpha \neq \beta$) from base of origin α to ultimate base of destination β per unit of time. We shall measure q_β^α in tons per hour. The problem is to find the most efficient program of operations to effect this transportation.

We shall consider two kinds of operations, flying planes and loading and unloading. Two costs are treated. One may be measured in money. That includes the costs of flight, which are primarily costs of fuel and personnel, and the costs of loading and unloading, which are costs of personnel and use of equipment. The second cost is in terms of the number of planes used. We take the bases (with their capacities) as given.

The problem is to determine the "efficient points" with respect to these two costs; that is, we find the set of programs such that we cannot change any program without increasing one cost or the other. We then know the program of minimum money cost for a given number of planes; vice versa, we know the program of minimum plane use for a given money cost.

The essential simplifying assumption is that we can treat the problem continuously. We shall assume that we can use any fraction of a plane in our program. We further assume that the situation is stationary; that is, that we can consider the problem as one of constant flows.

The cargo of a plane on a given hop will be denoted by a vector

$$q = (q_1, \ldots, q_B) \ ,$$

where q_b is the number of tons (measured in integers) of goods destined eventually for base b. A hop means a flight between any two bases in our program; that is, we do not need to consider a base which is only a refueling stop between two other particular bases.

One of our primary activities is flying a plane from base α to base β with a cargo q. The intensity $x_{\alpha\beta}(q)$ is measured in planes per hour. The cargo is restricted by $\sum_b q_b \leq K_{\alpha\beta}$, the maximum load for this flight, and $q_\alpha = 0$. The range of the indices α, β is restricted to exclude pairs of bases whose distance apart is too great to permit direct flight between them.

The products (inputs and outputs) are:

	Notation	Quantity
Fueled planes at base α with cargo q	${}_1P_\alpha(q)$	-1
Money	M	$-m_{\alpha\beta}(\sum q_b)$
Plane Use	P	$-t_{\alpha\beta}(\sum q_b)$
Unfueled plane at base β with cargo q	${}_0P_\beta(q)$	1

The money cost $m_{\alpha\beta}(\sum q_b)$ of one unit of the operation depends on the two bases and the total tonnage of the load. This is measured in dollars per plane. Thus M is measured in dollars per hour. The plane use cost of one unit of operation also depends on the two bases involved and the weight of the load; this is measured in hours. Thus P is measured in planes.

The other primary activity is unloading-loading. Let $\ell_\alpha(q, \bar{q})$ (measured in planes per hour) denote the intensity of the operation of taking a plane at base α with cargo q and changing it to a plane with cargo $\bar{q}$. This includes both loading and unloading. The products are:

	Notation	Quantity
Unfueled plane at base α with cargo q	${}_0P_\alpha(q)$	-1
Money	M	$-m_\alpha \sum\limits_{q_b > q_b} \left\| q_b - \bar{q}_b \right\|$ $-m'_\alpha \sum\limits_{q_b < \bar{q}_b} \left\| q_b - \bar{q}_b \right\|$ $-r_\alpha$
Plane Use	P	$-\max\left\{ t''_\alpha ,\ t_\alpha \sum\limits_{q_b > \bar{q}_b} \left\| q_b - \bar{q}_b \right\| + t'_\alpha \sum\limits_{q_b < \bar{q}_b} \left\| q_b - \bar{q}_b \right\| \right\}$
Fueled plane at base α with cargo $\bar{q}$	${}_1P_\alpha(\bar{q})$	1
goods at base α	C_α	$q - \bar{q}$.

The coefficient m_α is the cost of unloading one ton at α and m'_α is the cost of loading one ton at α. Similarly t_α and t'_α are the times of unloading and loading one ton at α, respectively: r_α is the cost of refueling at α; this is simply the cost of personnel and equipment, not of the fuel itself (the cost of fuel is included in $m_{\alpha\beta}(\sum q_b)$). t''_α is the time of refueling. The goods on the ground at α is a vector C_α each component of which is the amount of goods destined for a particular base.

Now let us consider the restrictions. The net flow of goods at base α must be

$$\sum_{q,\bar{q}} \ell_\alpha(q, \bar{q})(q - \bar{q}) = - q^\alpha$$

where $q^\alpha = (q_1^\alpha, \ldots, q_B^\alpha)$. This vector equation says that the net flow of goods destined for β at α is the amount specified in the problem as originating at β with ultimate destination α. Since we have not defined q_α^α, the equation only serves as a restriction for the B-1 components other than the α^{th}. We may take the α^{th} component of the above vector equation as defining q_α^α, which is, then, the flow of goods at α of goods destined for α. It may easily be verified that in a stationary state,

$$- q_\alpha^\alpha = \sum_{\substack{\beta=1 \\ \beta\neq\alpha}} q_\alpha^\beta \quad ;$$

i.e., the net flow of goods at α destined for α equals the total quantity of goods originating at other bases and ultimately destined for α.

The number of planes arriving at α with a given cargo must equal the number of planes subject to loading activities with that initial cargo; similarly, the number of planes subject to loading activities with a given final cargo must equal the number of planes leaving the field with that cargo. Thus

$$\sum_{\beta} x_{\beta\alpha}(q) = \sum_{\bar{q}} \ell_{\alpha}(q, \bar{q})$$

$$\sum_{\beta} x_{\alpha\beta}(\bar{q}) = \sum_{q} \ell_{\alpha}(q, \bar{q})$$

If the capacity of a base is limited, we have

$$\sum_{q,\bar{q}} \ell\alpha(q, \bar{q}) \max \left\{ t_{\alpha}'', \; t_{\alpha} \sum_{q_b > \bar{q}_b} \left| q_b - \bar{q}_b \right| + t_{\alpha}' \sum_{q_b < \bar{q}_b} \left| q_b - \bar{q}_b \right| \right\} \leq P_{\alpha}$$

where P_{α} is the maximum number of planes the base can handle.

The expense in money is

$$M = \sum_{\alpha, q, \bar{q}} \ell_{\alpha}(q, \bar{q}) \left\{ m_{\alpha} \sum_{q_b > \bar{q}_b} \left| q_b - \bar{q}_b \right. + m_{\alpha}' \sum_{q_b < \bar{q}_b} \left| q_b - \bar{q}_b \right| + r_{\alpha} \right\} + \sum_{\alpha, \beta, q} x_{\alpha\beta}(q) m_{\alpha\beta}(\Sigma\, q_b) \; .$$

The expense in number of planes used is

$$P = \sum_{\alpha,q,\bar{q}} \ell_\alpha(q,\bar{q}) \max \left\{ t''_\alpha,\ t_\alpha \sum_{q_b > \bar{q}_b} \left| q_b - \bar{q}_b \right| + t'_\alpha \sum_{q_b < \bar{q}_b} \left| q_b - \bar{q}_b \right| \right\}$$

$$+ \sum_{\alpha,\beta,q} x_{\alpha\beta}(q)\, t_{\alpha\beta}(\Sigma q_b).$$

It may be briefly noted that a requirement that airplanes not fly more than a fixed number of hours before being overhauled may be incorporated into the system with a slight change of symbols. In the activity of flying a plane from one base to another, we must specify the "age" of the airplane used as well as the other characteristics of the flight already noted. The intensity of the activity can be denoted by $x_{\alpha\beta}(q;a)$, where a is the number of hours the airplane used has flown since the last overhauling. The input of the airplane is designated by ${}_1P_\alpha(q;\ a)$, the output by ${}_0P_\beta(q;\ a + t_{\alpha\beta})$. The restriction that the airplanes must have an overhauling after A hours can be expressed by saying that we do not permit flying activities in which $a + t_{\alpha\beta} > A$. Similarly, an additional index a has to be added to the loading activity and to the inputs and outputs of airplanes there; for the loading activity, the input will be ${}_0P_\alpha(q;\ a)$, the output ${}_1P_\alpha(\bar{q};\ a)$. Finally, it is necessary to introduce an overhauling activity, taking an airplane at a base α of age a into a plane at the same base of age 0. This activity may also include loading and unloading at the same base. The inputs and outputs will be the same as the loading activity, except that the output will be ${}_1P_\alpha(\bar{q};\ 0)$, instead of ${}_1P_\alpha(\bar{q};\ a)$ and that terms depending only on α will be added to the time and money costs.

The intensity of this activity will be denoted by $v_{\alpha}(q, \bar{q})$. The condition that overhauling can only be done at certain bases can be expressed by saying that the subscript α can only range from 1 to $B' < B$.

je