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RESEARCH MEMORANDUM

THE EMISSIVE POWER OF IONIZED HYDROGEN GAS

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SUMMARY

The emissive power per atom of hydrogen is computed for spontaneous emission into the continuous spectrum lying between the L and M edges.

The results are quoted for temperatures between 3 and 30 ev and densities between 10^{15} and 10^{17} atoms per cubic centimeter.

It is shown that under the stated thermodynamic conditions the complicating effect of free electron screening and bound electron screening may be neglected. Consequently the results are obtained from forms suitable for hand computation.

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I. THE EMISSION AND ABSORPTION COEFFICIENT OF HYDROGEN

We are interested in computing the energy per second lost to the continuous radiation spectrum by a mass of hydrogen gas. The gas is in a temperature range (3-30 ev) and a density range (10^{15} - 10^{17} atoms/cc) which leaves it almost completely ionized. We shall confine our interest to radiation emitted into the spectral region between the L and M shells (i.e., $h\nu \sim 1.5$ to 3 ev).

If the dimensions of the emitting object are large compared with the radiation mean free path, blackbody conditions will prevail. However we shall find that the radiation mean free path is long, so that for most practical problems we must compute in detail the spontaneous emission arising from all the relevant bound-free and free-free transitions. We shall neglect the line spectrum because of the small line widths.

Let the energy spontaneously emitted per atom per second into the energy interval $d(h\nu)$ be given by $E_\nu d(h\nu)$. We can relate E_ν to the inverse absorption length $\mu(\nu)$ by using the method of detailed balancing. Equivalently, we can look at the situation that exists for radiation equilibrium. The energy absorbed per second per unit volume is $\mu(\nu) I(\nu) 4\pi d\nu$, where $I(\nu) d\mathcal{L}$ is the blackbody radiation flux into a solid angle $d\mathcal{L}$. However, because of the induced emission, the net energy absorbed becomes $(1 - e^{-h\nu/kT}) \mu(\nu) 4\pi I(\nu) d\nu$. At equilibrium the net energy absorbed must just equal the spontaneous emission. If ρ_N is the number of atoms per unit volume, we have that

$$\rho_N E_\nu h d\nu = (1 - e^{-h\nu/kT}) \mu(\nu) I(\nu) 4\pi d\nu. \quad (1)$$

We now note that the (spontaneous) emissive power depends only on the thermodynamic state of the matter, and is independent of the spectral distribution of the radiation present. Consequently Eq. (1) holds even if radiation equilibrium does not prevail. Insertion of the Planck value for $I(\nu)$ then yields

$$E_{\nu} = \frac{8\pi}{\rho_N} \frac{\nu^3 \mu(\nu)}{c^2} e^{-h\nu/kT} . \quad (2)$$

The absorption coefficient $\mu(\nu)$ is a thermodynamic average of the absorption processes arising from all of the electron states (both bound and free) that are present in the hydrogen gas. Harris Mayer⁽¹⁾ finds it convenient to define a reduced absorption coefficient, D , by the equation

$$\mu(\nu) = AD \left(\frac{kT}{h\nu}\right)^3 . \quad (3)$$

A is a characteristic inverse length given by

$$A = \frac{2^4}{3\sqrt{3}} \frac{he^2}{mc} \frac{1}{kT} \rho_N . \quad (4)$$

It follows that the emissive power can be written as

$$E_{\nu} = \frac{2^3 \alpha^4 c}{3\sqrt{3} \pi a_0} e^{-h\nu/kT} (kT)_R^2 D . \quad (5)$$

Here, α is the fine structure constant and $(kT)_R$ is the temperature measured in Rydbergs.

Harris Mayer⁽¹⁾ gives a general expression for D, from which it follows that, if the hydrogen gas is almost completely ionized,

$$D = \frac{\rho_N^4 \pi^{3/2} a_0^3}{(kT)_R^{5/2}} \bar{g}_{ff}(h\nu) + \sum \frac{N^b}{N} \frac{1}{n} \left(\frac{I_n}{kT}\right)^2 g_{bf}(h\nu) . \quad (6)$$

The g_{bf} 's and \bar{g}_{ff} are the bound-free and free-free Gaunt factors; they are dimensionless numbers related to the dipole matrix elements for bound-free and free-free transitions. Tables of hydrogenic Gaunt factors have been computed by Karzas and Latter.^(2,3) In Eq. (6), a_0 is the radius of the first Bohr orbit, $\frac{N^b}{N}$ is the fraction of hydrogen nuclei having an electron in the b^{th} bound state, n is the quantum number corresponding to the b^{th} bound state, and I_n is the ionization energy of the n^{th} shell.

We now merely have to compute E_ν by making use of Eq. (6). However we shall first dispose of the following questions: (1) What is the effect of electron screening on the Gaunt factors? (2) Are the number of bound electrons small compared to the number of frees? (3) Will the negative hydrogen ion, H^- , contribute to the reduced absorption coefficient?

II. FREE ELECTRON SCREENING

The Gaunt factors which are tabulated^(2,3) are computed from Coulombic matrix elements. The presence of free electrons will distort the potential seen by the electrons emitting radiation. Since the free-free process involves an electron remote from the nucleus we can expect that free-electron screening will have its greatest effect on the free-free Gaunt factor.

It has been shown elsewhere⁽⁴⁾ that the relative change, F, produced in \bar{g}_{ff} by the free-electron screening is given by

$$F = \frac{\bar{g}_{ff} - g_0}{\bar{g}_{ff}} \frac{2l}{(kT)_R} . \quad (7)$$

As before, \bar{g}_{ff} is the temperature averaged free-free Gaunt factor; $g_0(h\nu)$ is the free-free Gaunt factor for scattering which leaves the electron in a final state of zero kinetic energy; l is a_0 divided by the Debye length

$$l = a_0 \left(\frac{8\pi e^2 \rho_N}{kT} \right)^{1/2} . \quad (8)$$

If we evaluate Eq. (7) at the lowest temperature (3 ev) and the highest density ($1 \times 10^{17} \text{ cm}^{-3}$), we find that over the frequency range that interests us

$$F < 4.1 \times 10^{-3} . \quad (9)$$

We regard this error to be small and conclude that free electron screening has a negligible effect on the Gaunt factors.

III. THE NUMBER OF BOUND ELECTRONS

Equation (6) has been specialized to include the assumption that the H atoms are almost all ionized. The errors made are of the order of the ratio of the number of bound electrons to the number of free electrons. We now need to verify the assumption that this ratio is a small number. Now the number density of bound electrons is just equal* to the number density of all the un-ionized H atoms. If $[N_b]$, $[N_f]$ and $[H^+]$ are, respectively, the number densities of bound electrons, free electrons and H^+ ions, we can write that

$$\frac{[N_b]}{[N_f]} = \frac{[N_b]}{[H^+]} = \frac{[N_f]}{\left(\frac{2\pi mkT}{h^2}\right)^{3/2}} \sum_{n=1}^{n_0} n^2 e^{E_n/kT} . \quad (10)$$

E_n is the binding energy of the n^{th} shell. The sum terminates at $n = n_0$, which is the level at which E_n vanishes.

Following Harris Mayer,⁽¹⁾ we write an approximate expression for the binding energy of the n^{th} level.

$$E_n = \frac{1}{n^2} - \frac{18}{5} \left(\frac{a_0}{a_z}\right) \text{ Rydbergs.} \quad (11)$$

The second term in Eq. (11) arises from the interaction of the atom with the free electrons that are present. This term is density dependent, since a_z is related to the atomic volume according to the following definition

$$\rho_N = \frac{3}{4\pi a_z^3} . \quad (12)$$

* We make the a priori assumption that the concentration of H^- ions is small.

We have omitted from Eq. (11) a negligible term of the form

$$\frac{|r^2|_{nl}}{a_z^2} \left(\frac{a_o}{a_z}\right).$$

Rather than evaluate the sum in Eq. (10), we use the following inequalities

$$\sum_{n=1}^{n_o} n^2 e^{E_n/kT} < \sum_{n=1}^{n_o} n^2 e^{\frac{1}{n^2(kT)_R}} < e^{\frac{1}{(kT)_R}} \sum_{n=1}^2 n^2 + e^{\frac{1}{9(kT)_R}} \sum_{n=3}^{n_o} n^2 \quad (13)$$

We will approximate the last sum in Eq. (13) by an integral and conclude that

$$\sum_{n=1}^{n_o} n^2 e^{\epsilon_n/kT} < e^{1/(kT)_R} \left\{ 5 + \frac{1}{3} \left(\frac{5a_z}{18a_o}\right)^{3/2} e^{-8/(9kT)_R} \right\} \quad (14)$$

We will use the inequality of Eq. (14) and evaluate Eq. (10) at the lowest temperature and highest density to conclude that

$$\frac{[N_b]}{[N_f]} < 5.2 \times 10^{-3}. \quad (15)$$

For our purposes this number is indeed negligible and we may, in general, take

$$[H^+] = \rho_N = [N_f]. \quad (16)$$

IV. THE H⁻ ION: BOUND-ELECTRON SCREENING

In evaluating the sum in Eq. (6) we shall regard the H⁻ ion as an ionic species distinct from the H atom. Clearly, bound-free processes from states of the H atom are uncomplicated by bound electron screening; here the hydrogenic Gaunt factors are rigorously applicable. Bound electron screening must be considered only for bound-free transitions from the H⁻ ion and for free-free scatterings from the H atom cores. We shall show that these last two processes make a negligible contribution to the reduced absorption coefficient.

In considering free-free absorption, we first remark that the cross section for scattering from an H atom must be less than that for the same processes from an (unshielded) H⁺ ion, since the cross section varies as the square of the effective field, and the effective field of the neutral atom is less than that of the ion in all regions. Moreover, from Eqs. (15) and (16)

$$\frac{[H]}{[H^+]} < 6 \times 10^{-3} . \quad (17)$$

A fortiori, the contribution to the free-free process from the (screened) H atom is negligible.

From Eq. (6), we see that the contribution to the bound-free absorption from the H⁻ ion, is proportional to

$$\frac{2 [H^-]}{\rho_N} \left\{ \frac{I_{H^-}}{kT} \right\}^2 g_{H^-} ,$$

where I_{H^-} is the ionization energy (0.055 Rybergs) of H^- , and the factor of two comes from the two (equivalent) electrons. The contribution from the $n = 3$ state of H is proportional to

$$\frac{[\text{H}_{n=3}]}{\rho_N} \frac{1}{3^5} \frac{1}{(kT)^2} g_3.$$

The ratio of these two is

$$r = \frac{[\text{H}^-]}{[\text{H}_{n=3}]} \cdot 2 \cdot 3^5 (I_{\text{H}^-})^2 \frac{g_{\text{H}^-}}{g_3}, \quad (18)$$

Now if we look at the partition function we can show that

$$\frac{[\text{H}^-]}{[\text{H}_{n=3}]} = \frac{1}{4 \cdot 3^2} \frac{[N_f]}{\left(\frac{2\pi mkT}{h^2}\right)^{3/2}} e^{(I_{\text{H}^-} + \frac{8}{9})/kT} = \frac{e^{(I_{\text{H}^-} - \frac{1}{9})/kT}}{4 \cdot 3^2} \frac{[\text{H}_{n=1}]}{[\text{H}^+]} \quad (19)$$

Consequently, since $I_{\text{H}^-} = .055 R_y$,

$$r < (3I_{\text{H}^-})^2 (1.5) \frac{[\text{H}_{n=1}]}{[\text{H}^+]} \frac{g_{\text{H}^-}}{g_{\text{H}_3}} < (.17)^2 \frac{g_{\text{H}^-}}{g_{\text{H}_3}} (1.5) \frac{[\text{H}]}{[\text{H}^+]} < 9(.17)^2 \times 10^{-3} \frac{g_{\text{H}^-}}{g_{\text{H}_3}}. \quad (20)$$

If we make the rough approximation that the Gaunt factors are unity, we have

$$r < 3 \times 10^{-4}. \quad (21)$$

This number is small enough to swamp any uncertainties in the Gaunt factors, and we conclude that the bound-free process from the H^- ion is also negligible.

V. THE EVALUATION OF D

In evaluating D, we note that since the only bound states we need now consider are those of the H atom, we can write

$$\frac{N_b}{N} = \frac{[H_n]}{[H^+]} = \frac{n^2 \rho_N}{\left(\frac{2\pi m k T}{h^2}\right)^{3/2}} e^{E_n/kT} \quad (22)$$

$[H_n]$ is the number density of H atoms in the n^{th} quantum state. The sum in Eq. (6) will terminate when E_n vanishes. The lowest value of n corresponds to an absorption edge; we will start the sum at n equal to 3.

If we use Eq. (22), we can write Eq. (6) in the form

$$D = \frac{3\sqrt{\pi}}{(kT)_R^{5/2}} \left(\frac{a_o}{a_z}\right)^3 \left\{ \bar{g}_{ff} + \frac{2}{(kT)_R} \sum_3^{n_o} \frac{e^{E_n/kT}}{n^3} g_n \right\}, \quad (23)$$

where

$$E_{n_o} = 0 = \frac{1}{n_o^2} - \frac{18}{5} \frac{a_o}{a_z} \quad (24)$$

By using the Euler-Maclaurin sum formula, and replacing sums by integrals in appropriate places, the expression for D may be further simplified to the approximate form,

$$D = \frac{3\sqrt{\pi}}{(kT)_R^{5/2}} \left(\frac{a_o}{a_z}\right)^3 \left\{ \bar{g}_{ff} + \left[\bar{g}_{bf} - \frac{18}{5} \left(\frac{a_o}{a_z}\right) \frac{g_o}{(kT)_R} \right] e^{-\frac{18}{5} \left(\frac{a_o}{a_z}\right) \frac{1}{(kT)_R}} \right\}, \quad (25)$$

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where

$$\bar{\epsilon}_{bf} = \frac{2}{(kT)_R} \sum_{\frac{3}{2}}^{\frac{9}{2}} \frac{e^{-\frac{1}{n^2(kT)_R}}}{n^3} g_n(h\nu) + \frac{11g_0}{1000(kT)_R} \cdot \quad (26)$$

Equation (25) is in a form particularly convenient for hand computation, since now the complicated volume dependence has been separated out. Using Eqs. (5) and (25) the emissive power has been computed over the previously discussed range of parameters, and the results are presented in the accompanying graphs.

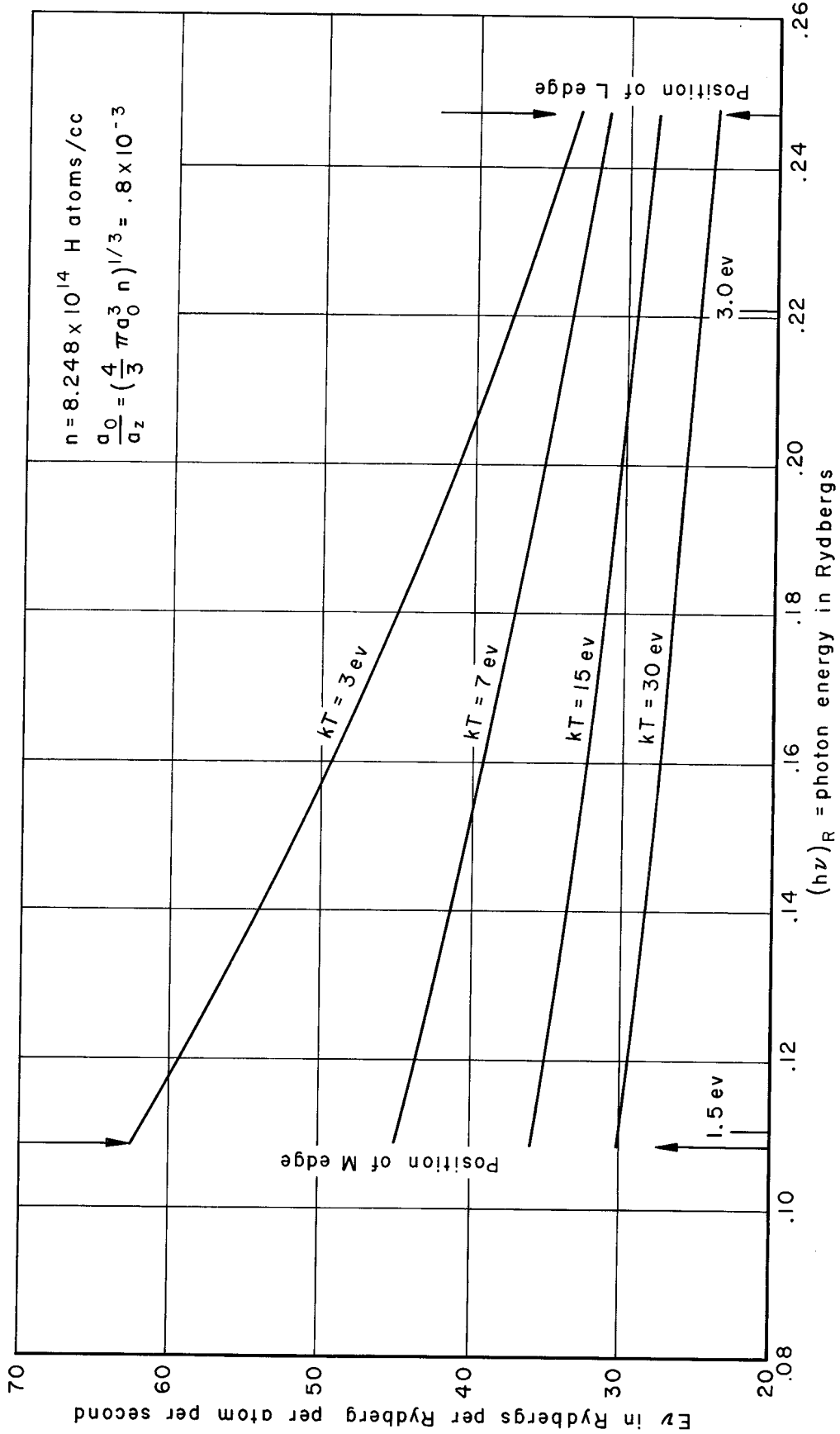


Fig. 1 — Continuous emissive power of H

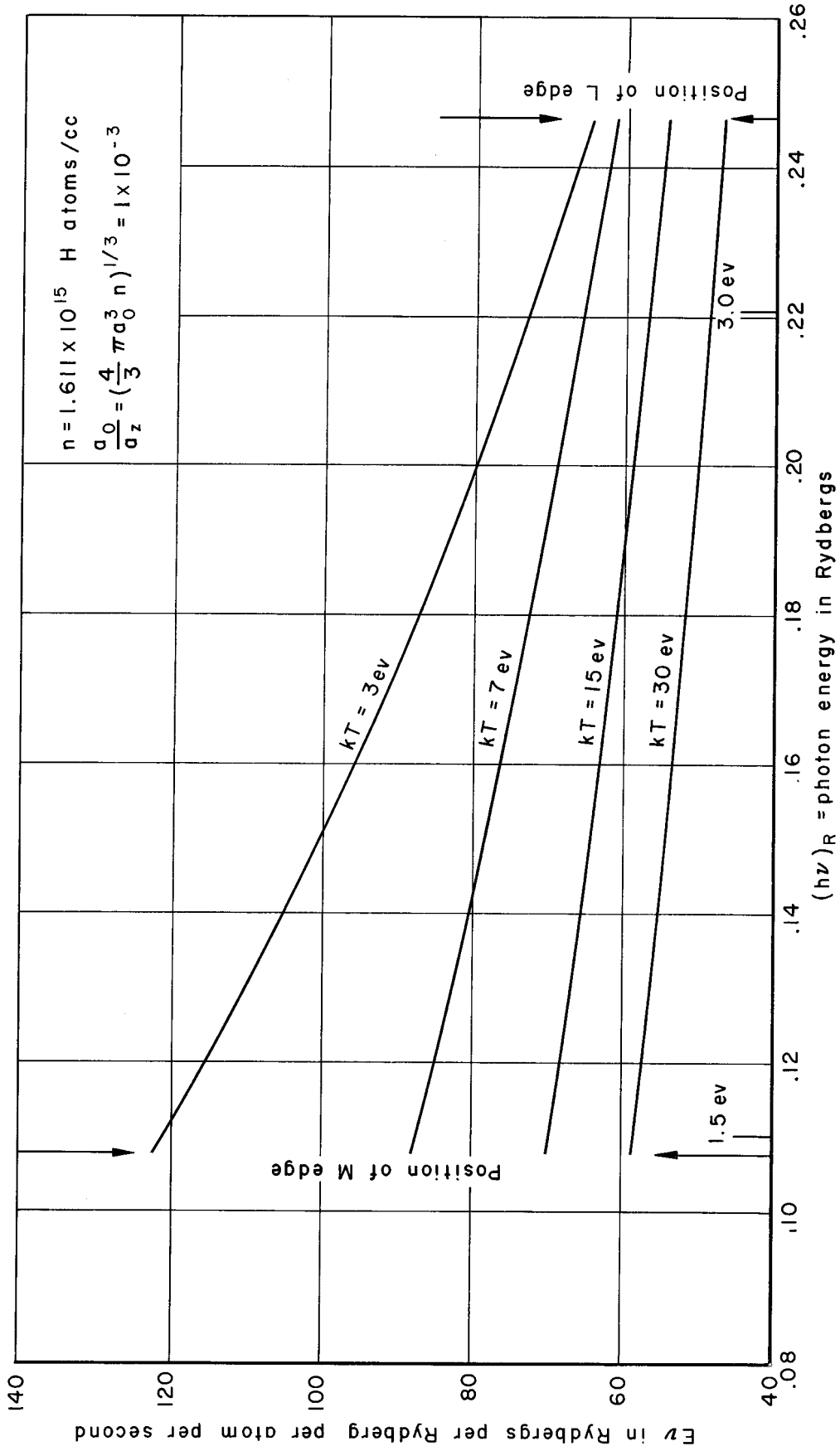


Fig. 2 — Continuous emissive power of H

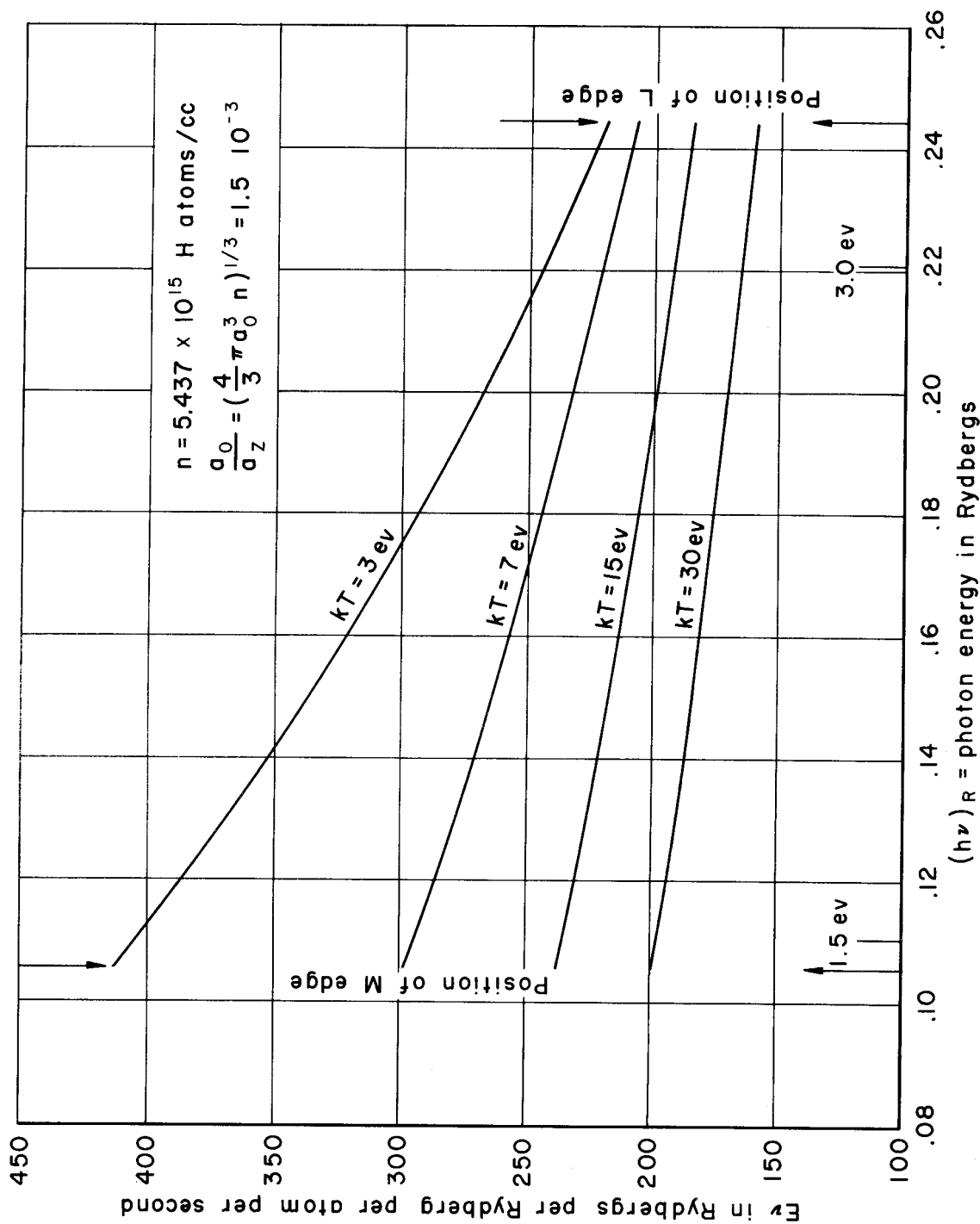


Fig. 3 — Continuous emissive power of H

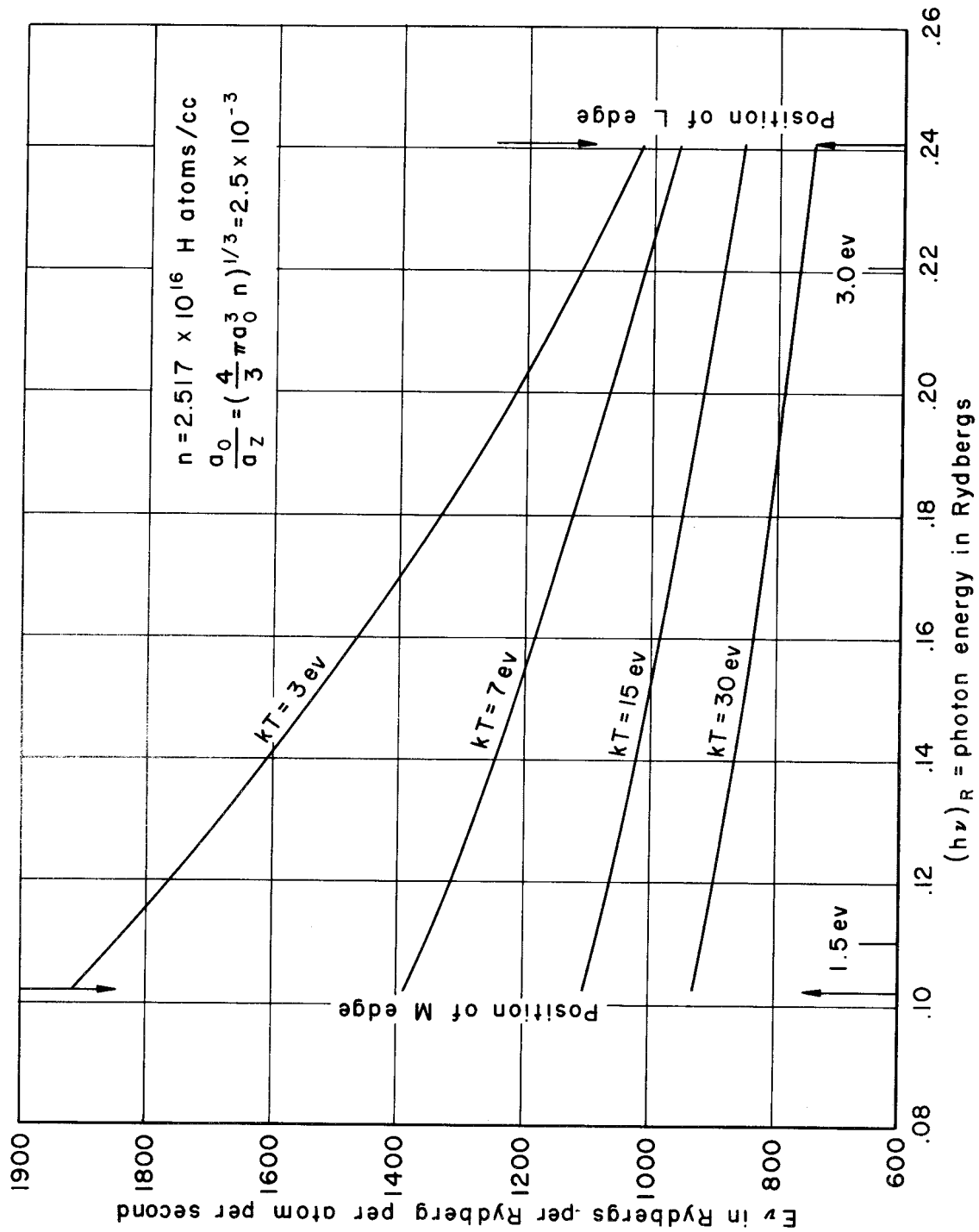


Fig. 4 — Continuous emissive power of H

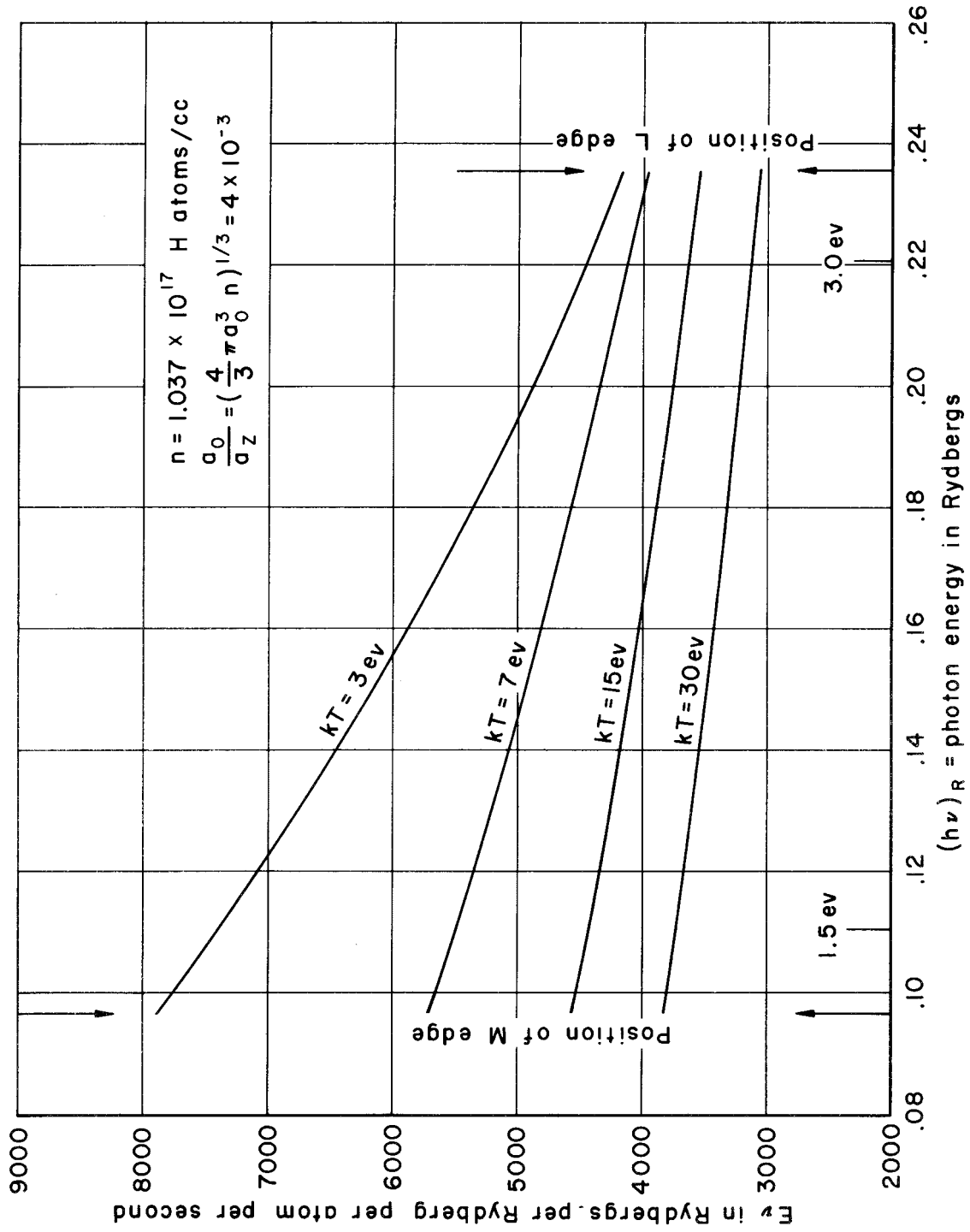


Fig. 5 — Continuous emissive power of H

VI. THE RADIATION MEAN FREE PATH

The data on emissive power presented here may be used to determine the energy lost by a radiating mass of hydrogen gas if the gas is transparent to its own radiation. Neglecting stimulated emission, we can compute

$$\lambda = 1/\mu , \tag{27}$$

the mean free path for absorption.* If we use Eq. (2) we find that

$$\lambda = 1/\mu = \frac{8\pi\nu^3 e^{-h\nu/kT}}{\rho_N c^2 E_\nu} . \tag{28}$$

If we evaluate Eq. (28) at the lowest frequency (1.5 ev), the lowest temperature ($kT = 3$ ev), and the highest density ($a_o/a_z = 4 \times 10^{-3}$) we find that

$$\lambda > 11 \text{ meters.} \tag{29}$$

Objects of suitably small linear dimensions will radiate according to the emissive power here computed.

* At these low frequencies Compton scattering may be neglected.

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