U. S. AIR FORCE

PROJECT RAND

RESEARCH MEMORANDUM

VALUES OF LARGE GAMES - I: A LIMIT THEOREM

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RM-2648

November 2, 1960

Assigned to______________________________

This research is sponsored by the United States Air Force under contract No. AF 49(638)-700 monitored by the Directorate of Development Planning, Deputy Chief of Staff, Development, Hq USAF.

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1700 MAIN ST. • SANTA MONICA • CALIFORNIA

SUMMARY

This paper gives an analysis of what happens in a weighted majority game when a block of votes is broken up and distributed among a large number of players. It is shown that the value of the game to the other players converges to a limit as the size of the largest fragment tends to zero. An explicit expression is given for the limit.

CONTENTS

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1. GENERAL INTRODUCTION

Institutions having large numbers of competing participants are common in political and economic life; examples are markets, exchanges, corporations (from the shareholders' viewpoint), Presidential nominating conventions, legislatures, etc. They can all be regarded as n-person games, yet game theory has not been able so far to produce much in the way of fundamental principles of "mass competition" that might help to explain how they operate in practice. Perhaps some radical theoretical innovations are required, aimed at the special problems posed by such "populous" games; nevertheless, it seems worth while to spend a little effort looking at the behavior of existing n-person solution concepts, as n becomes very large.[1]

Since "cooperative" phenomena are involved, there are vast numbers of player combinations that must be taken into account, in one way or another. Effective techniques of aggregation or approximation will be indispensable if we hope to penetrate significantly beyond the highly symmetrical cases that have so far been solved. The symmetric value, introduced by one of the authors in [2] , seems to be the most promising solution concept for a starter, since it is at heart an average, with

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[1]Kuhn and Tucker list fourteen outstanding research problems in their Preface to the first volume of Contributions to the Theory of Games. The eleventh urges us "to establish significant asymptotic properties of n-person games, for large n" ([1] , p. xii).

comparatively smooth analytical properties. Accordingly, the series of notes (with varying authorship) of which this is the first will be devoted to an investigation of the values of various classes of large games. At first the emphasis will be on the so–called weighted majority games, partly because of the interesting comparison between the values and the voting weights of the players. Later, other types of games will be considered. A notable feature will be the introduction, in various guises, of infinite–person games.

In this first note, after some basic preliminaries, we proceed to analyze what happens in a (finite–person) weighted majority game when a block of votes is broken up and spread over more and more players. The result — i.e., the existence of limit values under appropriate conditions — is recorded below in Theorem 1 and its corollaries. In the second installment of the series, the corresponding infinite–person games will be introduced, in which some of the votes are assigned to an uncountable "ocean" — or continuum — of players. Later papers will take up countably infinite games, bi– and multi–cameral games, and so on. The results of numerical computations of the values of some particular large games will also be given.

2. GAMES AND VALUES

We begin with a very general and abstract definition of "game," wide enough to include all or most of the more special and concrete versions that will come up in this paper and its sequels. A Boolean ring is given, denoted by $\mathcal{R}$. In general, $\mathcal{R}$ may be infinite. Its elements S, T, etc., are interpreted as sets of players. Its atoms (if any) correspond to the identifiable individual players. A real-valued function v(S), called the characteristic function, is given on $\mathcal{R}$; it is unrestricted[2] except for the condition v(0) = 0. It can be thought of as the expected amount of money, or utility, that the set of players S can obtain from the game by forming a coalition and playing optimally. The mechanics of these strategic processes will not concern us; for our purpose a game is no deeper than its characteristic function.[3] Thus, we shall often refer to a function v as a "game."

Any $N \in \mathcal{R}$ such that

$$v(S) \equiv v(N \cap S) \qquad \text{for all } S \in \mathcal{R},$$

is termed a carrier of v. It follows that the intersection of two carriers of v is again a carrier of v, and that v(S) = 0 if S does not intersect every carrier of v.

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[2]In most applications v will be superadditive: $S \cap T = 0$ implies $v(S) + v(T) \leq v(S \cup T)$; this property, however, plays no role in our present work and we do not assume it.

[3]A sort of voting mechanism will presently be introduced, but only as an aid in describing (and motivating) a particular class of characteristic functions, not as a strategic element.

If v is a game with carrier N, the *dual game* v^* is defined by

$$(2.1) \qquad v^*(S) = v(N) - v(N - S), \qquad \text{all } S \in \mathcal{R} .$$

This definition is actually independent of what carrier N is used. Clearly, we have $(v^*)^* = v$. A game v that is self–dual, $v^* = v$, is called "constant–sum" (or, sometimes, "strong") in the literature.

In a game with a finite carrier N, the *value* to a player i can be defined by the formula,

$$(2.2) \qquad \phi_i = \sum_{S \subseteq N - \{i\}} \frac{|S|\,!\,(|N| - |S| - 1)!}{|N|\,!} \left[v(S \cup \{i\}) - v(S) \right] .$$

(We use $|X|$ to denote the cardinal number of the set X.) As with (2.1), this definition is independent of the particular carrier used. The value to all players outside N is zero, and the sum of the values to the players comprising N is precisely v(N). Combining (2.2) and (2.1), we see that *the values of a game and its dual are the same*. We shall sometimes treat the value as a function of sets on $\mathcal{R}$ and write

$$\phi(S) \equiv \sum_{i \in S \cap N} \phi_i .$$

For the present we shall not attempt to extend the "value" concept to games not having finite carriers.

Formula (2.2) can be deduced from some simple axioms (see [2]), which lend intuitive support to the interpretation of ϕ_i as a "value."

But it is often more convenient to regard (2. 1) as arising from a probabilistic model, as follows: Let the players of N be arranged in order at random, with each possible ordering having probability $1/|N|!$. Let P_i denote the set of players that precede the given player, i, in the ordering, and let him be awarded the amount $v(P_i \cup \{i\}) - v(P_i)$ — namely, the gain that he brings to the coalition consisting of his predecessors. Then it is easy to calculate that his expected gain under this scheme is precisely ϕ_i, as defined by (2. 2).

In the special case of a <u>simple game</u>, in which v is monotonic and assumes only the values 0 and 1, there is exactly one player in each ordering that receives a nonzero gain. We shall call him the <u>pivot</u> of the ordering. Thus, in a simple game, a player's value is equal to his probability of being pivotal.

The so–called "weighted majority games" are a special class of simple games. They will be denoted by [c; w] , where c is a real number and w is a measure on $\mathcal{R}$. We may interpret w(S) as the number of votes of the coalition S, and c as the number of votes needed to "win." In terms of the characteristic function, then, we have

$$(2.3) \qquad v(S) = \begin{cases} 0 & \text{if } w(S) < c, \\ 1 & \text{if } w(S) \geq c. \end{cases}$$

Any carrier of w is also a carrier of v. In the case of a finite (or countable) carrier N, we shall sometimes use a more explicit symbol for the game, of the form $[c; w_1, w_2, \ldots]$. Here the players in N are assumed to have been matched with the natural numbers 1, 2, ...; and w_i is an abbreviation of $w(\{i\})$.

A possible variant of $[c; w]$ would be a game that might be denoted by $(c; w)$, in which strict inequality, $w(S) > c$, is required for S to "win." This modification produces nothing new in the finite-carrier case, since $(c; w)$ is identical to $[c + \epsilon; w]$ for sufficiently small positive ϵ. But in general the two formulations cover somewhat different classes of games, because of the possibility that $w(\mathcal{K})$ may have a point of accumulation at c. Their value theories, however, are perfectly interchangeable via the duality relations:

$$(2.4) \qquad \begin{cases} [c; w] = (w(N)-c; w)^*, \\ (c; w) = [w(N)-c; w]^*. \end{cases}$$

We can therefore restrict our attention to the original formulation without loss of generality.

As might be expected, the weights and the values in a weighted majority game are closely related. Both ϕ and w induce the same ranking on the players, except that two players with unequal weights $w_i \neq w_j$ may be tied in value $\phi_i = \phi_j$. (See [2], Example 5.) Also, it is not hard to show that a player's value is a monotonic, nondecreasing function of his weight if the rest of the weights are held fixed and the quota, c, is either held fixed or adjusted to keep the ratio $c/w(N)$ constant. Indeed, since the value purportedly depicts the distribution of "power" in a voting system (see [3]), we should not be surprised to find ϕ and w roughly proportional in most cases.

The following simple examples are designed to show how "rough" this proportionality can be, and to suggest some of the other pitfalls that will beset our attempts to discover a smooth, limiting relationship between ϕ and w.

	WEIGHTED MAJORITY GAME	VALUES
1	[8; 3, 5, 7]	All equal
2	[c; 2, 2, ..., 2, 1] (c even)	Last player zero
3	[c; 2, 2, ..., 2, 1] (c odd)	All equal
4	[7; 5, 1, 1, ..., 1] (9 players)	In ratio 10:1:1 ... :1

To prepare for the main theorem of this paper, let us go more deeply into the class of games typified by game 4, above:

(2.5) $$[c;\ a,\ 1,\ 1,\ \ldots,\ 1] \qquad (n + 1 \text{ players}).$$

Note that if the players are arranged in order at random, the first player will occupy each position with equal probability, and we can therefore find his value by merely counting the number of pivotal positions and dividing by n + 1. Assume for the moment that $a \leq c \leq n$. Then, if a is an integer, we get simply

$$(2.6) \qquad \phi_1 = \frac{a}{n+1},$$

and hence, for the other players,

$$(2.7) \qquad \phi_2 = \ldots = \phi_{n+1} = \frac{n+1-a}{n(n+1)},$$

since the values must add up to 1. The major player thus receives a fraction $\frac{a}{n+1}$ of the total <u>value</u> somewhat greater than his fraction $\frac{a}{n+a}$ of the total <u>weight</u>. This advantage dwindles away, however, as n increases. In other words, letting $n \to \infty$ in (2.5) yields just the obvious result: power proportional to voting strength.

A more interesting result appears if we keep constant the major player's fraction of the total weight, while the number of "minor" players increases. Let us recast (2.5) in the following form:

$$(2.8) \qquad [c; w_1, \frac{\alpha}{n}, \frac{\alpha}{n}, \ldots, \frac{\alpha}{n}] \qquad (n + 1 \text{ players}).$$

The inequality condition assumed in the preceding paragraph then becomes[4]

$$(2.9) \qquad w_1 \leq c \leq \alpha.$$

If w_1 happens to be an integral multiple of α/n then (2.8) reduces directly to the preceding case, and we get

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[4]We shall call this the <u>interior</u> case.

$$\phi_1 = \frac{w_1 n}{\alpha(n + 1)} .$$

If not, the error is not more than $1/(n + 1)$. Thus, in any case, we have

$$\phi_1 = \frac{w_1}{\alpha} + 0(\frac{1}{n}). \tag{2.10}$$

This time, we see that the major player's "advantage" does persist in the limit, since his share of the total weight is $w_1/(\alpha + w_1)$, which is less than w_1/α. Curiously, the result (2.10) is independent of c.

The cases excluded by (2.9) — the noninterior cases — are hardly more complicated. In order to combine them concisely in a single expression, we introduce the following notation, which will be repeatedly useful:

$$\langle x \rangle = \text{median of } (0, x, 1) = \begin{cases} 0 & \text{if } x \leq 0, \\ x & \text{if } 0 \leq x \leq 1, \\ 1 & \text{if } x \geq 1. \end{cases} \tag{2.11}$$

The general formula for the value of the game (2.8) to the first player, if w_1 is an integer multiple of α/n, then turns out to be

$$\phi_1 = \left\langle \frac{cn}{\alpha(n + 1)} \right\rangle - \left\langle \frac{(c - w_1)n}{\alpha(n + 1)} \right\rangle \tag{2.12}$$

(we omit the derivation). Thus, asymptotically,

$$\phi_1 = \left\langle \frac{c}{\alpha} \right\rangle - \left\langle \frac{c - w_1}{\alpha} \right\rangle + 0(\frac{1}{n}) . \tag{2.13}$$

It is precisely in the interior case (2.9) that the brackets $\langle \ \rangle$ can be removed, permitting cancellation of the terms involving c.

A solution could be worked out similarly for the case of two major players, etc. Instead, in the next section, we go immediately to an arbitrary number of major players, and at the same time abandon the symmetry of the minor players, stipulating only that their weights become arbitrarily small.

3. THE LIMIT THEOREM

Consider a sequence of $(m + n_\ell)$-person weighted majority games

$$\Gamma_\ell = [c; w_1, \ldots, w_m, a_{1,\ell}, \ldots, a_{n_\ell,\ell}], \qquad \ell = 1, 2, \ldots, \tag{3.1}$$

such that

$$\sum_{j=1}^{n} a_{j,\ell} = \alpha, \qquad \ell = 1, 2, \ldots, \tag{3.2}$$

α being a positive constant, and such that

$$\max_j a_{j,\ell} \equiv a_{\max,\ell} \to 0 \qquad \text{as } \ell \to \infty. \tag{3.3}$$

These conditions of course entail $n_\ell \to \infty$. (Note that the minor players do not retain their identities from game to game in the sequence.) Let $\phi_{i,\ell}$ denote the value of Γ_ℓ to the i–th major player, $i = 1, \ldots, m$.

We shall employ the following special notations, in addition to the brackets $< >$ defined above in (2.11): M for the set $\{1, \ldots, m\}$ of major players; N_ℓ for the set $\{m + 1, \ldots, m + n_\ell\}$ of minor players; M_i for the set $M - \{i\}$; s for $|S|$; and w(S) for $\sum_{i \in S} w_i$. The subscript

"ℓ" will be suppressed through most of the proof of the theorem.

Theorem 1. *For each major player* $i \in M$, *the values of the games* (3.1), *subject to* (3.2) *and* (3.3), *converge to a limit that is independent of the* $a_{j,\ell}$. *Specifically,*

$$(3.4) \qquad \phi_{i,\infty} = \sum_{S \subseteq M_i} \int_{\left\langle \frac{c-w(S\cup\{i\})}{\alpha} \right\rangle}^{\left\langle \frac{c-w(S)}{\alpha} \right\rangle} t^s (1-t)^{m-s-1} dt.$$

For an estimate of the rate of convergence, we have

$$(3.5) \qquad | \phi_{i,\ell} - \phi_{i,\infty} | = 0(\sqrt[3]{a_{max,\ell}}).$$

Remark: The proof will actually provide a somewhat sharper estimate than (3.5), namely

$$0(\frac{1}{n_\ell}) + 0(\sqrt[3]{n_\ell \sigma^2}),$$

where σ is the standard deviation of the $\alpha_{i,\ell}$. Thus, when the minor players have equal weights, a rate of convergence matching that of (2.13) is assured, even with many major players.

Proof of the Theorem. Let $P_{i,\ell}$ be the set of predecessors of the major player i in a random ordering of $M \cup N_\ell$. Then we have

$$\phi_{i,\ell} = \text{Prob}\,\{c-w_i \leq w(P_{i,\ell}) < c\}.$$

If we separate the "major" and "minor" contingents in $P_{i,\ell}$, this can be written as

$$(3.6) \qquad \phi_{i,\ell} = \sum_{S \subseteq M_i} \sum_{k=0}^{n_\ell} \beta_\ell(S,\ k) \text{ Prob } \{c - w_i \leq w(S) + H_{k,\ell} < c\},$$

where $\beta_\ell(S,\ k)$ is a certain combinatorial coefficient (evaluated below) representing the probability that both $P_{i,\ell} \cap M = S$ and $|P_{i,\ell} \cap N_\ell| = k$, and where $H_{k,\ell}$ is the random variable obtained by summing k terms of the series $a_1 + a_2 + \ldots + a_{n_\ell}$, chosen at random. Since the outer summation of (3.6) now corresponds to (3.4) and involves a fixed (with respect to ℓ), finite number of terms, we shall confine our attention to the inside part, namely,

$$(3.7) \qquad \sum_{k=0}^{n} \beta(S,\ k) \text{ Prob } \{c - w_i \leq w(S) + H_k < c\},$$

with S treated as a constant. (The subscript "ℓ" will henceforth be suppressed.)

Suppose for the moment that we could replace the random variable H_k by its mean value $\overline{H}_k = k\alpha/n$. Then (3.7) would take the form of a simple summation $\Sigma_k\ \beta(S,\ k)$, where the range of k is given by the conditions

$$(3.8) \qquad \begin{cases} 0 \leq k \leq n, \\ \frac{n}{\alpha}\,[c - w_1 - w(S)] \leq k < \frac{n}{\alpha}\,[c - w(S)] . \end{cases}$$

To evaluate this sum we must determine $\beta(S,\ k)$ explicitly. We first compute it exactly (line 2, below), and then reformulate it as an approximate function of k/n:

$$\beta(S, k) = \sum_{\substack{T \subseteq N \\ |T| = k}} \text{Prob}\{P_i = T \cup S\} = \binom{n}{k} \frac{(s+k)!\,(m+n-s-k-1)!}{(m+n)!}$$

$$= \frac{1}{n}\left(\frac{(k+s)!}{k!\,n^s}\right)\left(\frac{(n-k+m-s-1)!}{(n-k)!\,n^{m-s-1}}\right)\left(\frac{n!\,n^m}{(n+m)!}\right)$$

$$= \frac{1}{n}\left(\frac{k^s}{n^s} + 0(\tfrac{1}{n})\right)\left(\frac{(n-k)^{m-s-1}}{n^{m-s-1}} + 0(\tfrac{1}{n})\right)\left(1 + 0(\tfrac{1}{n})\right)$$

$$= \frac{1}{n}\left(\frac{k}{n}\right)^s\left(1 - \frac{k}{n}\right)^{m-s-1} + 0(\tfrac{1}{n^2}) .$$

The asymptotic behavior of the error term here is uniform in k; hence summing on k over the range (3.8) leaves an error that is at worst $0(1/n)$. Keeping within $0(1/n)$, we can pass from the sum on k to an integral on a continuous variable t, corresponding to k/n, as follows:

$$\Sigma_k\ \beta(S, k) \approx \int_{t_1}^{t_2} t^s(1-t)^{m-s-1}dt.$$

Inserting $t = k/n$ into (3.8), we find our limits of integration are

$$t_1 = \left\langle \frac{1}{\alpha}\,[c - w(S \cup \{i\})] \right\rangle,\quad t_2 = \left\langle \frac{1}{\alpha}\,[c - w(S)] \right\rangle .$$

These are exactly the limits of integration stipulated in the statement of the theorem. Since $1/n \leq a_{max}/\alpha$, the error committed so far is well within the stated tolerance (3.5).

In order to complete the proof of the theorem, it remains only to justify the use of the mean value $\overline{H}_k$ in place of H_k. Define, for real x,

$$\epsilon_k(x) = \text{Prob}\,\{H_k < x\} - \text{Prob}\,\{\overline{H}_k < x\}.$$

(The last term is of course always either 0 or 1.) As with any random variable with given mean and variance, we have[5]

$$(3.9) \qquad |\epsilon_k(x)| \leq \frac{\sigma_k^2}{\sigma_k^2 + (x-\overline{H}_k)^2},$$

where σ_k^2 denotes the variance of H_k. Now

$$\sigma_k^2 = \frac{k(n-k)}{(n-1)}\,\sigma_1^2 \leq n\sigma_1^2 = \Sigma\, a_j^2 - \alpha^2/n.$$

Denote the last expression by θ, thus: $\theta = \Sigma\, a_j^2 - \frac{\alpha^2}{n}$. We shall weaken the estimate (3.9) to the more useful form

$$(3.10) \qquad |\epsilon_k(x)| \leq \min\left[1, \frac{\theta}{(x-\overline{H}_k)^2}\right].$$

Now, the precise error incurred in replacing H_k by $\overline{H}_k$ in (3.7) is

$$\Sigma_k\, \beta(S,\ k)\,[\epsilon_k(c-w_i-w(S)) - \epsilon_k(c-w(S))],$$

the summation to be taken over the range (3.8). It will clearly be sufficient for our purpose if we can find a suitable asymptotic estimate, uniform in x, for the simpler quantity,

[5]This bound is derived from considering the "worst" case — a distribution concentrated at the two points x and $\overline{H}_k - \sigma_k^2/(x-\overline{H}_k)$.

$$\epsilon(x) = \sum_{k=0}^{n} \beta(S, k) \, |\epsilon_k(x)|.$$

Noting that $\beta(S, k) \leq 1/(m+n)$ for all S, k, and applying (3.10), we have

$$|\epsilon(x)| \leq \frac{1}{m+n} \sum_{k=0}^{n} \min \left[1, \frac{\theta}{(x-\overline{H}_k)^2}\right].$$

If we omit those terms for which $|x-\overline{H}_k|$ is relatively small, say less than some preassigned δ, then the other terms are all dominated by θ/δ^2, and their sum is $\leq (n+1)\theta/\delta^2$. On the other hand, the contribution from the omitted terms is no greater than the number of such terms, which does not exceed $2n\delta/\alpha + 1$. Hence, for any $\delta > 0$ that we may choose, we have

$$|\epsilon(x)| \leq \frac{1}{m+n} \left[\frac{(n+1)\theta}{\delta^2} + \frac{2n\delta}{\alpha} + 1\right]$$

uniformly in x. A good choice for δ is $\theta^{1/3}$; this gives us $0(\theta^{1/3}) + 0(1/n)$ as an error estimate. The latter term is tolerable, as we saw before; as for the other, we have

$$\theta = \Sigma a_j^2 - \frac{\alpha^2}{n} < \Sigma a_j^2 \leq \Sigma a_j a_{max} = \alpha a_{max},$$

which provides the $0(\sqrt[3]{a_{max}})$ of the theorem. Q. E. D.

4. REMARKS AND ADDITIONAL RESULTS

For one major player, m = 1, the formula of Theorem 1 agrees with our previous result (2.13). If m = 2, then (3.4) reduces to

$$(4.1) \quad \left\langle \frac{c}{\alpha} \right\rangle - \left\langle \frac{c-w_1}{\alpha} \right\rangle - \frac{1}{2}\left\{ \left\langle \frac{c}{\alpha} \right\rangle^2 - \left\langle \frac{c-w_1}{\alpha} \right\rangle^2 - \left\langle \frac{c-w_2}{\alpha} \right\rangle^2 + \left\langle \frac{c-w_1-w_2}{\alpha} \right\rangle^2 \right\}$$

for $\phi_{1,\infty}$, and similarly for $\phi_{2,\infty}$. In the "interior" case, $w(M) \leq c \leq \alpha$, however, all brackets can be removed, and after much cancellation we obtain

$$(4.2) \qquad \phi_{1,\infty} = \frac{w_1}{\alpha} - \frac{w_1 w_2}{\alpha^2}, \qquad \phi_{2,\infty} = \frac{w_2}{\alpha} - \frac{w_1 w_2}{\alpha^2}.$$

Note that c disappears entirely. The corresponding interior–case formulas for larger values of m will be presented in a later paper.

An inspection of (3.4) reveals that $\phi_{i,\infty}$ is unchanged if c is replaced by $\alpha + w(M) - c$. Hence, by duality (see (2.4)), we have the following result:[6]

Corollary 1. Theorem 1 remains valid in all respects if the "[ ]" games of (3.1) are replaced by the corresponding "( )" games.

The theorem also remains valid if the major weights, the total minor weight, and the winning quota are allowed to vary (convergently) during the passage to the limit:

Corollary 2. Consider the sequence of games

$$(4.3) \qquad H_\ell = [c_\ell;\, w_{1,\ell}, \ldots, w_{m,\ell}, a_{1,\ell}, \ldots, a_{n_\ell,\ell}],$$

[6] This corollary is also apparent from the fact that the distinction between "$\leq$" and "$<$" in (3.6) above, and elsewhere, plays no part in the proof.

where $c_\ell \to c$, $w_{i,\ell} \to w_i$ for each $i \epsilon M$, and

$$(4.4) \qquad \sum_j a_{j,\ell} \equiv \alpha_\ell \to \alpha > 0$$

as $\ell \to \infty$. Then, subject to condition (3.3), the values of the major players converge to the limit given by (3.4).

Proof. It is not sufficient just to point out that (3.4) is a continuous function of α, c, and the w_i. One can go through the proof of Theorem 1, however, and show that the various "error" terms are all independent of these parameters, or can be made so. Indeed, the convergence to the limit (3.4) is uniform in the parameters if we exclude a neighborhood of $\alpha = 0$. This, together with the continuity of (3.4), is sufficient. Of course, the estimate (3.5) of the rate of convergence would have to be modified to take account of the additional variables.

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