U. S. AIR FORCE

PROJECT RAND

RESEARCH MEMORANDUM

VALUES OF LARGE GAMES, II: OCEANIC GAMES

J. W. Milnor
L. S. Shapley

RM–2649

February 28, 1961

Assigned to ________________________

This research is sponsored by the United States Air Force under contract No. AF 49(638)-700 monitored by the Directorate of Development Planning, Deputy Chief of Staff, Development, Hq USAF.

This is a working paper. It may be expanded, modified, or withdrawn at any time. The views, conclusions, and recommendations expressed herein do not necessarily reflect the official views or policies of the United States Air Force.

The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

SUMMARY

A value theory is developed for voting games in which a sizable fraction of the total vote is controlled by a few major players and the rest is distributed among a continuous infinity of individually insignificant minor players. The latter are referred to collectively as an "ocean," to suggest the total lack of order or cohesion that is assumed.

CONTENTS

SUMMARY iii

Section
1. THE CONCEPT OF OCEANIC GAME 1
2. DEFINITIONS 2
3. CONTINUITY OF THE VALUE IN THE LIMIT 6
4. VARIABLE QUOTAS. THE OCEANIC VALUE. ADDED PLAYERS 9
5. A RECURSION FOR THE INTERIOR CASE 16
6. GENERALIZED OCEANIC GAMES 22

APPENDIX 25

REFERENCES 27

VALUES OF LARGE GAMES, II: OCEANIC GAMES

1. THE CONCEPT OF OCEANIC GAME

The first note in this series* developed the idea of a sequence of weighted majority games in which certain fixed fractions of the voting strength are held by a few "major" players, while the rest is scattered among a growing number of increasingly insignificant "minor" players. It was shown that the game values to the major players tend to limits that are independent of the particular way in which the minor weights go to zero. An explicit expression for the limits was obtained. (For ease of reference, this fundamental limit theorem is reproduced in the appendix of the present research memorandum.)

The fact that at least one kind of solution — i.e., the value — is convergent in such a sequence of games suggests that we might profitably deal with the limit of the sequence as a game in its own right. The limit game would of necessity have infinitely many players: a finite discrete set corresponding to the major players, plus a continuum of infinitesimal minor players. We shall refer to the latter as an "ocean," to emphasize the almost total absence of order or cohesion. The voting power of the "oceanic" players will be expressed as a measure, defined on the measurable subsets of the ocean. Any payoffs or value allocations to the oceanic players will be represented similarly in measure-theoretic

*See the references at the end of this research memorandum.

terms; individual oceanic players will not be considered.

In an oceanic game we would hope to be able to define the major–player values in such a way that continuity is preserved with respect to the values of the finite approximants; i.e., we would expect that

$$(1.1) \qquad \lim_{\ell \to \infty} (\text{val}\,\Gamma_\ell) = \text{val}\,(\lim_{\ell \to \infty} \Gamma_\ell).$$

That our definition, given below in Sec. 2, fulfills this hope is confirmed by Theorem 1, proved in Sec. 3. In the rest of this paper we proceed to reap some of the benefits of dealing directly with an infinite–person game, instead of with a sequence of finite approximants, and derive some interesting properties of the value–solutions. In particular, the cumbersome general formula of [1] is greatly simplified in Sec. 5 for the important special class of "interior" games — those in which the ocean by itself is a winning coalition. A generalized definition of oceanic game is discussed in Sec. 6. The computed values of some oceanic games of particular interest will be presented in subsequent notes of this series (see, e.g., [2]).

2. DEFINITIONS

Following the general procedure for defining a game, as set forth in reference [1], take $\mathcal{R}$ to be the Boolean ring generated by the subsets of the finite set $M = \{1, \ldots, m\}$ together with the Lebesgue–measurable subsets of the real unit interval $I = [0, 1]$. Let real numbers $w_1, \ldots, w_m \geq 0$ be given, and write $w(S)$ for $\Sigma_S w_i$. Define a "vote" measure u on $\mathcal{R}$ by

$$u(R) = w(R \cap M) + \alpha\mu(R \cap I), \qquad R \in \mathcal{R}, \tag{2.1}$$

where μ denotes Lebesgue measure and α is a positive constant. Then the weighted majority game $[c; u]$, with majority quota $c \geq 0$, is called an <u>oceanic game</u>, and will be denoted by the special symbol

$$[c; w_1, \ldots, w_m; \alpha]. \tag{2.2}$$

We see that the total weight of the ocean I is α, and that a coalition wins if and only if its fraction of the ocean, plus its contingent of major players, "weighs" at least c.

How may we define the value of such a game? The direct formula, applicable to finite-person games,* is not readily extended to the present case. We therefore resort to the "pivotal player" approach, wherein a player's value is taken to be the probability that, in a random ordering of all the players, he and his predecessors together have enough votes to win, but his predecessors alone do not. In the finite case this is equivalent to the direct definition. We shall use this approach repeatedly in constructing value theories for infinite-person games.

---

*Equation (A.4) in the appendix.

The notion of a perfectly random shuffling of the continuum of oceanic players, even without the major players, is not an easy thing to formulate precisely.* Fortunately, because of the symmetry of the present case, the problem can be sidestepped. We must only be sure to insert the major players into the ocean in a properly random fashion — the ocean having previously been ordered in some fixed way.

Accordingly, let $x_1, \ldots, x_m$ be independent random variables distributed uniformly on the unit interval I. Then for any measurable set A we have

$$(2.3) \qquad \text{Prob}\,\{x_i \in A\} = \mu(A).$$

Let P(x) denote the set of major players $i \in M$ such that $x_i < x$. Define the "predecessors" of a major player i to be the finite set $P(x_i)$ together with the oceanic interval $[0, x_i)$. Then define ϕ_i, the <u>value of the game to player i,</u> to be the probability that

$$(2.4) \qquad w(P(x_i)) + \alpha x_i \leq c \leq w(P(x_i)) + w_i + \alpha x_i.$$

As for the typical oceanic player, if his position in the ocean is represented by the real number x, then <u>his</u> predecessors will comprise the set $P(x) \cup [0, x)$, and we might arbitrarily call him pivotal if

$$w(P(x)) + \alpha x = c.$$

---

*The general question of random orderings of infinite sets will be taken up in a later paper of this series.

Under this artificial definition, the combined value, Φ, of all the oceanic players just complements the values of the major players:

$$\Phi + \phi(M) = 1. \tag{2.5}$$

We might equally well take (2.5) as the definition of Φ.* Of course the value of any particular oceanic player is zero.

If we examine the cumulative weight function (see Fig. 1)

$$f(x) = w(P(x)) + \alpha x,$$

we see that it is monotonic increasing (strictly) with discontinuous jumps at the points $x_1, \ldots, x_m$ and with a constant slope, α, in between. This function is of course a random variable. If the graph happens to intersect the c level, then there is an oceanic pivot; if not, then the major player responsible for the jump past that level is the pivot.

A second way to view the situation geometrically is to regard $(x_1, \ldots, x_m)$ as a random point in the m–dimensional unit cube I^m. (See Fig. 2.) Let A_i denote the subset of I^m in which the inequalities (2.4) are satisfied. The sets $A_1, \ldots, A_m$ are obviously disjoint, except for some overlapping boundaries of measure zero. The

---

*This relationship would fail in the "null game," $c > w(M) + \alpha$, in which there are no winning coalitions and all values are zero. We therefore assume that $0 \leq c \leq w(M) + \alpha$.

major-player values, $\phi_1, \ldots, \phi_m$, are by definition just the m-dimensional volumes of $A_1, \ldots, A_m$, respectively. The oceanic value Φ is the volume of "no man's land" (shaded in the figure). The numerical values given in Fig. 2 may be compared with the corresponding numbers for the similar (and easy-to-compute) 12-person game: [8; 4, 1, 1, ..., 1], which are

$$\phi_1 = .333, \quad \phi_2 = .061, \quad \sum_3^{12} \phi_i = .606.$$

As the theorem of the next section will show, further refinement of the minor players' weights would produce an even better fit.

3. CONTINUITY OF THE VALUE IN THE LIMIT

We shall now state and prove the continuity theorem stated symbolically in (1.1).

Theorem 1. *Let $\phi_{i,\ell}$ denote the value to player $i \leq m$ of the $(m+n_\ell)$-person game $[c_\ell;\ w_{1,\ell}, \ldots, w_{m,\ell},\ a_{1,\ell}, \ldots, a_{n_\ell,\ell}]$, and let ϕ_i denote the value to major player i of the oceanic game $[c;\ w_1, \ldots, w_m;\ \alpha]$. Then*

$$\phi_i = \lim_{\ell \to \infty} \phi_{i,\ell}, \tag{3.1}$$

provided that, as $\ell \to \infty$, we have

$$c_\ell \to c, \quad w_{i,\ell} \to w_i, \quad \sum_j a_{j,\ell} \to \alpha, \quad \max_j a_{j,\ell} \to 0. \tag{3.2}$$

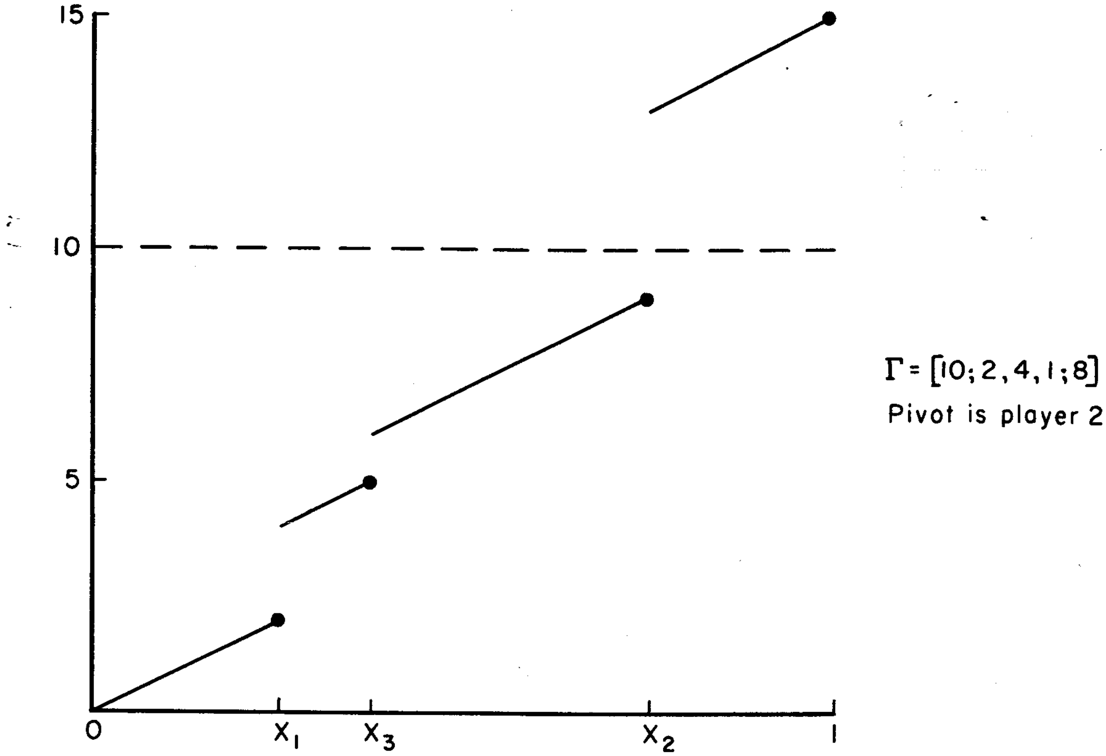

Fig. 1—Cumulative voting strength after a random ordering

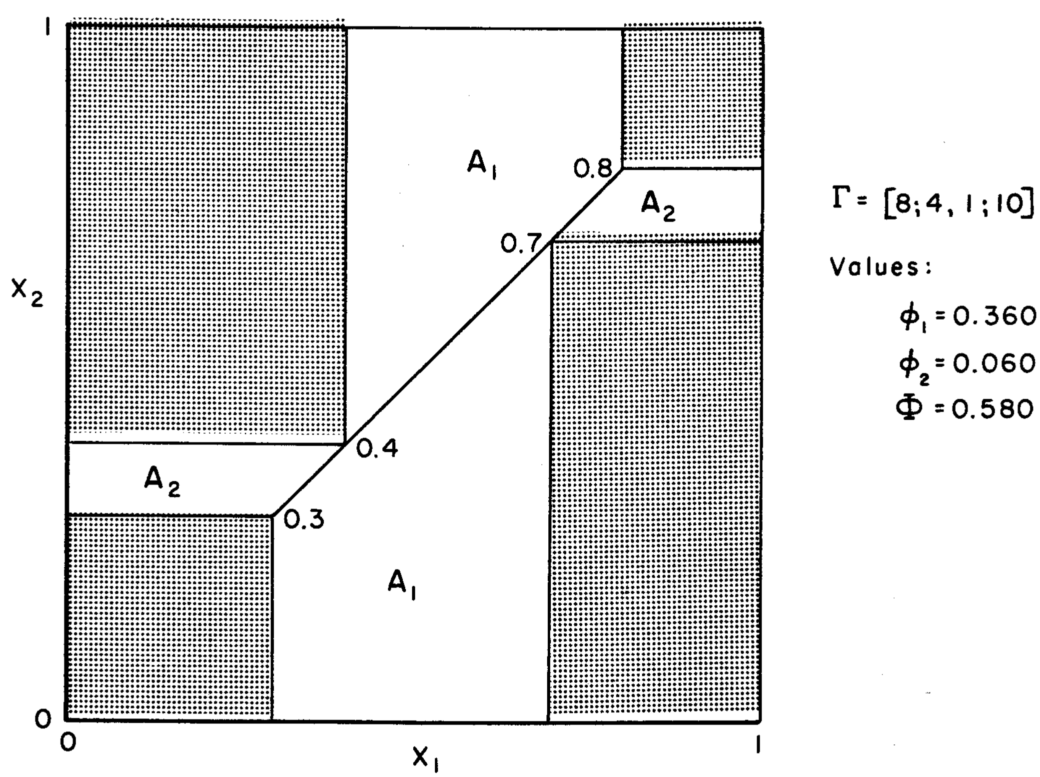

Fig. 2—Typical partition of I^2

Proof. Consider a fixed $i \in M$ and let S be a subset of M not containing i. Let $B_{i,S}$ denote the set of points $(x_1, \ldots, x_m)$ in I^m such that

$$(3.3) \qquad \begin{cases} x_j < x_i & \text{for all } j \in S, \\ x_j \geq x_i & \text{for all } j \in M - S. \end{cases}$$

Every point of I^m is in exactly one of these sets (given a fixed i), namely the set $B_{i, P(x_i)}$. Recalling our definition of A_i (see Fig. 2), we have

$$(3.4) \qquad \phi_i = \mu^m(A_i) = \sum_{S \subseteq M - \{i\}} \mu^m(A_i \cap B_{i,S}),$$

where μ^m denotes m–dimensional Lebesgue measure. Consider a particular set $A_i \cap B_{i,S}$. For a given value of x_i, the other coordinates range independently over one or the other of the intervals $[0, x_i)$ and $[x_i, 1]$, and the $(m-1)$–dimensional measure of the cross section is $x_i^s(1 - x_i)^{m-s-1}$. Hence we have

$$(3.5) \qquad \mu^m(A_i \cap B_{i,S}) = \int_{t_1}^{t_2} x_i^s(1 - x_i)^{m-s-1}\, dx_i.$$

Here the limits of integration depend on the pivot inequalities (2.4). Taking $P(x_i) = S$ in the latter, and including (via the bracket notation*)

---

*We recall that $\langle x \rangle$ denotes the median of 0, x, and 1.

the over—all range restriction $0 \leq x_i \leq 1$, we find that

$$(3.6) \qquad t_1 = \left\langle \frac{c - w(S \cup \{i\})}{\alpha} \right\rangle, \qquad t_2 = \left\langle \frac{c - w(S)}{\alpha} \right\rangle.$$

Assembling (3.4), (3.5), and (3.6) gives us precisely the expression derived in [1] for the limit of the finitely defined values $\phi_{i,\ell}$.* This completes the proof of Theorem 1.

4. VARIABLE QUOTAS. THE OCEANIC VALUE. ADDED PLAYERS

Let us now consider the majority quota c of the oceanic game (2.2) as a variable, and write

$$(4.1) \qquad \Gamma(y) = [y;\ w_1, \ldots, w_m;\ \alpha].$$

Define the function

$$(4.2) \qquad F(y) = \min \{x \mid w(P(x)) + \alpha x \geq y\},$$

where $0 \leq y \leq w(M) + \alpha$. This function is essentially the inverse of the function f(x) illustrated in Fig. 1. It gives the location in the ordered ocean of the pivot of the game with quota y. F is continuous, nondecreasing, and piecewise linear, with slope alternating between 0 and $1/\alpha$. (See Fig. 3.)

---

*Equation (A.3) in the appendix.

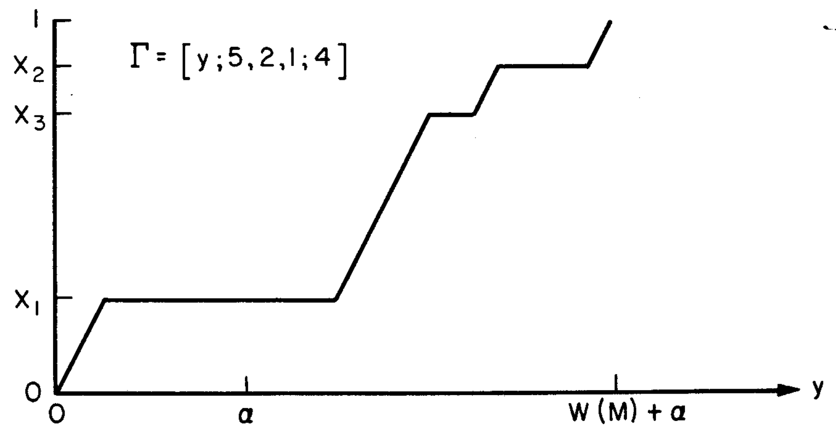

Fig. 3—A typical F(y)

Let $\phi_i(y)$ denote the value of the game $\Gamma(y)$ to the i–th major player. Let $\Phi(y)$ denote the value to the ocean. Applying the remarks that accompany Fig. 1, we see that

(4.3) $$\phi_i(y) = \text{Prob}\ \{F(y) = x_i\}, \qquad i = 1, \ldots, m.$$

Also, since F(y) is differentiable at any particular y with probability 1, we have

(4.4) $$\sum_1^m \phi_i(y) = \text{Prob}\ \{F'(y) = 0\}$$

and

(4.5) $$\Phi(y) = \text{Prob}\ \{F'(y) = 1/\alpha\}.$$

Let

$$E\{\cdots\} = \int_{I^m} \cdots dx_1 \ldots dx_m$$

denote the "expected value" operator with respect to our basic random variables $x_1, \ldots, x_m$. Then (4.5) can be rewritten

(4.6) $$\Phi(y) = \alpha E\{F'(y)\}.$$

But "E" commutes with the derivative. Hence the oceanic value is essentially just the slope of the function $E\{F\}$.

Theorem 2. The combined value of the oceanic players in $\Gamma(y)$ is given by

(4.7) $$\Phi(y) = \alpha \frac{d}{dy} E\{F(y)\}.$$

We easily verify that $E\{F\}$ is always continuous and has a continuous first derivative. Typical examples of $E\{F\}$ for $m = 1$ are shown in Fig. 4. Note that Φ tends to be smaller for central values of y, especially if the major player is very powerful, as in the second case illustrated.

A simple consequence of (4.3) — (4.7) is worth separate mention, because it helps clarify the distinction between voting strengths (as measured by the weights) and "power" (as measured by the values).

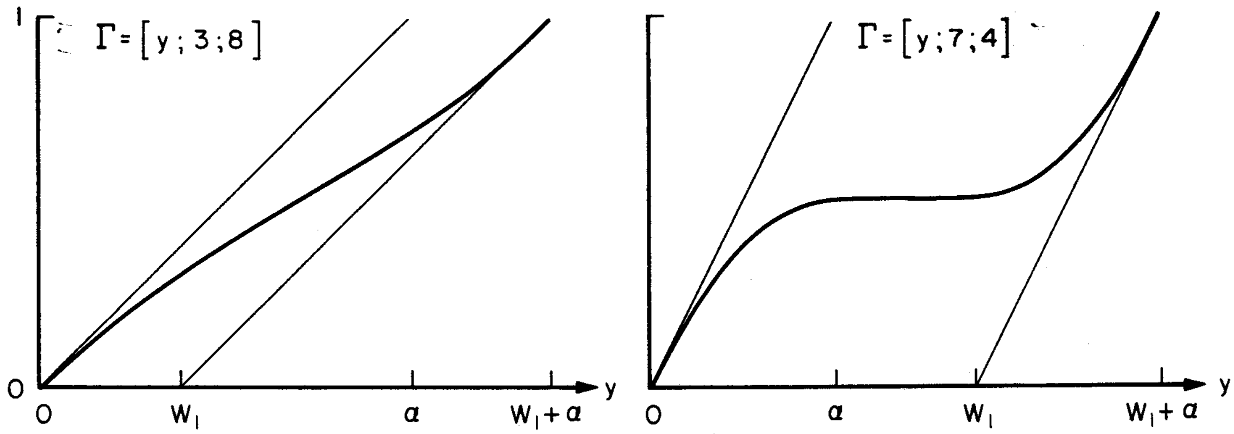

Fig. 4 — Some typical $E\{F(y)\}$ for m = 1

Theorem 3. The value of any major player (or of the ocean) in the game $\Gamma(y)$, averaged over all possible quotas y from 0 to $W = w(M) + \alpha$, is precisely equal to his (its) fraction of the total weight.

By the symmetry of F this theorem remains true if y is averaged only over the interval (W/2, W]. This frees us from having to consider "improper" games whose low quotas permit several coalitions to "win" simultaneously.*

Note that as y approaches W (the "unanimous" game) the values of all players tend to equalize. This means that the major players become very weak while the ocean becomes all–powerful. To counterbalance this,

*The counterpart of Theorem 3 for finite weighted majority games will be found in [3], page 16.

Theorem 3 implies that for each major player there will be a smaller quota that gives him a compensatingly higher value, relative to his voting strength. A detailed study of the case $m = 2$ (see [2]) indicates that the quota $y = W/2$ (the simple majority case) is usually favorable to the major players in this sense, but not always.

Now let us consider the oceanic game $\Gamma^+(y)$ obtained by adding a new major player to the game $\Gamma(y)$ of (4.1), thus:

$$\Gamma^+(y) = [y; w_1, \ldots, w_m, w_{m+1}; \alpha]. \tag{4.8}$$

Fix a point $(x_1, \ldots, x_m)$ in I^m. By (4.2), this point defines a particular function $F(y)$ with respect to the game $\Gamma(y)$. Now choose x_{m+1} at random from I^1, and use it to define the corresponding function $F^+(y)$ with respect to $\Gamma^+(y)$. Graphically (see Fig. 5), F^+ differs from F only by having a

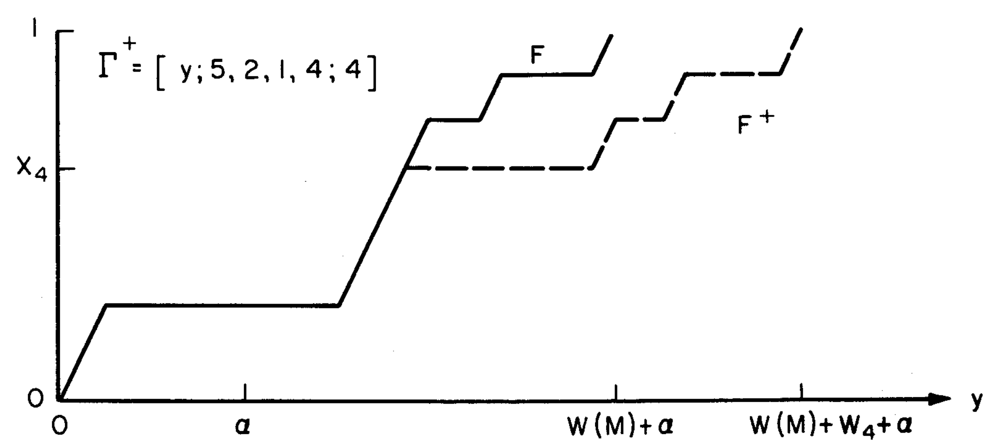

Fig. 5 — Effect of added player on F(y)
(Compare Fig. 3)

horizontal segment of length w_{m+1} inserted at the x_{m+1} level, with everything to the right moved bodily over to accommodate the insertion. Let $0 \leq z \leq w(M) + w_{m+1} + \alpha$. In order that $F^+(z) = x_{m+1}$ (the pivot condition for player $m+1$ in $\Gamma^+(z)$), we must have

(4.9) $$F(z - w_{m+1}) \leq x_{m+1} \leq F(z).$$

The probability of (4.9) with respect to the single random variable x_{m+1} is simply $F(z) - F(z - w_{m+1})$, provided we assume both arguments of F in (4.9) to be within the domain of definition — i.e., provided we assume that

(4.10) $$w_{m+1} \leq z \leq w(M) + \alpha .$$

More generally, it is $F(t_2) - F(t_1)$, with

(4.11) $$\begin{cases} t_1 = \max (z - w_{m+1}, 0), \\ t_2 = \min (z, w(M) + \alpha) . \end{cases}$$

Integrating this probability over the other random variables $x_1, \ldots, x_m$ gives us an expression for the value to the $(m+1)$–st player in the game $\Gamma^+(z)$, as follows:

$$\phi^+_{m+1}(z) = \operatorname*{Prob}_{I^{m+1}} \{F^+(z) = x_{m+1}\} = \operatorname*{E}_{I^m} \{F(t_2) - F(t_1)\} .$$

Theorem 2 now comes into play, enabling us to replace $E\{F\}$ by the integral of Φ. The following useful lemma results:

Lemma. The added player's value in the game $\Gamma^{+}(z)$ is related to the oceanic values in the games $\Gamma(y)$ by

$$\phi^{+}_{m+1}(z) = \frac{1}{\alpha}\int_{t_1}^{t_2} \Phi(y)\,dy, \tag{4.12}$$

the limits being given by (4.11). If (4.10) holds, this reduces to

$$\phi^{+}_{m+1}(z) = \frac{1}{\alpha}\int_{z-w_{m+1}}^{z} \Phi(y)\,dy. \tag{4.13}$$

This lemma makes it easy to prove the next theorem, which states the not–surprising fact that a very small major player is almost indistinguishable (in value) from a piece of "ocean" of the same size.

Theorem 4. In the general oceanic game, if a major player's weight tends to zero, his value–per–vote ratio ϕ_i/w_i approaches the value–per–vote ratio Φ/α of the oceanic players.

Proof. It suffices to establish the principle for the added player in (4.8). Differentiating (4.13) gives us

$$\frac{d\phi^{+}_{m+1}(z)}{dw_{m+1}} = \frac{1}{\alpha}\,\Phi(z - w_{m+1}) \tag{4.14}$$

(subject to (4.10)). Continuity of the values with respect to w_{m+1} makes (4.10) inoperative in the limit as $w_{m+1} \to 0$, and also removes the distinction between Φ and Φ^{+} in the limit. The result is now apparent.

5. A RECURSION FOR THE INTERIOR CASE

The lemma proved in the last section is important because it relates the values of $(m+1)$—person oceanic games to the values of m—person oceanic games. It opens the possibility of computing values recursively, from the ground up. (Or, should we say, from the water up?) We could hope to start with games that have no major players and alternately apply (4.13) and (2.5).* An encouraging sign is the absence of case distinctions that proliferate at each stage of the recursion. So long as we can stick to (4.13) in place of (4.12), we need only deal with a single analytic function at each step. On the other hand, the repeated integrations may prove to be cumbersome, and the repeated application of condition (4.10) threatens to impose ever—narrowing restrictions on the generality of the result. Happily, neither of these difficulties assumes serious proportions.

Let us first consider the effect of the conditions (4.10). Suppose that we are attempting to derive recursively the value formulas for the m—major—player game

$$[c;\ w_1,\ \ldots,\ w_m;\ \alpha], \tag{5.1}$$

using (4.13) but not (4.12). It is evident that the y's that appear in the integrals at the first stage of the iteration ($m = 0$ in the lemma) will collectively cover the entire range from $c - w(M)$ to c. This range will

---

*Strangely, no such recursive method for computing values has ever been found for finite games.

have to be included in the domain of definition of the innermost F function. There being no major players, this domain is simply the interval $[0, \alpha]$. Thus we must have, as a necessary condition,

$$0 \leq \{c - w(M), \; c\} \leq \alpha, \tag{5.2}$$

or

$$w(M) \leq c \leq \alpha. \tag{5.3}$$

The reader may recall that (5.3) defines what was called in [1] the <u>interior case</u>. It may be characterized verbally by saying that the ocean is a winning coalition while the set of major players is a losing coalition. Interior games constitute a not inconsiderable fraction of the totality of oceanic games. They are in a sense the furthest removed from the finite (nonoceanic) case, in that the weight of the ocean is bounded below but not above.

We have seen that (5.3) is necessary to the success of the recursion based on (4.13) and (2.5). That it is also sufficient follows rather easily. We need only observe that if $\Gamma^+(z)$ is an interior game, then all games $\Gamma(y)$ called for in the application of (4.13) are interior as well. Hence the single "interior" hypothesis (5.3) efficiently guarantees the satisfaction of all conditions of type (4.10) that will arise.

When we begin to carry out the iterated integrations, a pleasant surprise greets us. We discover that in an interior oceanic game <u>the</u>

values are independent of the majority quota. To see why this is so, note that equation (4.13) has the following property: If $\Phi(y)$ is independent of y, then $\phi_i^+(z)$ is independent of z, for every i. But the underlined statement is true trivially for $m = 0$. Thus the integration is trivial, and we have the following result.

Theorem 5. If the games

$$\Gamma = [c; w_1, \ldots, w_m; \alpha]$$

and

$$\Gamma^+ = [d; w_1, \ldots, w_{m+1}; \alpha]$$

are interior, then

$$\phi_{m+1}^+ = \frac{1}{\alpha} \Phi w_{m+1} . \tag{5.4}$$

Corollary. (Compare Theorem 4.) In an interior oceanic game, a major player's value–per–vote ratio is equal to the ocean's value–per–vote ratio with that player eliminated, thus:

$$\frac{\phi_i[c; w_1, \ldots, w_i, \ldots, w_m; \alpha]}{w_i} = \frac{\Phi[c; w_1, \ldots, 0, \ldots, w_m; \alpha]}{\alpha} . \tag{5.5}$$

Coupling (5.4) and (2.5), we can proceed to calculate the interior–case value formulas for small values of m. An inspection of Table 1 makes it apparent that one obtains a sequence of symmetric, multilinear polynomials in the w_i, of a fairly simple, quasi–homogeneous form. We should be able

Table 1

VALUES OF INTERIOR OCEANIC GAME

m	ϕ_1*	Φ
0	—	1
1	$\frac{w_1}{\alpha}$	$\frac{\overline{w}_1}{\alpha}$
2	$\frac{w_1\overline{w}_2}{\alpha^2}$	$\frac{1}{\alpha^2}\left(\overline{w}_1\overline{w}_2 + w_1 w_2\right)$
3	$\frac{w_1}{\alpha^3}\left(\overline{w}_2\overline{w}_3 + w_2 w_3\right)$	$\frac{1}{\alpha^3}\left(\overline{w}_1\overline{w}_2\overline{w}_3 + w_1 w_2 \overline{w}_3 + w_1 \overline{w}_2 w_3 + \overline{w}_1 w_2 w_3 - 2w_1 w_2 w_3\right)$

*Other major-player values similar.

to write down the general formula as soon as we discover what coefficients to attach to the various products of the w_i and their "complements" $\overline{w}_i = \alpha - w_i$.

Accordingly, let $\pi(S)$ be defined, for subsets S of M, as the product $u_1 u_2 \ldots u_m$, where u_i is w_i/α for $i \in S$ and $\overline{w}_i/\alpha$ for $i \in M-S$. Let $\pi_i(S)$ be defined similarly for subsets S of $M - \{i\}$. The sought-for coefficients may be defined by the recurrence

(5.6) $$a_n = 1 - n a_{n-1},$$

with $a_0 = 1$, giving us the sequence

(5.7) $a_0 = 1, \quad a_1 = 0, \quad a_2 = 1, \quad a_3 = -2, \quad a_4 = 9, \quad a_5 = -44, \quad \ldots .$

We see that the a_i are alternately positive and nonpositive. Moreover, we have

(5.8) $$a_n = n! \left[\frac{1}{n!} - \frac{1}{(n-1)!} + \cdots + (-1)^n \right] .$$

From this we see that $|a_n|$ is the nearest integer to $n!/e$, for all $n > 0$.*

Theorem 6. In the interior oceanic game

$$[c;\, w_1, \ldots, w_m;\, \alpha], \qquad w(M) \leq c \leq \alpha,$$

the value to the i-th major player is

(5.9) $$\phi_i = \frac{w_i}{\alpha} \sum_{S \subseteq M - \{i\}} a_s \pi_i(S),$$

and the combined value of the oceanic players is

(5.10) $$\Phi = \sum_{S \subseteq M} a_s \pi(S),$$

where s denotes the number of elements in S.

---

*Another incidental fact: $|a_n|$ is the number of "total disarrangements" of n objects (permutations without fixed points).

(In other words, the coefficient of any term in Table 1 and its extension is simply a_s/α^m, where s is the number of unbarred w_i factors. The fact that $a_1 = 0$ gives an accidental extra simplicity to the formulas for small m.)

Proof. Table 1 suffices to start the induction. The inductive step involves checking to see that the formulas given satisfy both (5.4) and (2.5). The former is immediate. As for the latter, we have

$$\sum_{i \in M} \phi_i = \sum_{i \in M} \frac{w_i}{\alpha} \sum_{S \subseteq M - \{i\}} a_s \pi_i(S)$$

$$= \sum_{i \in M} \sum_{\substack{S \subseteq M \\ S \ni i}} a_{s-1} \pi(S).$$

Each $S \subseteq M$ appears in the double sum exactly s times, once for each of its elements i. Hence we continue

$$\sum_{i \in M} \phi_i = \sum_{S \subseteq M} s\, a_{s-1} \pi(S)$$

$$= \sum_{S \subseteq M} (1 - a_s) \pi(S) \qquad \text{(using (5.6))}$$

$$= \sum_{S \subseteq M} \pi(S) - \sum_{S \subseteq M} a_s \pi(S)$$

$$= 1 - \Phi,$$

as was to be shown.

6. GENERALIZED OCEANIC GAMES

It is natural to try to generalize the definitions of Sec. 2 to a game in which the voting strength of the ocean is distributed inhomogeneously. Equation (2.1), defining the oceanic game, would be replaced by

$$(6.1) \qquad u(R) = w(R \cap M) + \nu(R \quad I), \qquad \text{all } R \in \mathcal{R},$$

where ν is some more or less arbitrary finite measure on I.

It is evident that any atoms that may occur in ν will be essentially indistinguishable from major players, and vice versa.* We could therefore dispense entirely with major players in our formulation, if we wish, and simply define a generalized oceanic game as an arbitrary weighted majority game [c; u] (see [1]) in which the ring $\mathcal{R}$ of player sets consists of the measurable subsets of the unit interval I.**

If there are only a finite number of atoms in ν, however, then it is convenient to reverse the above procedure and pull the atoms out as explicit major players. This enables us to restrict our attention to nonatomic measures ν in (6.1). A nonatomic measure can always be represented as the limit of a sequence of finite step–functions, with the weight of the largest step going to zero. The inhomogeneous oceanic game can therefore be represented as the limit of a sequence of finite games, fulfilling the

---

*A major player can be inserted into the ocean at any point having zero ν–measure, by adding the obvious step–function to ν.

**One might prefer to withhold the adjective "oceanic" unless the spectrum of μ is a continuum, or has the power of the continuum.

hypotheses of the fundamental limit theorem (see the appendix). While we have not yet defined the value for the inhomogeneous case, we can nevertheless conclude that there is a unique extension of the finite value definition that is continuous (in the sense of (1.1) or Theorem 1), at least in so far as the major players are concerned. For the present discussion we shall take this continuous extension as the definition of value, since we are not yet equipped to derive the value from a random-ordering principle and then prove it continuous.

Under this definition, the first point of interest is the observation that the values to the major players in an inhomogeneous game are the same as in the corresponding homogeneous game. One merely sets $\alpha = \nu(I)$. The irregularities in voting strength in the ocean have no effect on the major players.

We can also determine without much difficulty the distribution of value within an inhomogeneous ocean. Not surprisingly, it turns out to be directly proportional to the measure ν:

$$\phi(S) = \frac{\nu(S)}{\nu(I)} \Phi, \qquad \text{all measurable } S \subseteq I. \tag{6.2}$$

To prove this, consider an S that has a rational fraction of the total measure, say $\nu(S) / \nu(I) = r/s$, with r and s integers. Since ν is continuous, S can be partitioned into subsets $A_1, \ldots, A_r$, and $I - S$ into subsets $B_1, \ldots, B_{s-r}$, all of which have ν-measure exactly equal to $\nu(I)/s$. In the finite, $(m+s)$-person game obtained by considering these subsets as individual players, the values of S and I are clearly in the desired ratio r/s, by symmetry. But the partition just described can be refined indefinitely, preserving symmetry. Therefore we obtain (6.2) in the limit. The extension to irrational—fractional subsets S is immediate.

We note that a game with two or more homogeneous oceans is a special case of the inhomogeneous theory we have just presented. Our results show that multiple oceans can be "pooled" without affecting the major players' values.

If there are an infinite number of atoms in the measure ν of (6.1), then we are confronted with what amounts to a denumerable infinity of major players, with or without an additional continuous ocean. Consideration of this case must be deferred until we have developed an approach to the theory of countable—person games.

APPENDIX

We recapitulate here the main theorem of [1] in its most general form.

Consider the sequence $\{\Gamma_\ell \mid \ell = 1, 2, \dots\}$ of $(m+n_\ell)$–person weighted majority games:

(A.1) $$\Gamma_\ell = [c_\ell; w_{1,\ell}, \dots, w_{m,\ell}, a_{1,\ell}, \dots, a_{n_\ell,\ell}].$$

Let $M = \{1, \dots, m\}$ denote the set of the first m players of each of these games. Suppose that, as $\ell \to \infty$, we have

(A.2) $$\begin{cases} c_\ell \to c \geq 0, & \\ w_{i,\ell} \to w_i \geq 0, & \text{all } i \in M, \\ \sum_{j=1}^{n_\ell} a_{j,\ell} \to \alpha > 0, & \\ \max_j a_{j,\ell} \to 0. & \end{cases}$$

Let $\phi_{i,\ell}$ denote the value to player $i \in M$ of the game Γ_ℓ. Then the theorem states that $\phi_{i,\ell}$ converges to a limit that is independent of the $a_{j,\ell}$, given by the following expression:

(A.3) $$\phi_{i,\infty} = \sum_{\substack{S \subset M \\ i \notin S}} \int t^{|S|} (1-t)^{m-|S|-1} dt,$$

the integral being taken over the intersection of the two intervals $[0, 1]$ and

$$\left[\frac{c - w(S) - w_i}{\alpha}, \frac{c - w(S)}{\alpha} \right].$$

The expression (A. 3) is evidently continuous and homogeneous in the parameters c, $w_1, \ldots, w_m$, and α, and is in fact a piecewise polynomial of degree $\leq m$ in the variables c/α, $w_1/\alpha, \ldots, w_m/\alpha$. Its behavior as $\alpha \to 0$ is discontinuous in the other parameters. In this regard (A. 3) may be compared with the direct formula for the finite game $[c; w_1, \ldots, w_m]$, namely:

$$\text{(A. 4)} \qquad \phi_i = \sum_{\substack{S \subset M \\ i \notin S}} \frac{|S|!\,(m - |S| - 1)!}{m!} \left[v(S \cup \{i\}) - v(S) \right],$$

where $v(S) = 1$ if $w(S) \geq c$, and otherwise $v(S) = 0$. Indeed it is not hard to show that (A. 3) converges to (A. 4) as $\alpha \to 0$, the other parameters being held fixed.

REFERENCES

1. Shapley, L. S., and N. Z. Shapiro, Values of Large Games, I: A Limit Theorem, The RAND Corporation, Research Memorandum RM–2648, November 2, 1960.

2. Shapley, L. S., Values of Large Games, III: A Corporation with Two Large Stockholders, The RAND Corporation, Research Memorandum RM–2650, to be published.

3. Mann, Irwin, and L. S. Shapley, Values of Large Games, IV: Evaluating the Electoral College by Montecarlo Techniques, The RAND Corporation, Research Memorandum RM–2651 (ASTIA No. AD 246277), September 19, 1960.