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# THE PROBABILISTIC METHOD FOR PROBLEMS OF RADIATIVE TRANSFER: THE MARKOV PROPERTY OF RADIATIVE TRANSFER AND OF NEUTRON DIFFUSION

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The RAND Corporation
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## PREFACE

The theory of radiative transfer is important in meteorology, for example in the interpretation of radiation measurements from satellites. Some of the techniques can also be applied to studies of neutron transport.

The present study relates the theory of radiative transfer to basic methods in the theory of probability. It should be of interest to specialists in the theory of radiative transfer and to those concerned with applications of probability theory to physics.

## SUMMARY

On the basis of the stochastic model of multiple scattering of photons, we consider the diffuse reflection and transmission of a parallel beam of radiation by a finite, plane-parallel, non-emitting and homogeneous atmosphere with conservative and isotropic scattering. We assume that the stochastic process under consideration represents a homogeneous stationary evolution in a Markovian manner with respect to the optical depth.

First we derive the forward and the backward integro-differential equations for the emission probability distributions from the Chapman-Kolmogoroff equations. Then, starting with the Laplace transform of these equations, we obtain the S- and T-functions of S. Chandrasekhar for monodirectional illumination of the upper and the lower boundaries, depending on the optical depths $\tau_0$ and $\tau_1$ $(0 \leq \tau_0 < \tau_1)$. The results obtained with the aid of the forward equations reduce to those derived from the backward equations, because of the homogeneous optical properties of the medium. Some new functional equations for the source functions of the auxiliary equations are given.

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# THE PROBABILISTIC METHOD FOR PROBLEMS OF RADIATIVE TRANSFER: THE MARKOV PROPERTY OF RADIATIVE TRANSFER AND OF NEUTRON DIFFUSION

## 1. INTRODUCTION

It is well known that the Markov property has been considered in detail in the field of Brownian motion [9], turbulent diffusion [2], and others [6].

In a preceding paper [16], the author showed that Milne's problem in a semi-infinite, plane-parallel atmosphere with isotropic scattering can be solved, with the aid of the Markovian stochastic model of multiple scattering of photons, by using Chandrasekhar's idea [11], which is based on the principle of invariance arising from the asymptotic solution at infinity.

With the aid of the physical method based on the principle of invariance, the problem of the diffuse reflection and transmission of parallel rays by a finite, plane-parallel, homogeneous atmosphere was rigorously treated by Ambarzumian [1], Busbridge [7], and Chandrasekhar [10]. Furthermore, the mathematical discussion of the same problem in an inhomogeneous medium has been given by Bellman and Kalaba [3], [4], [5], Sobolev [14], Busbridge [8], and Ueno [20].

In the present paper we consider a stochastic process representing the evolution in a Markovian manner with respect to the one-dimensional parameter $\tau$ in radiative transfer and neutron diffusion. As a typical example we deal with the diffuse reflection and transmission of a parallel beam of

radiation by a plane-parallel and non-emitting atmosphere of finite optical thickness $\tau_1 - \tau_0$ $(0 \leq \tau_0 < \tau_1)$ with conservative and isotropic scattering, because the solutions of all other transfer problems in a similar flat layer can apparently be reduced to this one. In such a stochastic field the process is considered as denoting the evolution of the probability distribution of emission $p(\mu;\tau_0,\tau,\tau_1)$ with increasing optical depth $\tau$ $(0 \leq \tau_0 \leq \tau \leq \tau_1)$. Here $p(\mu;\tau_0,\tau,\tau_1)d\mu$ is the probability of finding a continuous stochastic parameter $\mu(\tau)$ between $\mu$ and $d\mu$ at the level $\tau$ where $\cos^{-1}\mu$ denotes the inclination to the outward normal of the surface $\tau = \tau_0$. It should be mentioned that $p(\mu;\tau_0,\tau,\tau_1)$ is a truncated function of $\tau$, i.e., $p(\mu;\tau_0,\tau,\tau_1)$ is finite for $0 \leq \tau_0 \leq \tau \leq \tau_1$ and $p(\mu;\tau_0,\tau,\tau_1) = 0$ for $\tau < \tau_0$ and $\tau_1 < \tau$.

In the present paper, we construct a Markovian stochastic model of multiple scattering of photons and assume the stationary and homogeneous character of the evolution with respect to $\tau$. Then we derive the forward and the backward integro-differential equations for the emission probability distributions from the Chapman-Kolmogoroff equations appropriate to the present case. Finally, starting with the Laplace transform of these Kolmogoroff-Feller equations [12], we obtain the scattering and the transmission functions which vary with the optical depths $\tau_0$ and $\tau_1$. A complete set of the integral equations for the S- and T-functions derived from

the forward integro-differential equations are equal to those for the S- and T-functions derived from the backward equations, because of the optical homogeneity of the medium. These equations show a Markovian property of multiple scattering of classical particles, i.e., photon and neutron, under the given boundary conditions.

It may be of interest to note that our above procedure is somewhat similar to that used by Feller [13] on boundaries and lateral conditions for the Kolmogoroff differential equations.

While the forward equation depends on the knowledge of what happens in the final infinitesimal interval, in formulating the backward equation we seek to show what happens in the initial infinitesimal interval. In other words, while the former involves differentiation with respect to $\tau_1$, the latter contains differentiation with respect to $\tau_0$. Mathematically speaking, the forward and the backward equations together can be derived from the auxiliary equations for monodirectional illumination of the upper and the lower boundaries, because the emission probability distributions depend on the optical depths $\tau_0$ and $\tau_1$. In what follows, for convenience the various quantities, i.e., the emission probability distribution, the auxiliary equation, the S- and T-functions, and the X- and Y-functions for monodirectional illumination of the upper and the lower boundaries, will be denoted respectively as the downward and the upward ones.

While the above Markovian stochastic model was first put forward by Ueno [16], the probabilistic idea in radiative transfer can be traced out in the method of invariance due to Ambarsumian [1] and Chandrasekhar [10]. The S- and T-functions of Chandrasekhar can be interpreted as the intensities of the probability currents, and furthermore the X- and Y-functions are equal to the probability distributions of emission from the bounding planes $\tau = \tau_0$ and $\tau = \tau_1$, respectively.

In later papers we shall reconsider the Markov property of radiative transfer, allowing for the inhomogeneity (see Ueno [21]), noncoherent scattering, and other geometries than the plane ones in the diffuse radiation field.

## 2. THE FORWARD AND THE BACKWARD INTEGRO-DIFFERENTIAL EQUATIONS

Assuming that the diffuse reflection and transmission of parallel rays by a finite flat layer with conservative and isotropic scattering represents a Markovian stochastic process that is stationary and homogeneous with respect to the optical depth $\tau$, we can physically define the emission probability distribution as follows: Let $p(\mu;\tau_0,\tau,\tau_1)$ be the probability that a photon absorbed at level $\tau$ (or $\tau_1 + \tau_0 - \tau$) will be re-emitted in the direction $+\mu$ (or $-\mu$) $(0 < \mu \leq 1)$ in the radiation emerging from the surface $\tau = \tau_0$ (or $\tau = \tau_1$). Furthermore, let $p(\mu;\tau_1,\tau,\tau_0)$ be the emergence probability of a photon absorbed at the level $\tau$ (or $\tau_1 + \tau_0 - \tau$) in the direction $-\mu$ (or $+\mu$) $(0 < \mu \leq 1)$ from the surface $\tau = \tau_1$ (or $\tau = \tau_0$). In a manner similar

to that used in the inhomogeneous case [20], we shall call $p(\mu;\tau_0,\tau,\tau_1)$ and $p(\mu;\tau_1,\tau,\tau_0)$ the downward and the upward emission probability distributions, respectively.

In a preceding paper [17], assuming the Markov property of the diffuse reflection and transmission of radiation for isotropic and coherent scattering in the conservative case, we derived the Kolmogoroff–Feller equation from the Chapman–Kolmogoroff equation, allowing for the leakage of the probability current through the bounding plane $\tau = \tau_1$ due to the transmission of radiation directed towards the level $\tau = \tau_1$.

In a manner similar to that used in probability theory [12], we find the Chapman–Kolmogoroff equation appropriate to the present case to be

$$p(\mu;\tau_0,\tau,\tau_1) = \int_0^1 \bar{p}(\mu';\tau_0,\tau-\Delta\tau,\tau_1)p(\mu/\mu';\Delta\tau,\tau_1)d\mu', \tag{2.1}$$

for all values of $\Delta$ between $\tau_0$ and $\tau_1$. In Eq. (2.1) $\bar{p}(\mu;\tau_0,\tau-\Delta\tau,\tau_1)$ and $p(\mu/\mu';\Delta\tau,\tau_1)$ are given by

$$\bar{p}(\mu;\tau_0,\tau-\Delta\tau,\tau_1) = p(\mu;\tau_0,\tau-\Delta\tau,\tau_1) - \Delta\tau \int_0^1 p(\mu";\tau_1,\tau,\tau_0)R_2(\mu/\mu")d\mu", \tag{2.2}$$

$$p(\mu/\mu';\Delta\tau,\tau_1) = R_1(\mu/\mu')\Delta\tau + \delta(\mu-\mu')\left\{1 - \Delta\tau \int_0^1 R(\mu"/\mu')d\mu"\right\}. \tag{2.3}$$

In the above, the probability distribution function $p(\mu_b/\mu_a;\tau_b-\tau_a,\tau_1)d\mu_b$ represents the probability that $\mu(\tau_b)$ lies between $\mu_b$ and $\mu_b + d\mu$, provided that $\mu(\tau_a) = \mu_a$ $(0 \leq \tau_0 \leq \tau_a < \tau_b \leq \tau_1)$. In (2.3), $\delta$ is the Dirac delta function, and $R_1$, $R_2$, and R-functions are provided by

$$R_1(\mu/\mu') = \frac{1}{2\mu'} X(\mu,\tau_1), \quad R_2(\mu/\mu') = \frac{1}{2\mu'} Y(\mu,\tau_1), \tag{2.4}$$

$$R(\mu/\mu') = R_1(\mu/\mu') + R_2(\mu/\mu'), \tag{2.5}$$

where

$$\begin{aligned} X(\mu,\tau_0) &= p(\mu;\tau_0,\tau_0,\tau_1) = X(\mu,\tau_1) = p(\mu;\tau_1,\tau_1,\tau_0), \\ Y(\mu,\tau_1) &= p(\mu;\tau_0,\tau_1,\tau_1) = Y(\mu,\tau_0) = p(\mu;\tau_1,\tau_0,\tau_0). \end{aligned} \tag{2.6}$$

The quantity $R(\mu/\mu')d\mu\, d\tau$ represents the conditional transition probability that, given $\mu'$, one finds $\mu$ in the range $(\mu, \mu + d\mu)$ through a parametric interval $d\tau$ of the optical depth.

The normalization condition for the conditional probability $R(\mu/\mu')$ in the conservative case is provided by

$$\mu \int_0^1 R(\mu'/\mu)d\mu' = 1. \tag{2.7}$$

In the nonconservative case the above integral is less than unity. While an explicit appeal to the K-integral in the conservative case is needed to resolve the ambiguity

implied in the integral equations (see Chandrasekhar [11]), the normalization condition (2.7) is also useful for the resolution of the arbitrariness of the solution (see Ueno [18], [19]).

Letting $\Delta\tau \to 0$, we obtain the stochastic integro-differential equation for $p(\mu;\tau_0,\tau,\tau_1)$

$$\frac{\partial p(\mu;\tau_0,\tau,\tau_1)}{\partial\tau} = -\frac{1}{\mu}\,p(\mu;\tau_0,\tau,\tau_1) + \frac{1}{2}\,X(\mu,\tau_1)\int_0^1 p(\mu';\tau_0,\tau,\tau_1)\,\frac{d\mu'}{\mu'} - \frac{1}{2}\,Y(\mu,\tau_1)\int_0^1 p(\mu';\tau_1,\tau,\tau_0)\,\frac{d\mu'}{\mu'}\,. \tag{2.8}$$

Similarly, we can obtain the other integro-differential equation for the upward emission probability distribution $p(\mu;\tau_1,\tau,\tau_0)$. We write

$$p(\mu;\tau_1,\tau,\tau_0) = \int_0^1 \bar{p}(\mu';\tau_1,\tau+\Delta\tau,\tau_0)\,p(\mu/\mu';\Delta\tau,\tau_0)\,d\mu', \tag{2.9}$$

where

$$\bar{p}(\mu;\tau_1,\tau+\Delta\tau,\tau_0) = p(\mu;\tau_1,\tau+\Delta\tau,\tau_0) - \Delta\tau\int_0^1 p(\mu";\tau_0,\tau+\Delta\tau,\tau_1)\,R_2(\mu/\mu")\,d\mu", \tag{2.10}$$

$$p(\mu/\mu';\Delta\tau,\tau_0) = p(\mu/\mu';\Delta\tau,\tau_1), \tag{2.11}$$

since the relative transition probabilities per unit change in the parameter $\tau$ are identical in the homogeneous case.

As $\Delta\tau \to 0$, we have

$$(2.12)\qquad \frac{\partial p(\mu;\tau_1,\tau,\tau_0)}{\partial\tau} = \frac{1}{\mu}\, p(\mu;\tau_1,\tau,\tau_0) - \frac{1}{2}\, X(\mu,\tau_1) \int_0^1 p(\mu';\tau_1,\tau,\tau_0)\, \frac{d\mu'}{\mu'} + \frac{1}{2}\, Y(\mu,\tau_1) \int_0^1 p(\mu';\tau_0,\tau,\tau_1)\, \frac{d\mu'}{\mu'} .$$

Next we derive the forward equations. The Chapman-Kolmogoroff equation appropriate to this case takes the form

$$(2.13)\qquad p(\mu;\tau_0,\tau,\tau_1) = \int_0^1 \overline{p}(\mu';\tau_0,\tau,\tau_1-\Delta\tau)\, p(\mu/\mu';\tau,\Delta\tau)\, d\mu',$$

where $0 \le \Delta\tau < \tau_1 - \tau_0$.

In (2.12) we put

$$(2.14)\qquad \overline{p}(\mu;\tau_0,\tau,\tau_1-\Delta\tau) = p(\mu;\tau_0,\tau,\tau_1-\Delta\tau) + \Delta\tau \int_0^1 p(\mu";\tau_1-\Delta\tau,\tau,\tau_0)\, R_2(\mu/\mu")\, d\mu",$$

and

$$(2.15)\qquad p(\mu/\mu';\tau,\Delta\tau) = \delta(\mu - \mu').$$

Then, letting $\Delta\tau \to 0$, we get

$$\frac{\partial p(\mu;\tau_0,\tau,\tau_1)}{\partial \tau_1} = \frac{1}{2} Y(\mu,\tau_1) \int_0^1 p(\mu';\tau_1,\tau,\tau_0) \frac{d\mu'}{\mu'} . \tag{2.16}$$

Similarly, we may start with another type of Chapman–Kolmogoroff equation for the upward emission probability distribution, namely

$$p(\mu;\tau_1,\tau,\tau_0) = \int_0^1 \overline{p}(\mu';\tau_1-\Delta\tau,\tau,\tau_0) p^*(\mu/\mu';\tau,\Delta\tau) d\mu', \tag{2.17}$$

where

$$p^*(\mu/\mu';\tau,\Delta\tau) = p(\mu/\mu';\Delta\tau,\tau_1). \tag{2.18}$$

Then as $\Delta\tau \to 0$, (2.17) becomes

$$\frac{\partial p(\mu;\tau_1,\tau,\tau_0)}{\partial \tau_1} = -\frac{1}{\mu} p(\mu;\tau_1,\tau,\tau_0) + \frac{1}{2} X(\mu,\tau_1) \int_0^1 p(\mu';\tau_1,\tau,\tau_0) \frac{d\mu'}{\mu'} . \tag{2.19}$$

On combining Eqs. (2.8) and (2.12) with Eqs. (2.16) and (2.19), respectively, we obtain

$$\frac{\partial p(\mu;\tau_0,\tau,\tau_1)}{\partial \tau} + \frac{\partial p(\mu;\tau_0,\tau,\tau_1)}{\partial \tau_1} = -\frac{1}{\mu} p(\mu;\tau_0,\tau,\tau_1) + \frac{1}{2} X(\mu,\tau_1) \cdot \int_0^1 p(\mu';\tau_0,\tau,\tau_1) \frac{d\mu'}{\mu'} , \tag{2.20}$$

$$(2.21)\qquad \frac{\partial p(\mu;\tau_1,\tau,\tau_0)}{\partial \tau} + \frac{\partial p(\mu;\tau_1,\tau,\tau_0)}{\partial \tau_1} = \frac{1}{2}\, Y(\mu,\tau_1) \int_0^1 p(\mu';\tau_0,\tau,\tau_1)\, \frac{d\mu'}{\mu'} .$$

Now we derive the backward equations corresponding to the forward equations (2.16) and (2.19). First we write

$$(2.22)\qquad p(\mu;\tau_0,\tau,\tau_1) = \int_0^1 \bar{p}(\mu';\tau_0+\Delta\tau,\tau,\tau_1)p(\mu/\mu';\tau,\Delta\tau)d\mu',$$

where $p(\mu/\mu';\tau,\Delta\tau)$ is equal to $p(\mu/\mu';\Delta\tau,\tau_1)$ in (2.3).

Then, as $\Delta\tau \to 0$,

$$(2.23)\qquad \frac{\partial p(\mu;\tau_0,\tau,\tau_1)}{\partial \tau_0} = \frac{1}{\mu}\, p(\mu;\tau_0,\tau,\tau_1) - \frac{1}{2}\, X(\mu,\tau_1) \int_0^1 p(\mu';\tau_0,\tau,\tau_1)\, \frac{d\mu'}{\mu'} .$$

Similarly, if we start with the Chapman-Kolmogoroff equation yielded by

$$(2.24)\qquad p(\mu;\tau_1,\tau,\tau_0) = \int_0^1 \bar{p}(\mu;\tau_1,\tau,\tau_0+\Delta\tau)p(\mu/\mu';\tau,\Delta\tau)d\mu',$$

where

$$(2.25)\qquad \bar{p}(\mu;\tau_1,\tau,\tau_0+\Delta\tau) = p(\mu'\tau_1,\tau,\tau_0+\Delta\tau) + \Delta\tau \int_0^1 p(\mu";\tau_0+\Delta\tau,\tau,\tau_1)R_2(\mu/\mu')\, \frac{d\mu'}{\mu'} ,$$

$$(2.26) \qquad p(\mu/\mu';\tau,\Delta\tau) = \delta(\mu - \mu'),$$

then, in the limit as $\Delta\tau \to 0$ we get

$$(2.27) \qquad \frac{\partial p(\mu;\tau_1,\tau,\tau_0)}{\partial \tau_0} = -\frac{1}{2} Y(\mu,\tau_1) \int_0^1 p(\mu';\tau_0,\tau,\tau_1) \frac{d\mu'}{\mu'} .$$

The combination of Eqs. (2.23) and (2.27) with Eqs. (2.8) and (2.12) provides the following equations:

$$(2.28) \qquad \frac{\partial p(\mu;\tau_0,\tau,\tau_1)}{\partial \tau} + \frac{\partial p(\mu'\tau_0,\tau,\tau_1)}{\partial \tau_0} = -\frac{1}{2} Y(\mu,\tau_1) \int_0^1 p(\mu';\tau_1,\tau,\tau_0) \frac{d\mu'}{\mu'} ,$$

$$(2.29) \qquad \frac{\partial p(\mu;\tau_1,\tau,\tau_0)}{\partial \tau} + \frac{\partial p(\mu;\tau_1,\tau,\tau_0)}{\partial \tau_0} = \frac{1}{\mu} p(\mu;\tau_1,\tau,\tau_0) - \frac{1}{2} X(\mu,\tau_1) \int_0^1 p(\mu';\tau_1,\tau,\tau_0) \frac{d\mu'}{\mu'} ,$$

$$(2.30) \qquad \frac{\partial p(\mu;\tau_0,\tau,\tau_1)}{\partial \tau} + \frac{\partial p(\mu;\tau_0,\tau,\tau_1)}{\partial \tau_0} + \frac{\partial p(\mu;\tau_0,\tau,\tau_1)}{\partial \tau_1} = 0,$$

$$(2.31) \qquad \frac{\partial p(\mu;\tau_1,\tau,\tau_0)}{\partial \tau} + \frac{\partial p(\mu;\tau_1,\tau,\tau_0)}{\partial \tau_1} + \frac{\partial p(\mu;\tau_1,\tau,\tau_0)}{\partial \tau_0} = 0.$$

While Eq. (2.23) is equal to Eq. (3.23) given by Busbridge [8], so far as we know, Eqs. (2.28) – (2.31) are

new. Furthermore, it is known that the above emission probability distribution fulfills the following downward and the upward auxiliary equations (see Ueno [10]):

$$(2.32) \qquad [1 - \bar{K}]_{\tau}\{p(\mu;\tau_0,t,\tau_1)\} = e^{-(\tau-\tau_0)/\mu},$$

$$(2.33) \qquad [1 - \bar{K}]_{\tau}\{p(\mu;\tau_1,t,\tau_0)\} = e^{-(\tau_1-\tau)/\mu},$$

where 1 is the identity operator and the truncated Hopf operator $\bar{K}$ is

$$(2.34) \qquad \bar{K}_{\tau}\{f(t)\} = \frac{1}{2}\int_{\tau_0}^{\tau_1} f(t)E_1(|t - \tau|)dt.$$

In (2.33) $E_1$ is the first exponential integral

$$(2.35) \qquad E_1(\tau) = \int_0^1 e^{-\tau/\mu}\,\frac{d\mu}{\mu}\,.$$

In the conservative case, Eqs. (2.32) and (2.33) show that $p(\mu;\tau_0,\tau,\tau_1)$ coincides with $p(\mu;\tau_1,\tau_0+\tau_1-\tau,\tau_0)$.

## 3. SCATTERING AND TRANSMISSION FUNCTIONS

Let the scattering and the transmission functions be denoted by

$$(3.1) \qquad S(\tau_0,\tau_1;\mu,\mu_0) = \int_{\tau_0}^{\tau_1} p(\mu_0;\tau_0,\tau,\tau_1)\, e^{-(\tau-\tau_0)/\mu}\, d\tau,$$

$$(3.2)\qquad S(\tau_1,\tau_0;\mu,\mu_0) = \int_{\tau_0}^{\tau_1} p(\mu_0;\tau_1,\tau,\tau_0)e^{-(\tau_1-\tau)/\mu}\,d\tau,$$

$$(3.3)\qquad T(\tau_0,\tau_1;\mu,\mu_0) = \int_{\tau_0}^{\tau_1} p(\mu_0;\tau_0,\tau,\tau_1)e^{-(\tau_1-\tau)/\mu}\,d\tau,$$

$$(3.4)\qquad T(\tau_1,\tau_0;\mu,\mu_0) = \int_{\tau_0}^{\tau_1} p(\mu_0;\tau_1,\tau,\tau_0)e^{-(\tau-\tau_0)/\mu}\,d\tau.$$

However, allowing for $p(\mu;\tau_1,\tau,\tau_0) = p(\mu;\tau_0,\tau_0+\tau_1-\tau,\tau_1)$ in the conservative case, we have

$$(3.5)\qquad S(\tau_0,\tau_1;\mu,\mu_0) = S(\tau_1,\tau_0;\mu,\mu_0),$$

$$(3.6)\qquad T(\tau_0,\tau_1;\mu,\mu_0) = T(\tau_1,\tau_0;\mu,\mu_0).$$

If the optical properties of the medium vary with the optical depth (see Busbridge [8], Ueno [20]), Eqs. (3.5) and (3.6) do not hold.

From (2.32) and (2.33) we get

$$(3.7)\qquad X(\mu,\tau_1) = 1 + \tfrac{1}{2}\int_{\tau_0}^{\tau_1} S(\tau_0,\tau_1;\mu,\mu')\,\frac{d\mu'}{\mu'},$$

$$(3.8)\qquad Y(\mu,\tau_1) = e^{-(\tau_1-\tau_0)/\mu} + \tfrac{1}{2}\int_{\tau_0}^{\tau_1} T(\tau_0,\tau_1;\mu,\mu')\,\frac{d\mu'}{\mu'},$$

taking into consideration the principle of reciprocity (see Chandrasekhar [11]) given by

$$(3.9)\qquad S(\tau_0,\tau_1;\mu,\mu_0) = S(\tau_0,\tau_1;\mu_0,\mu),$$

$$(3.10)\qquad T(\tau_0,\tau_1;\mu,\mu_0) = T(\tau_0,\tau_1;\mu_0,\mu).$$

On multiplying the forward equation (2.19) by $e^{-(\tau-\tau_0)/\mu_0}$ and $e^{-(\tau_1-\tau)/\mu_0}$ respectively, and integrating with respect to $\tau$ over $(\tau_0,\tau_1)$, from Eqs. (3.1) – (3.10), we have

$$(3.11)\qquad S(\tau_1,\tau_0;\mu,\mu_0) = \int_{\tau_0}^{\tau_1} X(\mu,t)X(\mu_0,t) \cdot \exp\{-(\tau_1 - t)(\tfrac{1}{\mu} + \tfrac{1}{\mu_0})\}dt,$$

$$(3.12)\qquad T(\tau_0,\tau_1;\mu,\mu_0) = \int_{\tau_0}^{\tau_1} X(\mu,t)Y(\mu_0,t)e^{-(\tau_1-t)/\mu}\,dt,$$

where

$$(3.13)\qquad X(\mu,\tau) = 1 + \tfrac{1}{2}\int_0^1 \frac{d\mu'}{\mu'}\int_{\tau_0}^{\tau} X(\mu,t)X(\mu',t) \cdot \exp\{-(\tau - t)(\tfrac{1}{\mu} + \tfrac{1}{\mu'})\}dt,$$

$$(3.14)\qquad Y(\mu,\tau) = e^{-(\tau-\tau_0)/\mu} + \tfrac{1}{2}\int_0^1 \frac{d\mu'}{\mu'}\int_{\tau_0}^{\tau} X(\mu,t)Y(\mu',t)e^{-(\tau-t)/\mu}\,dt.$$

Multiply the forward equation (2.16) by $e^{-(\tau-\tau_0)/\mu_0}$ and $e^{-(\tau_1-\tau)/\mu_0}$ respectively, and integrate with respect to $\tau$ over $(\tau_0,\tau_1)$. Then, by making use of Eqs. (3.1) – (3.10), we find

$$(3.15)\qquad S(\tau_0,\tau_1;\mu,\mu_0) = \int_{\tau_0}^{\tau_1} Y(\mu,t)Y(\mu_0,t)dt,$$

$$(3.16)\qquad T(\tau_1,\tau_0;\mu,\mu_0) = \int_{\tau_0}^{\tau_1} Y(\mu,t)X(\mu_0,t)e^{-(\tau_1-t)\mu_0}\,dt,$$

where

$$(3.17)\qquad X(\mu,\tau) = 1 + \frac{1}{2}\int_0^1 \frac{d\mu'}{\mu'}\int_{\tau_0}^{\tau} Y(\mu,t)Y(\mu',t)dt,$$

$$(3.18)\qquad Y(\mu,\tau) = e^{-(\tau-\tau_0)/\mu} + \frac{1}{2}\int_0^1 \frac{d\mu'}{\mu'}\int_{\tau_0}^{\tau} Y(\mu,t)X(\mu',t)e^{-(\tau-t)/\mu'}\,dt.$$

On the other hand, multiplying the modified forward equation (2.20) by $e^{-(\tau-\tau_0)/\mu_0}$ and $e^{-(\tau_1-\tau)\mu_0}$, and integrating over $\tau$ from $\tau_0$ to $\tau_1$, we get Eqs. (3.11) – (3.14). Furthermore, multiply the modified forward equation (2.21) by $e^{-(\tau-\tau_0)/\mu_0}$ and $e^{-(\tau_1-\tau)/\mu_0}$, and integrate with respect to $\tau$ over $(\tau_0,\tau_1)$. Then, we have Eqs. (3.15) – (3.18).

In a manner similar to that used for the forward equations, we shall derive the S– and T–functions expressed in terms of the X– and Y–functions from the backward equations.

Multiply (2.23) by $e^{-(\tau-\tau_0)/\mu_0}$ and $e^{-(\tau_1-\tau)/\mu_0}$ and integrate with respect to $\tau$ over $(\tau_0,\tau_1)$. Then we obtain

$$(3.19)\qquad S(\tau_0,\tau_1;\mu,\mu_0) = \int_{\tau_0}^{\tau_1} X(\mu,t)X(\mu_0,t) \cdot \exp\left\{-(t-\tau_0)\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)\right\}dt,$$

$$(3.20)\qquad T(\tau_1,\tau_0;\mu,\mu_0) = \int_{\tau_0}^{\tau_1} X(\mu,t)Y(\mu_0,t)e^{-(t-\tau_0)/\mu}\,dt,$$

where

$$(3.21)\qquad X(\mu,\tau) = 1 + \frac{1}{2}\int_0^1 \frac{d\mu'}{\mu'}\int_\tau^{\tau_1} X(\mu,t)X(\mu',t) \cdot \exp\left[-(t-\tau)\left(\frac{1}{\mu}+\frac{1}{\mu'}\right)\right]dt,$$

$$(3.22)\qquad Y(\mu,\tau) = e^{-(\tau_1-\tau)/\mu} + \frac{1}{2}\int_0^1 \frac{d\mu'}{\mu'}\int_\tau^{\tau_1} X(\mu,t)Y(\mu',t)e^{-(t-\tau)/\mu}\,dt.$$

On the other hand, multiplying Eq. (2.27) by $e^{-(\tau-\tau_0)/\mu_0}$ and $e^{-(\tau_1-\tau)/\mu_0}$, and integrating over $\tau$ from $\tau_0$ to $\tau_1$, we find

$$(3.23)\qquad S(\tau_1,\tau_0;\mu,\mu_0) = \int_{\tau_0}^{\tau_1} Y(\mu,t)Y(\mu_0,t)\,dt,$$

$$(3.24)\qquad T(\tau_0,\tau_1;\mu,\mu_0) = \int_{\tau_0}^{\tau_1} Y(\mu,t)X(\mu_0,t)e^{-(t-\tau_0)/\mu_0}\,dt,$$

where

$$(3.25)\qquad X(\mu,\tau) = 1 + \frac{1}{2}\int_0^1 \frac{d\mu'}{\mu'}\int_\tau^{\tau_1} Y(\mu,t)Y(\mu',t)\,dt,$$

$$(3.26)\qquad Y(\mu,\tau) = e^{-(\tau_1-\tau)/\mu} + \frac{1}{2}\int_0^1 \frac{d\mu'}{\mu'}\int_\tau^{\tau_1} Y(\mu,t)X(\mu',t)e^{-(t-\tau)/\mu'}\,dt.$$

Thus, allowing for $p(\mu;\tau_0,\tau,\tau_1) = p(\mu;\tau_1,\tau_0+\tau_1-\tau,\tau_0)$ and the principle of reciprocity given by (3.9) and (3.10), we see that the laws of diffuse reflection and transmission based on the forward integro-differential equations are equal to those given by the backward equations. Hence, the diffuse reflection and transmission of parallel rays by a finite atmosphere represents a Markovian evolutionary process that is reversible with respect to the optical depth.

On differentiating Eqs. (3.12) and (3.15), (3.16) with respect to $\tau_1$ and combining them appropriately, we obtain Eqs. (82), (83) in [11], Chap. VII. Eqs. (2.20) and (2.21) are similar to those provided by Sobolev [15] in the non-conservative case.

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