U. S. AIR FORCE

PROJECT RAND

RESEARCH MEMORANDUM

VILLE'S EXAMPLE OF A GAME WITHOUT
A STRATEGIC SADDLE-POINT

M.E. Dresher
J. McKinsey

RM-31

23 February 1948

Assigned to ______________________

The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

4)

VILLE'S EXAMPLE OF A GAME WITHOUT A STRATEGIC SADDLE-POINT

Melvin Dresher and J. C. C. McKinsey

In his paper on the theory of games, Jean Ville* proves that von Neumann's theorem on strategic saddle-points holds for games with continuum of strategies and continuous payoff functions. Ville also gives an example to show that if the payoff function is discontinuous, then the von Neumann theorem no longer holds. Since it was not immediately evident to us that the example actually failed to have a strategic saddle-point, it may be useful to adduce some arguments to establish this fact.

Consider the two-person game in which each player chooses a number in the closed interval (0,1). Let x and y represent the choices of the first and second players, respectively. Let K(x,y) represent the payment to the second player and be defined as follows:

$$K(x,y) = \begin{cases} 0 & \text{for } x = y \\ +1 & \text{for } x = 1,\ y < 1 \text{ and for } x < y < 1 \\ -1 & \text{for } y = 1,\ x < 1 \text{ and for } y < x < 1 . \end{cases}$$

Define Q(F,G) the expectation of the second player if the first player uses strategy F(x) and the second player uses strategy G(y). We shall write Q(x,G) for $Q(I_x,G)$ and Q(F,y) for $Q(F,I_y)$; Q(x,G) is the expectation of the second player if

*M. Jean Ville. Sur la théorie générale des jeux ou intervient l'habileté des joueurs. Traité du Calcul des Probabilités et de ses Applications. Emile Borel and collaborators.

the first player uses a pure strategy x and the second player uses a pure strategy G(y). When the second player uses a pure strategy, y, we have

(i) if $0 < y < 1$,

$$Q(F,y) = -1F(0) + 1[F(y-0)-F(0)] + 0\cdot[F(y)-F(y-0)] - 1[F(1-0)-F(y)] + 1[F(1)-F(1-0)]$$

$$Q(F,y) = F(y-0) - 2F(1-0) + F(y) + 1$$

(1) (ii) if $y = 0$,

$$Q(F,y) = 0\cdot F(0) - 1[F(1-0)-F(0)] + 1[F(1)-F(1-0)]$$

$$Q(F,Y) = 1 - 2F(1-0) + F(0)\ .$$

(iii) if $y = 1$,

$$Q(F,y) = -1F(0) - 1[F(1-0)-F(0)] + 0\cdot[F(1)-F(1-0)]$$

$$Q(F,Y) = -F(1-0)\ .$$

From (1) it follows that if $0 < y < 1$, then

$$Q(F,y) \geq 1 - 2[F(1-0)-F(y-0)]\ .$$

Now $F(y-0)$ is continuous from the left. Hence for any ϵ such that $0 < \epsilon < 1$, there exists a y such that $Q(F,y) \geq 1-\epsilon$.

Therefore, for any F and any ϵ, $(0 < \epsilon < 1)$, there exists a G, namely I_y, such that $Q(F,G) \geq 1-\epsilon$. In a similar way we can show that for any G and any ϵ $(0 < \epsilon < 1)$ there exists an F such that $Q(F,G) \leq -1+\epsilon$. It follows that

$$\max_G Q(F,G) \geq 1 - \epsilon$$

$$\min_F Q(F,G) \leq -1 + \epsilon \;.$$

Therefore we have

$$\min_F \max_G Q(F,G) \geq 1 - \epsilon$$

$$\max_G \min_F Q(F,G) \leq -1 + \epsilon \;.$$

Thus the $\max_G \min_F Q$ is not equal to $\min_F \max_G Q$.