MEMORANDUM
RM-3494-PR
MARCH 1963

ON THE NONNEGATIVITY OF GREEN'S FUNCTIONS

Richard Bellman

PREPARED FOR:
UNITED STATES AIR FORCE PROJECT RAND

The RAND Corporation
SANTA MONICA • CALIFORNIA

MEMORANDUM
RM-3494-PR
MARCH 1963

ON THE NONNEGATIVITY OF GREEN'S FUNCTIONS

Richard Bellman

This research is sponsored by the United States Air Force under Project RAND—contract No. AF 49(638)-700 monitored by the Directorate of Development Planning, Deputy Chief of Staff, Research and Development, Hq USAF. Views or conclusions contained in this Memorandum should not be interpreted as representing the official opinion or policy of the United States Air Force.

The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

PREFACE

Part of the Project RAND research program consists of basic supporting studies. The mathematical research presented here is concerned with boundary conditions of ordinary differential equations, which arise in a variety of physical problems.

SUMMARY

In previous papers we discussed various methods of establishing the nonnegativity of the Green's function associated with the ordinary differential equation

$$u'' + q(x)u = f(x), \quad u(0) = u(1) = 0.$$

In this paper we wish to present another method which has certain merits. It clarifies the role played by the characteristic values of the associated Sturm-Liouville problem and it indicates how useful it may be to study the behavior of the solution of $Lu = v$, where L is a linear operator, by means of the limiting behavior of the solution of

$$\frac{\partial u}{\partial t} = Lu - v,$$

as $t \to \infty$. This method has been used by Arrow and Hearon to study the inverse of input-output matrices.

CONTENTS

ON THE NONNEGATIVITY OF GREEN'S FUNCTIONS

1. INTRODUCTION

In previous papers [1], [2], [3], we discussed various methods of establishing the nonnegativity of the Green's function associated with the ordinary differential equation

$$(1.1) \qquad u'' + q(x)u = f(x), \quad u(0) = u(1) = 0.$$

In this paper we wish to present another method which has certain merits. It clarifies the role played by the characteristic values of the associated Sturm-Liouville problem and it indicates how useful it may be to study the behavior of the solution of $Lu = v$, where L is a linear operator, by means of the limiting behavior of the solution of

$$(1.2) \qquad \frac{\partial u}{\partial t} = Lu - v,$$

as $t \to \infty$. This method has been used by Arrow and Hearon to study the inverse of input-output matrices; see [4].

2. NONNEGATIVITY OF SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

Consider the partial differential equation of parabolic type,

$$(2.1) \qquad u_t = u_{xx} + q(x)u - f(x),$$

with the initial condition $u(x,0) = h(x)$, with $h(x) \geq 0$, $0 \leq x \leq 1$, and $u(0,t) = u(1,t) = 0$. If we suppose that $q(x)$ is uniformly bounded, $0 \leq x \leq 1$, it is easy to show, under the hypotheses that $h(x)$, $f(x) \geq 0$, that $u(x,t) \geq 0$ for $t \geq 0$. If $q(x) \geq 0$, we use the finite difference approximation

$$(2.2) \qquad u(x,t+\Delta^2) = \frac{u(x+\Delta,t) + u(x-\Delta,t)}{2} + q(x)u(x,t)\Delta - f(x)\Delta,$$

to $0,\Delta^2,\ldots,$ which establishes inductively that $u(x,t) \geq 0$. As $\Delta \to 0$, the solution of the finite difference equation converges to that of the partial differential equation, thus establishing the required nonnegativity.

If $q(x)$ is not nonnegative, but bounded from below by a constant so that $M + q(x) \geq 0$, we write $u = e^{-Mt}w(x)$. Substituting, we obtain the equation

$$(2.3) \qquad w_t = w_{xx} + (q(x) + M)w - f(x)e^{-Mt},$$

which we can treat as in (2.2) to establish nonnegativity.

3. NONNEGATIVITY OF SOLUTION OF ORDINARY DIFFERENTIAL EQUATION

Returning to (2.1), let us allow t to become infinite. If all the characteristic values of the equation

$$(3.1) \qquad u_{xx} + q(x)u = \lambda u, \quad u(0) = u(1) = 0,$$

are negative, the solution of (2.1) converges as $t \to \infty$ to the solution of (1.1), establishing thereby the nonnegativity of the solution of (1.1).

4. EXTENSIONS

There is no difficulty in extending the proof to cover ordinary differential equations with more general boundary conditions and multidimensional partial differential equations of parabolic type.

REFERENCES

1. Bellman, R., "On the Nonnegativity of Green's Functions," Boll. d'Unione Matematico, Vol. 12, 1957, pp. 411-413.
2. Bellman, R., "On the Nonnegativity of the Heat Equation," Boll. d'Unione Matematico, Vol. 12, 1957, pp. 520-523.
3. Beckenbach, E. F., and R. Bellman, Inequalities, Ergebnisse der Math., Springer, Berlin, 1961.
4. Bellman, R., An Introduction to Matrix Analysis, McGraw-Hill Book Company, Inc., New York, 1960.