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INVARIANT IMBEDDING AND NONLINEAR FILTERING THEORY

R. E. Bellman, H. H. Kagiwada, R. E. Kalaba and R. Sridhar

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The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA • 90406

PREFACE AND SUMMARY

Suppose that a system is undergoing a process which we believe can be described by the differential equation

$$dx/dt = g(x, t).$$

On the time interval (0, T) we observe the function x, in a noisy manner, and denote this experimental function by the symbol y. We wish to determine the state of the system at time t = T which is such that J,

$$J = \int_0^T (x(t) - y(t))^2 \, dt,$$

is minimized. Many problems of orbit determination and adaptive control are of this type.

A solution is suggested in both the scalar and vector cases, which makes use of certain ideas from the theory of invariant imbedding, and some numerical examples are provided.

I. INTRODUCTION

Recent years have seen the production of many significant advances in the theory of optimal filtering. Let us cite the contributions of Wiener,[1] Middleton,[2] Swerling,[3] Pugachev,[4] Kalman and Bucy,[5] Ho,[6] and Cox.[7] For the most part investigators have dealt with linear systems and have assumed that a complete statistical knowledge of the state of affairs is available. The aim of the present note is to introduce a theory which is applicable to a wide variety of nonlinear estimation problems. It is based on the theory of invariant imbedding,[8-10] though no specialized knowledge on the part of the reader is assumed. Statistical assumptions are kept to a minimum, and there are immediate applications to problems of orbit determination[11] and identification problems in adaptive control.

II. FORMULATION

Let us consider a system which undergoes a process described by the nonlinear differential equation

$$\dot{x} = g(x,t) . \qquad (1)$$

We observe the process imperfectly; i.e., if

$$y(t) = \text{observed history of process on } 0 \le t \le T, \qquad (2)$$

then

$$y(t) - x(t) = \text{observational error}. \qquad (3)$$

Our aim is to make an estimate of the state of the system at time T on the basis of observations carried out on the interval $0 \le t \le T$. We shall make this estimate by selecting

$$x(T) = c \tag{4}$$

to be such that if $y(t)$ is the observed function, and the function $x(t)$ is determined on the interval $0 \le t \le T$ by the differential equation

$$\dot{x} = g(x,t) \tag{5}$$

and the condition

$$x(T) = c , \tag{6}$$

then the integral

$$J = \int_0^T (x(t) - y(t))^2 dt \tag{7}$$

is minimized.

The generalization of this problem for $x(t)$ in Eq. (1) and $y(t)$ in Eq. (2) being m-dimensional vectors is considered in the Appendix.

III. INVARIANT IMBEDDING

Let us recognize that the value of the integral in Eq. (7) is a function of T, the length of the time of observation, and c, the value assigned to $x(t)$ at time T,

$$\int_0^T (x(t) - y(t))^2 dt = f(c,T) . \tag{8}$$

Now we imbed the original process, involving fixed values of T and c, in a class of processes for which $0 \le T$ and $-\infty < c < +\infty$. Then we interconnect the costs, $f(c,T)$, for these processes. We see that

$$f(c,T + \Delta) = f(c - g(c,T)\Delta,T) + (y(T) - c)^2\Delta + o(\Delta), \tag{9}$$

or in the limit as $\Delta \to 0$,

$$f_T = (y(T) - c)^2 - g(c,T)f_c, \tag{10}$$

which is a linear first-order partial differential equation for the function $f(c,T)$.

To obtain an auxiliary condition on the function $f(c,T)$, we consider $f(c,0)$, which is the cost associated with estimating

$$x(0) = c \tag{11}$$

solely on the basis of the a priori information available. We might select the condition

$$f(c,0) = k(c - c_0)^2, \tag{12}$$

where c_0 is our best estimate of the quantity $x(0)$, and k is a measure of our confidence in that estimate. The linear partial differential Eq. (10) together with the condition in Eq. (12) determine the function $f(c,T)$.

Assume that we have been observing the process in the interval $0 \le t \le T_0$. Our estimate of the current state is the value of c which minimizes the cost function $f(c,T_0)$. In fact, we see that we would like to be able to determine the minimizing value of c for each value of $T \geq 0$. These are our estimates, e(T). These minimizing points satisfy the equation

$$f_c(e,T) = 0, \tag{13}$$

or

$$f_{cc}(e,T)de + f_{cT}(c,T)dT = 0, \tag{14}$$

$$\frac{de}{dT} = - f_{cT}(e,T)/f_{cc}(e,T). \tag{15}$$

Once the function y(t) has been observed on the interval $0 \leq t \leq T$, and the functions f(e,T), $f_{cT}(e,T)$ and $f_{cc}(e,T)$ have been determined, the Eq. (15) provides the optimal estimate of x(T), e(T). As an initial condition we might use the condition

$$e(0) = c_0. \tag{16}$$

IV. CONTINUATION OF THE ANALYSIS

Fortunately, we can continue our analysis much further. We take the partial derivatives of both sides of Eq. (10) to obtain the equation

$$f_{Tc} + g_c f_c + g f_{cc} = - 2(y(T) - c). \tag{17}$$

This yields

$$- f_{cT}/f_{cc} = (g_c f_c/f_{cc}) + g + (2(y - c))/f_{cc}. \tag{18}$$

Since

$$f_c(e,T) = 0, \tag{19}$$

we see that Eq. (15) becomes

$$\frac{de}{dT} = g(e,T) + (2/f_{cc}(e,T))\ (y - e). \tag{20}$$

If we introduce

$$2/f_{cc}(e(T),T) = q(T), \tag{21}$$

we see that Eq. (20) assumes a remarkable form

$$\frac{de}{dT} = g(e,T) + q(T)\ (y - e). \tag{22}$$

It states that the optimal estimate of the current state, e(T), is obtained by integrating a slightly modified form of the original system equation

$$\dot{x} = g(x,t), \tag{23}$$

the modification being a term q(T) (y - e), where y - e is the difference between the observed and estimated values and q is a weighting factor. This is an extension of well-known results for the linear case.(5)

V. EQUATION FOR THE WEIGHTING FACTOR

To derive an equation for the weighting factor

$$q(T) = 2/f_{cc}(e(T),T) \tag{24}$$

we differentiate Eq. (17) with respect to c, which yields the relation

$$(f_{cc})_T + g(f_{cc})_c = 2[1 - g_c f_{cc}] - g_{cc} f_c. \qquad (25)$$

Also we have

$$\frac{dq}{dT} = - 2f_{cc}^{-2} \, [f_{ccc} \frac{de}{dT} + f_{ccT}]. \qquad (26)$$

We can use Eqs. (24), (25), (13), and (22) to rewrite this equation in the form

$$\frac{dq}{dT} = - 2(q^2/4)[f_{ccc}\{g+q(y-e)\} + 2\{1-g_c(2/q)\} - gf_{ccc}] \qquad (27)$$

$$\frac{dq}{dT} = 2g_c q - q^2 - \frac{q^3}{2} f_{ccc}(y-e). \qquad (28)$$

We can again derive an equation for f_{ccc} by noting that

$$\frac{d}{dT} f_{ccc}(e(T),T) = f_{cccc} \cdot \frac{de}{dT} + f_{cccT} . \qquad (29)$$

Differentiating Eq. (17) with respect to c yields

$$f_{cccT} + g_c f_{ccc} + g \, f_{cccc} = - 2\{g_c f_{ccc} + g_{cc} f_{cc}\}$$ [redacted]

$$- g_{ccc} f_c - g_{cc} f_{cc} . \qquad (30)$$

Combining Eqs. (29) and (30) yields

$$\frac{d}{dT} f_{ccc} = f_{cccc} \{\frac{de}{dT} - g\} - 3\, g_c f_{ccc} - 3\, g_{cc} f_{cc}$$

$$= -\, 6 \frac{g_{cc}}{q} - 3\, g_c f_{ccc} + q(T) f_{cccc} (y-e) \quad . \tag{31}$$

VI. PRACTICAL CONSIDERATIONS

We note that Eq. (31) involves an additional unknown variable f_{cccc}. We can, of course, derive an additional differential equation for f_{cccc} which will again involve f_{ccccc}.

In practice, this refinement is quite unnecessary. We assert that the function $f(c,T)$ is representable in the form $a_0(T) + a_1(T)c + a_2(T)c^2$ in the neighborhood of the optimal estimate $e(T)$. This implies that f_{ccc} is negligible in the neighborhood of the optimal estimate.

With this assumption, the estimator equations are, from Eqs. (22) and (28),

$$\frac{de}{dT} = g(e,T) + q(T)\ (y-e) \tag{32}$$

and

$$\frac{dq}{dT} = 2\, g_c(e,T) q - q^2 . \tag{33}$$

VII. NUMERICAL RESULTS

To test the method we did some numerical experiments on a digital computer. We considered a system described by the nonlinear differential equation and initial condition

$$\dot{x} = - x + \epsilon x^3/3, \qquad 0 \le t \le 5, \tag{34}$$

$$x(0) = x_0. \tag{35}$$

We put $x_0 = 1$ and $\epsilon = 0.1$ and produced the solution, which is shown in Fig. 1. Next we produced values of

$$y(t) = x(t) + .5 \cos(60 t), \tag{36}$$

which is to serve as our "noisy" observation of the process $x(t)$. Then we tried several experiments to see how well and how rapidly we could determine the true curve $x(t)$ from the noisy observations. The equations being integrated are, according to Eqs. (32) and (33),

$$\frac{de}{dT} = - e + \epsilon \frac{e^3}{3} + q[y(T) - e], \qquad e(0) = e_0, \tag{37}$$

$$\frac{dq}{dT} = 2[\epsilon e^2 - 1]q - q^2, \qquad q(0) = q_0. \tag{38}$$

In the first set of experiments we put $q_0 = 0.2$ and tried three initial values for $e(0)$,

$$e(0) = 2\, x(0) = 2$$

$$e(0) = y(0) = 1.5$$

$$e(0) = 1.1\, x(0) = 1.1.$$

In all cases by time $t = 5$ the observations, represented by $y = y(t)$, have been filtered and $e(t) \cong x(t)$. Then we tried the same experiment but with $q_0 = q(0) = 20$. In this case excellent filtering is accomplished by time $t = 2$, as is shown in Fig. 2.

Additional experiments are being conducted at RAND and Purdue.

VIII. SYSTEM IDENTIFICATION

The approach presented here appears rather promising in many applications. As an example we cite the identification problem in adaptive control where one has to determine a running (on-line) estimate of certain plant parameters based on input-output observations before exerting further control. When suitably formulated, this problem takes the form:

Given the dynamics of the plant in the form

$$\dot{\underline{x}} = \underline{f}(t,\underline{x},\underline{u},\underline{a}) \quad , \tag{39}$$

where $\underline{x}$ is the state vector, $\underline{u}$ the control vector and $\underline{a}$ is an unknown parameter vector, determine the parameter vector $\underline{a}$ based on observations

$$\underline{y}(t) = \underline{g}(t,\underline{x}) + \text{observational error} \tag{40}$$

in order to minimize

$$\int_0^T \| \underline{y}(t) - \underline{g}(t,\underline{x}) \|_Q \, dt \quad , \tag{41}$$

where $\|\cdot\|_Q$ denotes a suitable semi-norm.

We note that this problem is exactly in the framework proposed in this note as soon as we adjoin the additional equation

$$\dot{\underline{a}} = 0 \quad . \tag{42}$$

It would be interesting to find out if it will be possible to extend this approach when Eq. (1) involves a random forcing term.

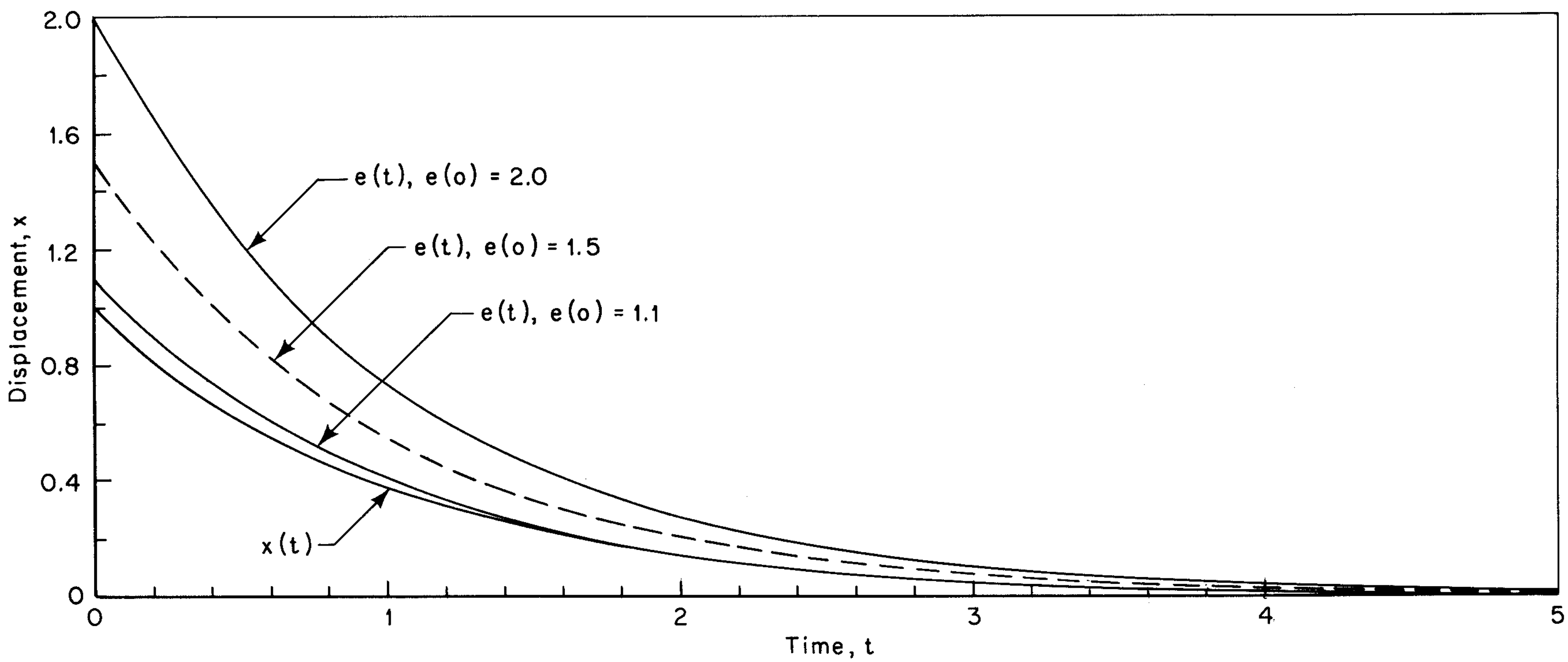

Fig. 1—Estimation of current state of nonlinear system: three trials with $q_0 = 0.2$

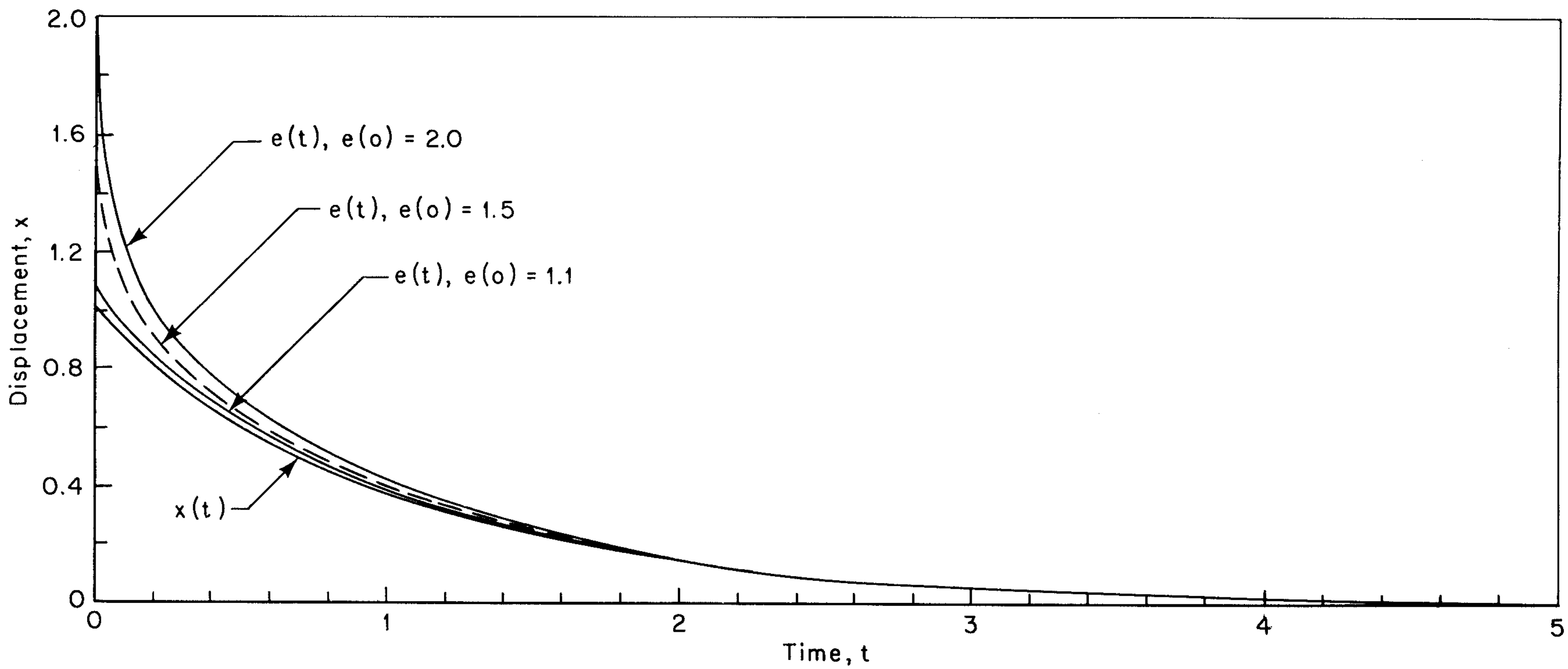

Fig. 2—Estimation of current state of nonlinear system: three trials with $q_0 = 20$

Appendix

SOME GENERALIZATIONS

Here we consider a system described by the nonlinear differential equation

$$\dot{x} = g(t,x) \quad , \tag{43}$$

where x and g are n-dimensional vectors with components $(x_1,x_2,\ldots,x_n)$ and $(g_1,g_2,\ldots,g_n)$. The observations are

$$y(t) = h(t,x) + \text{observational errors} \quad , \tag{44}$$

where y and h are m-dimensional vectors with components $(y_1,\ldots,y_m)$ and $(h_1,\ldots,h_m)$ with $m \leq n$ and $0 \leq t \leq T$.

As before, we wish to estimate

$$x(T) = c \quad , \tag{45}$$

where c has components $(c_1,\ldots,c_n)$ such that

$$J = \int_0^T \| y(t) - h(t,x) \|^2_{Q(t)} dt \tag{46}$$

is minimized.

In Eq. (46) $\| \cdot \|_{Q(t)}$ denotes an appropriately chosen quasi-norm. Using the invariant-imbedding approach and determining

$$f(c,T) = \int_0^T \| y(t) - h(t,x) \|^2_{Q(t)} dt \quad , \tag{47}$$

where $x(T) = c$, we obtain the partial differential equation

$$f_T + \langle \nabla_c f, g(T,c) \rangle = \| y(T) - h(T,c) \|^2_{Q(T)} . \tag{48}$$

In Eq. (48), $\nabla_c f$ represents the n-dimensional vector with components $\frac{\partial f}{\partial c_i}$ and < , > represents the Euclidean inner product.

If e(T) is the optimal estimate at time T, then as in Eq. (15), e(T) satisfies the differential equation

$$\frac{de}{dT} = - [\nabla_{cc} f(e,T)]^{-1} (\nabla_c f)_T \quad , \tag{49}$$

where e(T) is an n-dimensional vector with components $(e_1, \ldots, e_n)$.

In Eq. (49) $\nabla_{cc} f$ is the matrix

$$\left(\frac{\partial^2 f}{\partial c_i \partial c_j} \right) \qquad \begin{array}{l} i=1,2,\ldots,n \\ j=1,2,\ldots,n \end{array} \quad , \tag{50}$$

and $(\nabla_c f)_T$ is the vector $\frac{\partial}{\partial T} (\nabla_c f)$. As usual, $[\ ]^{-1}$ denotes the inverse of a matrix.

Taking the partial derivative with respect to c_i, i=1,2,...,n, in Eq. (48) yields

$$(\nabla_c f)_T + [\nabla_{cc} f]\, g + [g_c]\, (\nabla_c f) = - 2[h_c Q\, y(T) - h_c Q\, h(T,c)] \quad , \tag{51}$$

where

$$h_c = \left(\frac{\partial h_j}{\partial c_i} \right) \qquad \begin{array}{l} i=1,\ldots,n \\ j=1,\ldots,m \end{array} \tag{52}$$

and

$$[g_c] = \left(\frac{\partial g_j}{\partial c_i} \right) \qquad i,j=1,2,\ldots,n \quad . \tag{53}$$

Combining Eqs. (49) and (51) yields

$$\frac{de}{dT} = g(T,e) + 2[\nabla_{cc} f]^{-1} h_c(T,e) Q(T) [y(T) - h(T,e)] \quad . \tag{54}$$

Now define

$$q(T) = 2[\nabla_{cc} f]^{-1} \quad , \tag{55}$$

$$\frac{1}{2}\, q(T)\, [\nabla_{cc} f] = I \quad . \tag{56}$$

Taking the total derivative with respect to T in Eq. (55) yields

$$\frac{1}{2}\left[\frac{dq}{dT}\, [\nabla_{cc} f] + q(T)\, \frac{d}{dT}\, [\nabla_{cc} f]\right] = 0 \quad . \tag{57}$$

Hence

$$\frac{dq}{dT} = -\, q(T)\, \frac{d}{dT}\, [\nabla_{cc} f] \cdot q(T) \quad . \tag{58}$$

Taking the partial derivative with respect to c_i, $i=1,\ldots,n$ on both orders of Eq. (51) yields

$$\begin{aligned} &[\nabla_{cc} f]_T + g_c [\nabla_{cc} f] + [\nabla_{cc} f]\, g_c^T + p(c,T) \\ &= -\, 2[h_{cc} Q\, y(T) - h_{cc}\, Q\, h - h_c\, Q\, h_c^T] \end{aligned} \tag{59}$$

In Eq. (59) the matrix p(c,T) has elements which consist only of terms which have factors of the form $\partial f/\partial c_i$ or $\partial^3 f/\partial c_i \partial c_j \partial c_k$, and $h_{cc}\, Q\, h(T)$ is an n x n matrix with i^{th} column $\partial/\partial c_i\, h_c\, Q\, y(T)$. The term $h_{cc}\, Q\, h$ is similarly defined. The term g_c^T refers to the transpose of the matrix g_c.

Note that when c takes on its optimal estimate e, the terms in p(c,T) with factors of the form $\partial f/\partial c_i$ drop out, and by our previous assertion, the terms with factors $\partial^3 f/\partial c_i \partial c_j \partial c_k$ are negligible.

In Eq. (58) the term $\frac{d}{dT}\,[\nabla_{cc} f]$ can be written as

$$\frac{d}{dT}\,[\nabla_{cc} f] = [\nabla_{cc} f]_T + r(c,T) \quad . \tag{60}$$

In Eq. (60) the matrix r(c,T) has elements which consist only of terms which have factors of the form $\partial^3 f/\partial c_i \partial c_j \partial c_k$.

Combining Eqs. (58), (59), and (60) and dropping the matrices p(c,T) and r(c,T) yields

$$\frac{dq}{dT} = \text{[redacted]}[g_c q + q\ g_c^T] + q[h_{cc} Q(y-h)]q = q\ h_c Q\ h_c^T\ q \text{[redacted]} \quad . \tag{61}$$

Equation (54) can be rewritten [redacted]

$$\frac{de}{dT} = g + q\ h_c Q(y-h) \quad . \tag{62}$$

Equations (61) and (62) are the estimator equations.

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