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MAGNETOHYDRODYNAMIC-HYPERSONIC VISCOUS AND INVISCID FLOW NEAR THE STAGNATION POINT OF A BLUNT BODY

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PREFACE

The design of hypersonic vehicles has stimulated a large number of studies concerned with control techniques. This Memorandum considers one such method in which a magnetic field carried by the vehicle interacts with the shock-ionization produced by the vehicle. The study is an extension of RM-3970-PR, Magnetohydrodynamic-Hypersonic Viscous Flow near the Stagnation Point of a Blunt Body, January 1964; the boundary conditions used more closely approximate the physical situation.

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This study is part of continuing RAND research in theoretical fluid mechanics. The results will be useful as a guide in further work in this field.

SUMMARY

This Memorandum numerically investigates the hypersonic viscous and inviscid flow past a spherical body which contains a magnetic dipole field aligned with the flow. Local-similarity solutions of the flow field are assumed near the stagnation point. The solution found differs from previous solutions in that the magnetic field is defined on the body rather than at the shock. The numerical results indicate that the critical values of the magnetic interaction parameter no longer appear as in previous works and that the shock displacement is much larger than formerly found for magnetic interaction parameters less than one. The latter result suggests that a magnetic field is a more useful device for controlling the hypersonic flow about a body than previously thought.

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CONTENTS

INTRODUCTION

In studies of magnetohydrodynamic re-entry phenomena, the interaction between the magnetic field carried by the re-entry vehicle and the flow of the partially ionized gas surrounding the vehicle is given by the magnetic interaction parameter. This parameter is the product of the magnetic Reynolds number and the ratio of the magnetic pressures to the dynamic pressures. For a small magnetic interaction parameter the flow is essentially undisturbed by the magnetic field, and the induced magnetic field is negligible in comparison with the primary field. The magnetic field at any point in the flow can be assumed to be that of the primary field. By increasing the magnetic interaction parameter, i.e., increasing the conductivity (magnetic Reynolds number) or the applied magnetic field, the flow is no longer undisturbed. The effect which is of concern here is that the induced magnetic field is no longer negligible in comparison with the primary field.

An approach to simplifying the involved partial differential equations describing the hypersonic flow is to restrict the investigation to a local similarity solution. For further computational simplification the magnetic field strength at the shock is usually assumed as a boundary condition. Using this approach, Bush found that numerical integration becomes impossible for the inviscid case when the value of the magnetic interaction parameter exceeds a certain critical value.[1,2] More recently a similar result was found by Smith and Wu for the viscous case.[3]

In the present work the magnetic field is defined at the body where it is a natural characteristic of the problem. The solution then consists of solving ordinary differential equations by a quasilinearization technique which simultaneously satisfies boundary conditions on the shock and on the body.

LOCAL-SIMILARITY SOLUTION

The model used here is the same as that used previously in the literature, i.e., a steady-state constant-density viscous layer model. In the stagnation region, the equations of magnetohydrodynamics describing the flow can be written in the form(4)

$$\nabla \cdot \underline{v} = 0$$

$$\nabla \times \left[(\nabla \times \underline{v}) \times \underline{v} - \frac{\mu_e}{4\pi\rho} (\nabla \times \underline{H}) \times \underline{H} \right] = \nu \nabla^2 \nabla \times \underline{v}$$

$$\nabla \cdot \underline{H} = 0$$

$$\nabla \times \left[\nabla \times \underline{H} - 4\pi\sigma \mu_e \underline{v} \times \underline{H} \right] = 0 \qquad (1)$$

where

μ_e = magnetic permeability

σ = electrical conductivity

$\underline{H}$ = magnetic field intensity

ν = kinematic viscosity

∇^2 = Laplacian operator

$\underline{v}$ = velocity vector

ρ = density

The spherical body is assumed to contain a dipole magnetic field axisymmetric with the flow field. The postulated similarity solutions associated with the stagnation point flow have the following functional form

$$v_r = v_\infty \left[2f(\eta)/\eta^2\right] \cos\theta$$

$$v_\theta = -v_\infty \left[\frac{1}{\eta}\frac{df}{d\eta}\right] \sin\theta$$

$$H_r = H_o \left[2g(\eta)/\eta^2\right] \cos\theta$$

$$H_\theta = -H_o \left[\frac{1}{\eta}\frac{dg}{d\eta}\right] \sin\theta \qquad (2)$$

where the angle θ is measured from the axis of symmetry, v_∞ and $H_o = B/r_b^3$ are the free-stream velocity and reference dipole magnetic field at the body, respectively, and $\eta = r/r_b$ (r_b = body radius).

From Eqs. (1) and (2), using a right-handed spherical coordinate frame with $2 \sin\theta \simeq \sin 2\theta$, a system of ordinary differential equations can be found. The equation of motion reduces to

$$f \frac{d}{d\eta}\left[\frac{1}{\eta^2}\left(\frac{2f}{\eta^2} - \frac{d^2f}{d\eta^2}\right)\right] - M_p\, g \frac{d}{d\eta}\left[\frac{1}{\eta^2}\left(\frac{2g}{\eta^2} - \frac{d^2g}{d\eta^2}\right)\right] =$$

$$\frac{1}{2Re}\left[\left(\frac{1}{\eta}\frac{d}{d\eta}\frac{\eta^2 d}{d\eta} - \frac{2}{\eta}\right)\left(\frac{2f}{\eta^3} - \frac{1}{\eta}\frac{d^2f}{d\eta^2}\right)\right] \qquad (3)$$

and the induction equation becomes

$$\frac{2g}{\eta^2} - \frac{d^2g}{d\eta^2} = 2R_m \left[g\frac{df}{d\eta} - f\frac{dg}{d\eta} \right] \frac{1}{\eta^2} \quad (4)$$

where

$$M_p = \frac{\mu_e H_o^2}{4\pi\rho_\infty v_\infty^2} = \text{magnetic pressure number}$$

$$R_m = 4\pi\sigma \mu_e v_\infty r_b = \text{magnetic Reynolds number}$$

$$Q_m = M_p R_m = \text{magnetic parameter defined at the body}$$

$$Re = \frac{v_\infty r_b}{\nu} = \text{Reynolds number}$$

BOUNDARY CONDITIONS AT THE SHOCK

If the constant-density shock layer model is further simplified by assuming a thin viscous boundary layer, the boundary conditions of $f(\eta)$ at the shock, with $\eta_s = \frac{r_s}{r_b}$ (r_s = shock radius), are

$$f(\eta_s) = -\frac{\epsilon\eta_s^2}{2}, \quad \left(\frac{df}{d\eta}\right)_{\eta_s} = -\eta_s \quad (5)$$

where $\epsilon = \rho_\infty/\rho$, the ratio of density across the shock. The other boundary condition at the shock is based on the vorticity jump[4]

$$\left(\frac{d^2f}{d\eta^2}\right)_{\eta=\eta_s} = 2 - 2\epsilon - \frac{1}{\epsilon} - 4\frac{Q_m}{\eta_s^2\epsilon} g \left(2\epsilon\frac{dg}{d\eta} - \frac{g}{\eta}\right)_{\eta=\eta_s} \quad (6)$$

BOUNDARY CONDITIONS AT THE BODY

The flow is governed at the body by the condition of nonslip and zero radial velocity

$$f(1) = 0 \ , \qquad \left(\frac{df}{d\eta}\right)_{\eta=1} = 0 \tag{7}$$

In the inviscid solution the nonslip condition is removed. Two additional conditions at the body are based on the continuity of the magnetic field at the body. These are

$$g(1) = 1 \ , \qquad \left(\frac{dg}{d\eta}\right)_{\eta=1} = -1 \tag{8}$$

where $\eta=1$, at the body.

COMPUTATION PROCEDURE

A quasilinearization technique was used in the solution of the two-point boundary value problem. The viscous solution consisted of solving ordinary differential Eqs. (3) and (4), satisfying simultaneously three conditions on the shock and four conditions on the body. Since the shock stand-off distance is also unknown, a transformation to independent variable t

$$\eta = \left(\frac{r_s}{r_b} - 1\right) t + 1 \tag{9}$$

gives the integration range from the border to shock $0 \leq t \leq 1$, where the stand-off distance $\zeta(\zeta = r_s/r_b - 1)$ is an unknown. The differential equation

$$\frac{d\zeta}{dt} = 0 \tag{10}$$

is added to Eqs. (3) and (4). Equations (3), (4), and (9) reduce to a set of seven first-order ordinary differential equations and seven boundary conditions in the viscous solution. The initial value of ζ is determined by the boundary conditions.

In conjunction with the quasilinearization scheme it was found necessary to reorthogonalize at every other integration step. A summary description of the quasilinearization scheme is given in the Appendix, and a further example is found in Ref. 5. The Gram-Schmidt orthogonalization used is described in Ref. 6.

DISCUSSION OF RESULTS

The viscous Reynolds numbers Re used were 100, 1000, and ∞. The range of magnetic interaction parameters where solutions were possible was rather limited. At the larger magnetic interaction parameters the quasilinearization solution was not a solution of the original nonlinear differential equations, and was not included, However, a number of conclusions can be drawn from the solutions which were obtained.

The numerical calculations of stand-off distance as a function of interaction parameter are illustrated in Fig. 1. These results are in general agreement with previous works, i.e., the shock wave stand-off distance increases with increasing magnetic interaction parameter for a given viscous Reynolds number and for a constant magnetic interaction parameter the stand-off distance decreases with increasing viscous Reynolds number, as expected by physical reasoning.[3,4] There are possibly

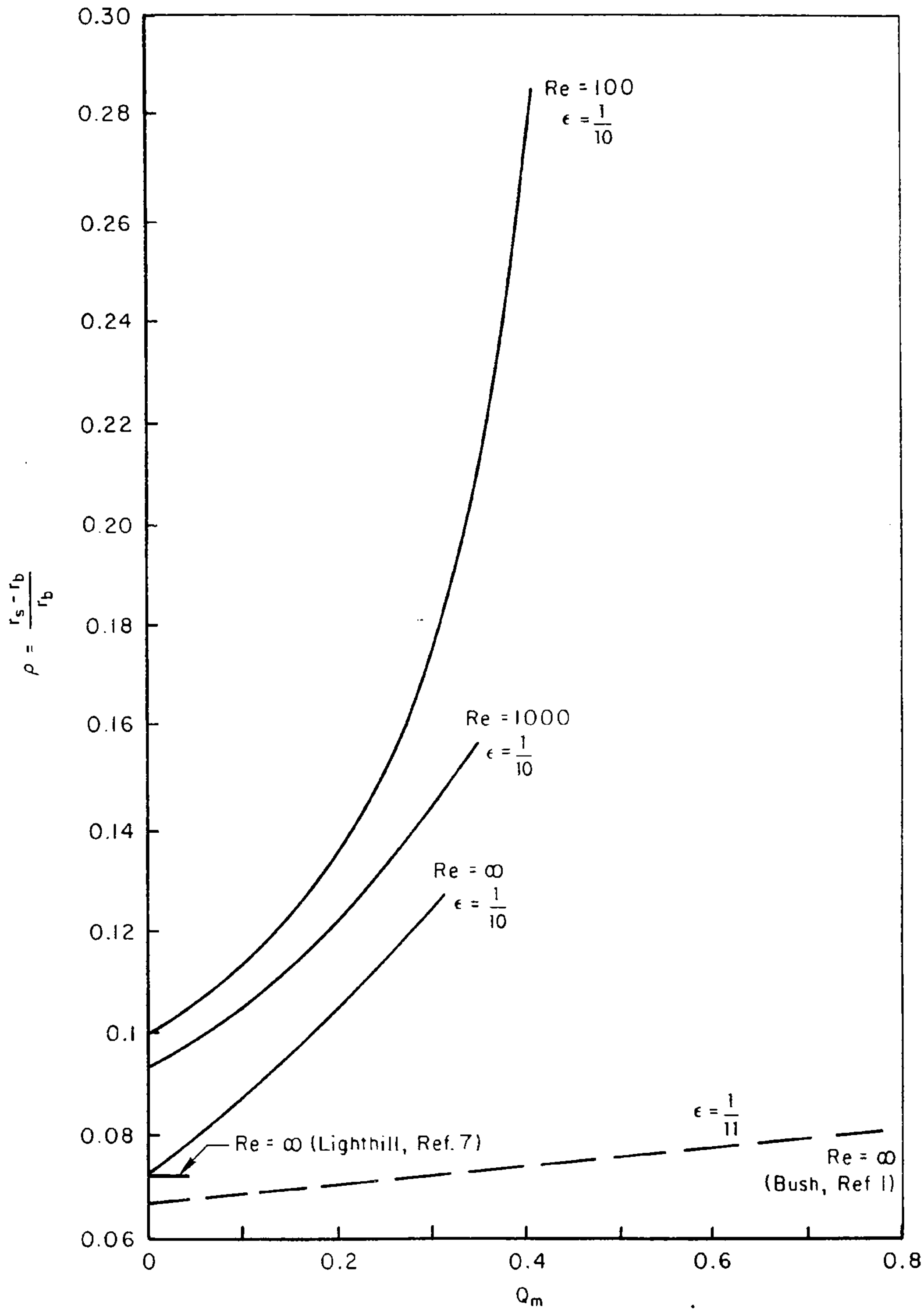

Fig. 1—Stand-off distance versus magnetic interaction parameter

two features of this plot which are new. In previous works a critical value of the interaction parameter appears at which the shock stand-off distance recedes to infinity asymptotically as the applied magnetic field is increased.[3] The critical intereaction parameter in the present plot no longer appears as in Ref. 3. Secondly, for $Q_m < 1$, the stand-off distance increases much more rapidly for increasing Q_m in this solution than in the work by Bush.[1] If these results are applicable, it would indicate that the magnetic field, even for $Q_m < 1$, is a more effective method of controlling the hypersonic flow than previously thought.

Table 1 gives the computed values of the ratio of the resultant magnetic field components at the shock to the dipole field at the body, and also the ratio of the undistorted dipole field at the shock to that at the body. It is evident from this Table that the stand-off distance depends essentially upon the product of R_m and M_p, i.e., Q_m. A similar result has appeared in previous calculations.[1,3,4] One further interesting feature of the tabulated results is that the resultant radial field H_r decreases much as a dipole field while the angular field component H_θ decrease is slightly greater. However, for a fixed Q_m, H_θ depends upon the relative values of R_m and M_p. It appears that the induced magnetic field is influenced mainly by the electrical conductivity. Hence a decrease in the conductivity (the induced field) should result in a H_θ more nearly approximated by a dipole field.

Table 1

R_m	M_p	$\frac{r_s - r_b}{r_s}$	$g(\eta_s)/\eta_s^2$	$-\frac{1}{\eta_s}\left(\frac{dg}{d\eta}\right)_{\eta_s}$	$\left(\frac{r_b}{r_s}\right)^3$
Re = 100					
0.0	0.0	0.1008	0.7497	0.7497	0.7497
1.0	0.1	0.1137	0.7272	0.6311	0.7239
1.0	0.2	0.1344	0.6884	0.5945	0.6850
1.0	0.3	0.1752	0.6195	0.5308	0.6161
1.0	0.4	0.2619	0.5002	0.4249	0.4977
0.1	1.0	0.1136	0.7243	0.7147	0.7240
0.2	1.0	0.1342	0.6860	0.6672	0.6853
0.3	1.0	0.1745	0.6181	0.5915	0.6171
Re = 1000					
0.0	0.0	0.0929	0.7660	0.7660	0.7670
1.0	0.1	0.1054	0.7435	0.6463	0.7402
1.0	0.2	0.1226	0.7103	0.6146	0.7068
1.0	0.3	0.1449	0.6704	0.5759	0.6667
Re = ∞					
0.0	0.0	0.0731	0.8093	0.8093	0.8094
1.0	0.1	0.0878	0.7799	0.6800	0.7767
1.0	0.2	0.1066	0.7410	0.6420	0.7378
1.0	0.3	0.1241	0.7074	0.6093	0.7039

Appendix

QUASILINEARIZATION

In the viscous solution, Eqs. (3) and (4) can be reduced to the solution of seven first-order nonlinear differential equations, subject to the seven boundary conditions in Eqs. (5), (6), (7) and (8)

$$\dot{x}_i = u_i(x, t) , \qquad i = 1, \cdots, 7 \qquad 0 \le t \le 1 \tag{11}$$

where t = 0 on the shock and t = 1 on the body. The quasilinearization scheme gives the n+1 approximation in terms of the n approximation as

$$\dot{x}_i^{n+1} = \sum_{j=1}^{7} d_{ij} (x^n, t) \, x_j^{n+1} + c_i(x^n, t) \tag{12}$$

where

$$d_{ij}(x^n, t) = \frac{\partial u_i(x^n, t)}{\partial x_j} \tag{13}$$

and

$$c_i(x^n, t) = u_i(x^n, t) - \sum_{j=1}^{7} d_{ij}(x^n, t) x^n \tag{14}$$

The linear equation (12) is solved by initially determining a particular solution and seven independent homogeneous solutions of the form

$$x_i^{n+1}(t) = p_i^{n+1}(t) + \sum_{j=1}^{7} a_j \, h_{ij}^{n+1}(t) \tag{15}$$

where the vector $\left\{p_i^{n+1}(t)\right\}$ is a particular solution and the column vectors $\left\{h_{ij}^{n+1}(t)\right\}$ for j=1, 2, ⋯, 7, are homogeneous solutions. The choice of the initial column vector $\left\{h_{ij}(0)\right\}$ has the $j^{\underline{th}}$ row unity and all others zero, and the vector $\left\{p_i(0)\right\} = 0$. The unknown constant multipliers a_j are found by substitution of the linear solution (15) into the linearized boundary conditions and solving for the a_j by matrix inversion.

There are two basic computational methods of solution.[5] The first method, outlined below, is used when the step size is sufficently large so as not to exceed the available storage for the particular and homogeneous solutions.

FIRST METHOD:

1) Initial approximation
   a) Integrate (12) and store $x_i^o(t)$ at each step in the interval $0 \le t \le 1$
2) $k^{\underline{th}}$ iteration
   a) Integrate and store $p_i(t)$, $h_{i1}(t)$, $\cdots$, $h_{i7}(t)$ in the interval $0 \le t \le 1$
   b) Find the constants a_1, $\cdots$, a_7
   c) Compute the $k^{\underline{th}}$ approximation

$$x_i^k(t) = p_i(t) + \sum_{j=1}^{7} a_j\, h_{ij}(t)$$

The second method, incorporating computer memory-saving techniques, is used when storage requirements become too great. Execution time is increased because of the larger system of differential equations, i.e., seven equations are added at each iteration.

SECOND METHOD:

1) Initial approximation

a) Integrate the nonlinearized equations for $x_i^o(t)$, $p_i^1(t)$, $h_{ij}^1(t)$ and compute the values for the first approximation $x_i^1(t)$

2) $k^{\underline{th}}$ iteration

a) Integrate $x_i^o(t)$, $x_i^1(t)$, $x_i^2(t)$, $\cdots$, $x_i^{k-1}(t)$, $p_i^k(t)$, $h_{ij}^k(t)$ in the interval $0 \le t \le 1$

b) Find the constants a_1, a_2, $\cdots$, a_7

c) Compute the $k^{\underline{th}}$ approximation

$$x_i^k(t) = p_i^k(t) + \sum_{j=1}^{7} a_j \, h_{ij}(t)$$

3) Final iteration

a) Integrate the nonlinear equations for $x_i^k(t)$

The second method was used to obtain the data given in this paper. The technique is discussed further in Ref. 5.

REFERENCES

1. Bush, W. B., "Magnetohydrodynamic-Hypersonic Flow Past a Blunt Body," Journal of the Aero/Space Sciences, Vol. 25, No. 11, November 1958, pp. 685-690, 728.

2. Bush, W. B., "A Note on Magnetohydrodynamic-Hypersonic Flow Past a Blunt Body," Journal of the Aero/Space Sciences, Vol. 26, No. 8, August 1959, pp. 536-537.

3. Smith, M. C., and C. S. Wu, "Magnetohydrodynamic-Hypersonic Viscous Flow Near the Stagnation Point of a Blunt Body," Journal of the Aero/Space Sceinces, Vol. 2, No. 5, May 1964, pp. 963-965.

4. Wu, C. S., "Hypersonic Viscous Flow Near the Stagnation Point in the Presence of a Magnetic Field," Journal of the Aero/Space Sciences, Vol. 27, No. 12, December 1960, pp. 882-893, 950.

5. Bellman, R. H., and R. E. Kalaba, Quasilinearization and Boundary Value Problems, American Elsevier Press, New York, December 1964.

6. Murdock, D. C., Linear Algebra for Undergraduates, John Wiley and Sons, New York, 1957, p. 34.

7. Lighthill, M. J., "Dynamics of a Dissociating Gas. Part I: Equilibrium Flow," Journal of Fluid Mechanics, Vol. 2, No. 1, January 1957, pp. 28-31.