ARPA ORDER NO. 189-61

MEMORANDUM
RM-4439-ARPA
JUNE 1965

A PRELIMINARY TREATMENT OF MOBILE SLBM DEFENSE: GAME THEORETIC ANALYSIS

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PREPARED FOR:
ADVANCED RESEARCH PROJECTS AGENCY

The RAND Corporation
SANTA MONICA • CALIFORNIA

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This research is supported by the Advanced Research Projects Agency under Contract No. SD-79. Any views or conclusions contained in this Memorandum should not be interpreted as representing the official opinion or policy of ARPA.

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Published by The RAND Corporation

PREFACE

This Memorandum is a product of a continuing study for the Advanced Research Projects Agency on defense against submarine-launched ballistic missiles.

SUMMARY

We analyze as a game the deployment of a defense system against submarine-launched ballistic missiles composed of mobile defense units capable of destroying a nearby submarine and its missiles at the time of launch. We assume that the ocean is divided into zones among which both the attacker and defender deploy their forces. The attacker then launches a mass attack against targets on the defender's land mass. Each defense unit can successfully destroy one submarine and its missiles in the same zone.

We first assume the payoff to the attacker to be the number of submarines which successfully launch their missiles. We solve the game with this payoff function when neither player has any information about the other's location, and obtain a partial solution when the defender has some information about the location of the attacker's submarines. We then assume the payoff to the attacker to be the number of zones from which at least one submarine successfully launches its missiles. We solve this game for certain values of the parameters involved when neither side has any information about the location of the other's forces.

CONTENTS

A PRELIMINARY TREATMENT OF MOBILE SLBM DEFENSE AS A GAME THEORETIC ANALYSIS

1. INTRODUCTION

We postulate a defense system against submarine-launched ballistic-missile (SLBM) attack. The system is composed of mobile defense units capable of destroying a nearby submarine and its missiles at the time of launch. We then analyze its deployment as a two-person, zero-sum game, i.e., one in which the attacker's gain is the defender's loss. Thus the attacker will try to maximize his expected payoff, while the defender will try to minimize it. (For a general discussion of two-person, zero-sum games, see [1].)

The game is played as follows. The ocean area of interest is divided into zones, whose size is determined by the capabilities of the defense unit. After both players have deployed their forces among the zones, the attacker launches a mass missile attack against targets on the defenders land mass. Each defense unit is capable of destroying one submarine and its missiles at the time of attack.

We first assume the payoff to the attacker to be the number of submarines which successfully launch their missiles, and solve the game when neither player has any information about the deployment of the other's forces. (For a variant of this game, in which the zones are of unequal value to the attacker and the defender may divide his forces arbitrarily, see [1], pp. 124-127.) We then introduce a simple detection system which gives the defender information about which zones contain submarines, but no information about how many submarines each zone contains. We solve this game when the detection probability is unity, and obtain a partial solution when the detection probability is less than unity.

We next assume the payoff to the attacker to be the number of zones from which at least one submarine successfully launches its missiles. We solve this game when the defender has no detection system and the parameters satisfy a particular constraint. We assume throughout that the attacker has no knowledge of the defender's deployment.

2. THE PAYOFF FUNCTION M(a,b) WITHOUT DETECTION

We assume that the ocean area of interest is divided into N zones, the defender has D defensive units, and the attacker has S submarines. The parameters N, D, and S are known to both players, but neither player knows the deployment of the other's forces. We shall see that each side may play optimally without knowing the strength of the opponent.in this case. The payoff to the attacker is the total number of submarines which successfully launch their missiles.

A strategy for the attacker is a vector $\underline{a} = (a_1,\ldots,a_N)$, with each a_i a nonnegative integer, such that $\sum_{i=1}^{N} a_i = S$. This strategy is interpreted as follows: The attacker divides his submarines into N groups, with a_i submarines in the i–th group (note that a_i may be zero), and distributes the N groups among the zones at random, so that each of the possible N! assignments of groups to zones is equally likely. (Note that with this convention, two strategies are the same if one is a permutation of the other.) A strategy for the defender is a vector $b = b_1, \cdots, b_N)$, with each b_j a non–negative integer, such that $\sum_{j=1}^{N} b_j = D$. The interpretation is analogous to the previous one. We need only consider random assignments of groups to zones, since any bias by either player toward a particular zone would tend to favor the other player. Let A be the set of strategies for the attacker and B be the set of strategies for the defender.

If the attacker uses a and the defender uses b, the expected payoff is given by

$$(1) \quad M(a,b) = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \max(a_i - b_j, 0).$$

For computational reasons, it will be more convenient to work with the attacker's loss, i.e., the number of sub–marines destroyed by the defender, rather than his payoff. Let $L(a_i, b_j)$ denote his expected loss from the i–th group of submarines and the j–th group of defense units, $L(a_i, b)$ his loss from the i–th group of submarines, $L(a, b_j)$ his loss from the j–th group of defense units, and $L(a,b)$ his total loss when he uses a and the defender uses b. Then

$$L(a_i,b_j) = \frac{\min(a_i,b_j)}{N},$$

$$L(a_i,b) = \sum_{j=1}^{N} L(a_i,b_j),$$

$$L(a,b_j) = \sum_{i=1}^{N} L(a_i,b_j), \tag{2}$$

$$L(a,b) = \sum_{i=1}^{N} \sum_{j=1}^{N} L(a_i,b_j).$$

It is clear that

$$M(a,b) = S-L(a,b). \tag{3}$$

Throughout the remainder of this and the next section, proofs are given in terms of the loss function $L(a,b)$. The corresponding statements in terms of the payoff function $M(a,b)$ follow immediately from (3).

For any strategy a, let $m(a) = \max\limits_{1\leq i\leq N} a_i$ denote the largest component of a, and let a^* be the strategy for the attacker such that $m(a) = S$, i.e., the strategy which places all submarines in the same zone. Let b^* be the strategy for the defender such that $b_j^* = m(b^*)$ or $m(b^*) - 1$, i.e., the strategy which distributes the defensive units uniformly over the zones.

Theorem 1. _The value of the game is_ $v = \max(S-D/N,0)$, _and_ a^* _and_ b^* _are optimal_. _Furthermore_

$$(4) \qquad M(a,b^*) = \min_{b \in B} M(a,b),$$

<u>for all</u> $a \in A$, <u>and if</u> $v > 0$, <u>then for any</u> $a \neq a^*$, $M(a,b^*) < v$.

<u>Proof</u>. For any $a \in A$, $b \in B$, and $1 \leq i \leq N$, we have

$$L(a_i,b) = \sum_{j=1}^{N} \frac{\min(a_i,b_j)}{N} \leq \min(a_i, \frac{D}{N}).$$

If $a_i \geq m(b^*)$, then $\min(a_i,b_j^*) = b_j^*$ for all j, while if $a_i < m(b^*)$, then $\min(a_i,b_j^*) = a_i$ for all j. Thus

$$L(a_i,b^*) = \min(a_i, \frac{D}{N}) \geq L(a_i,b),$$

for all i. Summing over i, we have $L(a,b^*) \geq L(a,b)$ for all a and b, and (4) follows from (3). Either $\min(a_i,D/N) = a_i$ for all i, or $\min(a_i,D/N) = D/N$ for at least one i; hence

$$(5) \qquad L(a,b^*) = \sum_{i=1}^{N} \min(a_i, \frac{D}{N}) \geq \min(S, \frac{D}{N}).$$

If $D/N < S$, then equality holds if and only if $a_i = S$ for some i and $a_i = 0$ otherwise, i.e., $a = a^*$. This completes the proof.

Thus, according to Theorem 1, the optimal strategy for the attacker is a^*, which places all submarines in a single zone, and the optimal strategy for the defender is b^*, which spreads the defensive units over the zones as uniformly as possible. If the defender uses b^*, this minimizes the expected payoff to the attacker regardless of what strategy the attacker uses. If $v > 0$, this minimum expected payoff is strictly less than v unless the attacker uses a^*, in which case it is v, so that a^* is the unique optimal strategy for the attacker.

For any strategy a, let $n(a) = \sum_{i=1}^{N} \min(a_i,1)$ denote the number of nonzero components of a. Let $A_k = \{a\epsilon A \mid n(a) = k\}$. Let $a^k \epsilon A_k$ be the strategy such that $a_1^k = S-k+1$; $a_i^k = 1$, $2 \leq i \leq k$; $a_i = 0$, $k < i \leq N$, (i.e., $a^* = a^1$). For future use we need the following lemma.

<u>Lemma 1</u>. <u>For any</u> k,

$$L(a^k,b^*) = \min\left(\frac{Dk}{N},\ k-1+\frac{D}{N}, S\right),$$

$$= \min_{a\epsilon A_k} L(a,b^*).$$

The proof follows easily from (5).

Thus, if for any reason the attacker must spread his submarine among k zones, i.e., use some $a\epsilon A_k$, then the strategy which minimizes his expected losses, and thus maximizes his expected payoff, is a^k—the strategy which places one submarine in each of k–1 zones and the remaining S–k+1 submarines in a single zone. It can also be seen from (5) that if $D/N > 1$, i.e., if the defender has more than one defense unit per zone, this is the unique strategy in A_k for which this minimum is attained, while if $D/N \leq 1$, then all strategies in A_k are equally good against b^*, so that $M(a,b^*) = M(a^k,b^*)$ for all $a\epsilon A_k$.

If $D/N \geq S$, the defender may destroy all the attacking submarines, and there is nothing more to be said. If however, $D/N < S$, which is a much more interesting situation, then b^*, while optimal for the defender, has the disadvantage that it minimizes the attacker's expected payoff by assigning a large probability to a minimum number of submarines destroyed and probability zero to any larger number. It might be—again, for reasons extraneous to the game—that the defender would prefer to trade this for some probability of destroying a larger number of submarines. Theorem 2 asserts that he may do this and still achieve the value of the game.

<u>Theorem 2</u>. <u>Any</u> b <u>for which</u> $m(b) \leq S$ <u>is optimal</u>. <u>Furthermore, unless</u> $m(b) = D = S$, $M(a,b) < v$ <u>for all</u> $a \neq a^*$.

<u>Proof</u>. For each j, either $\min(a_i, b_j) = a_i$ for all i, in which case $L(a,b_j) = S/N \geq b_j/N$ or $\min(a_i, b_j) = b_j$ for at least one j. Hence

$$L(a,b_j) = \sum_{i=1}^{N} \frac{\min(a_i,b_j)}{N} \geq \frac{b_j}{N} ,$$

with equality if and only if $a = a^*$ or $b_j = S$. Summing over j, we get

$$L(a,b) = \sum_{j=1}^{N} L(a,b_j) \geq \frac{D}{N} ,$$

with equality when $a \neq a^*$ if and only if $b_j = S$ for some j, and $b_j = 0$ otherwise, i.e., $m(b) = D = S$.

According to Theorem 2, anything the defender does is optimal, so long as he does not waste defense units by putting more in a single zone than the attacker has submarines there. This is because the decreased probability of destroying at least one submarine, which results from placing defensive units in fewer zones, is balanced by the increased probability of destroying a larger number when they are found. Furthermore, the only case in which the attacker is not penalized if he uses any strategy other than a^* is the case when both sides have the same number of units ($S = D$), and the defender places all defensive units in the same zone ($m(b) = S$). (Note that while a strategy b' with $m(b') \leq S$ will always guarantee no more than v successful launches, it is not necessarily best against any strategy a, so that in general, Eq. (4) with b^* replaced by b' will be false.)

A random strategy for the defender is a rule which picks a strategy out of the set of strategies according to some probability distribution, and any random strategy which selects only optimal strategies is also optimal. Hence we have the following corollary.

Corollary. Any random strategy for the defender which randomizes over a set $B' \subset B$ such that $m(b) \leq S$ for all $b \in B'$ is optimal.

3. THE PAYOFF FUNCTION M(a,b) WITH DETECTION

If N is large, the defender must maintain a large number of defense units to defend against a small number of submarines, which might make the defensive system quite expensive. It is possible that he would then desire to invest in a detection system which would provide him with some information about the location of the attacker's submarines. We now analyze the game with the same payoff function and with a detection system in each zone which g gives an alarm with probability p_1 if at least one sub is present, and gives a false alarm with probability $p_2 < p_1$ if no sub is present. (Since p_1 is the probability of detection or false alarm, if the true detection probability is p, then $p_1 = p_2 + p - pp_2$.) The system gives no information about the number of submarines present, and the probability of detection is not increased by the presence of multiple submarines.

We do not solve the game, but find sets of strategies for both players which dominate all others. We do obtain a solution for the case $p_1 = 1$.

The game is played as follows. The attacker selects $a \epsilon A$, and distributes his submarines among the zones, as before. Then T alarms occur, where T is a random variable with values $0 \leq t \leq N$. The defender then selects $b \epsilon B$, depending on T, and distributes $b_1, \ldots, b_T$ randomly among the zones which gave alarms, and $b_{T+1}, \ldots, b_N$ among the zones where no alarms occurred.

A strategy for the defender is thus a function f from $\{0,1,\ldots,N\}$ to B, i.e., a rule which selects the sub-strategy $b = f(t)$ that the defender will use in the above manner when $T = t$. Let F denote the set of strategies for the defender. B is now the set of substrategies for the defender. Let $M(a,b|t)$ be the expected payoff to the attacker if the attacker uses a, $T = t$, and the defender uses b, and let $p(t,n(a)) = P(T = t|$ attacker uses a). Our assumptions about the detection system imply that this probability depends only on $n(a)$; in fact, for $0 < p_2 < p_1 < 1$, T is the sum of two binomial random variables with parameters $n(a), p_1$ and $N-n(a)$, p_2 respectively. Hence

$$p(t,k) = \sum_{\ell=0}^{t} \binom{k}{\ell} \binom{N-k}{t-\ell} p_1^{\ell} (1-p_1)^{k-\ell} p_2^{t-\ell} (1-p_2)^{N-k-t+\ell} .$$

The payoff $M(a,f)$ is then given by

$$(6) \quad M(a,f) = \sum_{t=0}^{N} M(a,f(t)|t)\, p(t,n(a)).$$

For some fixed t and any $b \epsilon B$, let $\overline{b} = (b_1,\ldots,b_t,0,\ldots,0)$ and $\underline{b} = (0,\ldots,0,b_{t+1},\ldots,b_n)$, i.e., $\overline{b}$ is the portion of the substrategy b which is played against the zones with alarms, and $\underline{b}$ is the portion played against the game without alarms. Let $B_d = \{b| \sum_{j=1}^{t} b_j = d\}$, i.e., the set of substrategies which play d units against the zones with alarms, and let $b^d \epsilon B_d$ be the substrategy for which $b_j^d = m(\overline{b}^d)$ or $m(\overline{b}^d)-1$, $1 \leq j \leq t$ and $b_j^d = m(\underline{b}^d)$ or $m(\underline{b}^d) - 1$, $t + 1 \leq j \leq N$, i.e., the strategy which plays uniformly over each set of zones. These definitions all depend on t, but the particular t will be clear from the context, so no confusion will result if t is not indicated.

For any function δ from $\{0,1,\ldots,N\}$ to $\{0,1,\ldots,D\}$, i.e., any rule which assigns to each integer t between 0 and N an integer d between 0 and D, let $F_\delta = \{f\epsilon F | f(t)\epsilon B_{\delta(t)}\}$ be the set of strategies for the defender which play $\delta(t)$ defense units against the zones with alarms when $T = t$, and let f^δ be the strategy such that $f^\delta(t) = b^{\delta(t)}$. Let $A_k = \{a | n(a) = k\}$ as before. Theorem 3 states that the optimal strategy for the attacker is a mixture of strategies a^k and the optimal strategy for the defender is a mixture of strategies f^δ.

<u>Theorem 3</u>. <u>For any</u> a <u>and any</u> δ,

$$M(a,f^\delta) = \min_{f\epsilon F_\delta} M(a,f),$$

<u>and for any</u> k <u>and any</u> f^δ,

$$M(a^k,f^\delta) = \max_{a\epsilon A_k} M(a,f^\delta).$$

<u>Proof</u>. Let $L(a_i,b_j|t)$ be the attacker's expected losses from a_i and b_j if he uses a, $T = t$, and the defender uses b. Let $q(t,n(a))$ be the conditional probability that a specific zone containing at least one submarine gives an alarm, given that the attacker uses a and $T = t$. Routine calculations with conditional probabilities yield

$$q(t,k) = \frac{\sum_{\ell=1}^{t} \binom{k-1}{\ell-1}\binom{N-k}{t-\ell} p_1^{\ell}(1-p_1)^{k-\ell}\, p_2^{t-\ell}(1-p_2)^{N-k-t+\ell}}{p(t,k)}$$

For any t,d, $b \in B_d$, and $a \in A$,

$$L(a_i,b_j|t) = q(t,n(a))\,\frac{\min(a_i,b_j)}{t} \quad \text{if } 1 \leq j \leq t,$$

$$= [1-q(t,n(a))]\,\frac{\min(a_i,b_j)}{N-t} \quad \text{if } t+1 \leq j \leq N.$$

Summing $i = 1,\ldots,N$ and $j = 1,\ldots,t$, we obtain

$$(7) \quad L(a,\bar{b}|t) = q(t,n(a)) \sum_{i=1}^{N} \sum_{j=1}^{t} \frac{\min(a_i,b_j)}{t}.$$

If $n(a) \leq t$, then except for the factor $q(t,n(a))$, this is the expected loss to the attacker from the strategy which splits the submarines into the same groups as a, when the defender uses the strategy $(b_1,\ldots,b_t)$ in the game without detection, with t zones, d defense units, and S submarines. If $n(a) > t$, this interpretation is not possible, but the calculations are still the same. Hence by the argument used to prove (4),

$$L(a, \bar{b}^d|t) = \max_{b \in B_d} L(a,\bar{b}|t),$$

and by Lemma 1,

$$L(a^k,\bar{b}^d|t) = \min_{a \in A_k} L(a,\bar{b}^d|t).$$

Similar formulae hold for $L(a,\underline{b}|t)$, and hence

$$M(a,b^d|t) = \min_{b\in B_d} M(a,b|t),$$

and

$$M(a^k,b^d|t) = \max_{a\in A_k} M(a,b^d|t).$$

This, together with (6), completes the proof.

Thus, according to Theorem 3, for a given rule δ which tells the defender how many defense units to play against the zones that give alarms, he can minimize the attacker's expected payoff by deploying $\delta(t)$ units uniformly over the zones with alarms and the remaining $D - \delta(t)$ uniformly over the zones without alarms. If the attacker is going to spread his submarines among k zones, he can maximize his expected payoff by using strategy a^k. The optimal strategies will thus be random strategies which are a mixture of f^δ strategies for the defender and of a^k strategies for the attacker. The particular random strategies which should be used depend on p_1 and p_2, and we can make no general statement about them. If, however, the probability of detection is one, we obtain the following result.

Theorem 4. *If* $p_1 = 1$, *then*

$$\text{(8)} \quad v = \max_{1 \le k \le S} \sum_{t=k}^{N} \binom{N-k}{t-k} p_2^{t-k} (1-p_2)^{N-t} M(a^k, b^D | t),$$

where

$$\text{(9)} \quad M(a^k, b^D | t) = \max(S - \tfrac{Dk}{t},\ S-k+1 - \tfrac{D}{t},\ 0).$$

An optimal strategy for the attacker is a^K, *where* K *is that* k *which maximizes the right-hand side of (8). An optimal strategy for the defender is* $f^* \equiv b^D$, i.e., *play uniformly on those zones with alarms.*

Proof. Since $p_1 = 1$, $q(t,k) = 1$ for $t \ge k$, and $M(a, b^d | t)$ is a nonincreasing function of d, hence $M(a,b|t) \ge M(a,b^D|t)$ for all a, b, and t; and f^* is optimal for the defender.

Thus

$$\text{(10)} \qquad v = \max_{a \in A} M(a, f^*) = \max_{1 \le k \le s} M(a^k, f^*) ,$$

where the second equality follows from Theorem 3, and the optimal strategy for the defender is a^K where K is the value of k which maximizes (10). If $p_1 = 1$, $p(t,k) = 0$ for $t < k$ and $p(t,k) = \binom{N-k}{t-k} p_2^{t-k} (1 - p_2)^{N-t}$ for $t \ge k$; hence (10) is equivalent to (8). Equation (9) follows from Lemma 1. This completes the proof.

If the false alarm rate is zero, we obtain the following stronger result.

<u>Corollary</u>. <u>If</u> $p_1 = 1$ <u>and</u> $p_2 = 0$, <u>then</u>

$$K = [\sqrt{D}] \text{ or } [\sqrt{D}] + 1 \text{ and } v = \max(S-K+1-D/K, 0).$$

([ ] denotes integer part.)

<u>Proof</u>. If $p_1 = 1$ and $p_2 = 0$, then

$$M(a^k, f^*) = M(a^k, f^*|k) = \max(S-D, S-k+1-D/k, 0).$$

This result follows from treating k as a continuous variable and maximizing $M(a^k, f^*)$.

In other words, if the probability of detection is one, the defender can minimize the attacker's payoff by deploying all his defense units uniformly over the zones with alarms, while the attacker must spread his submarines over K zones in order to increase the number of alarms and cause the defender to spread his forces more thinly. The number of zones over which the attacker must spread (K) will increase as the false alarm probability decreases, and will achieve its maximum value when the false alarm probability is zero.

Even with an alarm system of the type postulated, the attacker still maximizes his payoff by bunching. He is very likely to lose the single submarines, and his payoff comes from one large group. He is forced to deploy the single submarines only to increase the defender's uncertainty concerning the location of the large group.

4. THE PAYOFF FUNCTION $M'(a,b)$ WITHOUT DETECTION

We now consider the game without detection when the payoff to the attacker is the number of zones from which at least one submarine successfully launches its missiles. The game is played in the manner described at the beginning of Sec. 2, but the expected payoff to the attacker is now given by

$$M'(a_i,b_j) \begin{cases} 1/N \text{ if } a_i > b_j, \\ 0 \text{ otherwise,} \end{cases}$$

$$M'(a,b) = \sum_{i=1}^{N} \sum_{j=1}^{N} M'(a_i,b_j).$$

We will analyze a generalized game which has the property that any pair of strategies in the submarine game with the payoff function $M'(a,b)$ corresponds to a pair of strategies in generalized game. We will solve the generalized game, and whenever there exist strategies in the submarine game which correspond to the optimal strategies in the generalized game, we will have a solution to the submarine game with payoff function $M'(a,b)$.

We define the generalized game as follows. A strategy for the attacker is a vector $x = (x_0,\ldots,x_S)$ with $x_i \geq 0$, $\sum_{i=0}^{S} x_i = 1$, and $\sum_{i=1}^{S} ix_i = \frac{S}{N}$. A strategy for the defender is a vector $y = (y_0,\ldots,y_D)$ with $y_j \geq 0$, $\sum_{j=0}^{D} y_j = 1$, and $\sum_{j=1}^{D} j\, y_j = D/N$. The payoff to the attacker is given by

$$(11)\quad M''(x,y) = \sum_{i=1}^{S} \sum_{j=0}^{i-1} x_i y_j = \sum_{j=0}^{D} \sum_{i=j+1}^{S} x_i y_j .$$

We may interpret the game in the following way. The attacker must choose a random variable X with values $0,1,\ldots,S$, and expected value S/N, such that $P(X = i) = x_i$. The defender must choose a random variable Y with values $0,1,\ldots,D$, and expected value D/N, such that $P(Y = j) = y_j$. The payoff is $M''(x,y) = P(X > Y)$.

Let $L = \min(S, [\frac{2D}{N} + 1])$. (As before, [ ] denotes integer part.) We consider only the case when

$$(12)\qquad D/N < S \leq \frac{N(L+1)}{2} .$$

The constraint (12) gives the range of S for which our solution is valid. As was previously remarked, if $S \leq D/N$, the game is trivial and the value of the game is zero. If $S > N(L+1)/2$, then x^* is not a probability distribution, so our solution is not valid. We have not studied the game in this case, since this corresponds to the case when the defender wishes to defend against a large number of submarines with a small number of defense units, and this does not appear to be of interest.

Let x^* be the strategy for the attacker given by

$$x_o^* = 1 - \frac{2S}{N(L+1)},$$

$$x_i^* = \frac{2S}{NL(L+1)}, \qquad 1 \leq i \leq L,$$

$$x_i^* = 0, \qquad L < i \leq S,$$

and let y^* be the strategy for the defender given by

$$y_j^* = (1 - \frac{D}{NL})\ (\frac{2}{L+1}), \qquad 0 \le j < L,$$

$$y_L^* = 1 - \frac{2(NL-D)}{N(L+1)},$$

$$y_j^* = 0, \qquad L < j \le D.$$

The constraint (12) and the definition of L insure that x^* and y^* are indeed strategies.

<u>Theorem 5</u>. <u>If</u> N,D, <u>and</u> S <u>satisfy (12), then the value of the generalized game is</u>

$$v'' = (1 - \frac{D}{NL})\ \frac{2S}{N(L+1)},$$

<u>and</u> x^* <u>and</u> y^* <u>are optimal</u>.

<u>Proof</u>. If $j \ge L$ then $L-j \le 0$, hence for any y

$$\begin{aligned} M''(x^*, y) &= \sum_{j=0}^{D} \sum_{i=j+1}^{S} x_i^* y_j \\ &= \frac{2S}{N(L+1)} \left(\sum_{j=0}^{L-1} y_j\ (\frac{L-j}{L}) \right) \\ &\ge \frac{2S}{N(L+1)} \left(\sum_{j=0}^{D} y_j\ (\frac{L-j}{L}) \right) \\ &= \frac{2S}{N(L+1)} \quad (1 - \frac{D}{LN}) \\ &= v''. \end{aligned}$$

If $L < i \leq S$, then $\frac{2i}{L+1}(1 - \frac{D}{LN}) \geq 1$. Hence for any x,

$$M''(x,y^*) = \sum_{i=1}^{S} \sum_{j=0}^{i-1} x_i y_j^*$$

$$= \sum_{i=1}^{L} x_i \frac{2i}{L+1}(1 - \frac{D}{NL}) + \sum_{i=L+1}^{S} x_i$$

$$\leq (1 - \frac{D}{NL}) \sum_{i=1}^{S} \frac{2i}{L+1} x_i$$

$$= (1 - \frac{D}{NL}) \frac{2S}{N(L+1)}$$

$$= v''.$$

Thus for any x and y,

$$M''(x,y^*) \leq v'' \leq M''(x^*,y) ,$$

which proves the theorem.

In the submarine game, a random strategy α for the attacker is a convex combination of elements of A, i.e.,

$\alpha = \sum_{\nu=1}^{m} \lambda_\nu a_\nu$, for some m, where $\lambda_\nu \geq 0$, $a_\nu \epsilon A$ for $1 \leq \nu \leq m$,

and $\sum_{\nu=1}^{m} \lambda_\nu = 1$. The interpretation of α is that the attacker uses a_ν with probability λ_ν. Corresponding to α is a probability distribution $x(\alpha)$ which is a strategy in the generalized game, such that for $0 \leq i \leq S$, $x_i(\alpha)$ is the probability that a given zone contains i submarines. Similarly, a random strategy β for the defender is a convex combination of elements of B. Corresponding to β is a strategy $y(\beta)$ in the generalized game, such that for $0 \leq j \leq D$, $y_j(\beta)$ is the probability that a given zone contains j defense units. $M'(\alpha,\beta)$ is the expected number of zones from which at least one submarine successfully fires if the attacker uses α and the defender uses β, hence $M'(\alpha,\beta) = N M''(x,(\alpha),y(\beta))$. Thus we have the following corollary to Theorem 5.

Corollary. *If the attacker has a random strategy* α^* *such that* $x(\alpha^*) = x^*$, *and the defender has a random strategy* β^* *such that* $y(\beta^*) = y^*$, *then* α^* *and* β^* *are optimal and the value of the submarine game with payoff function* $M'(a,b)$ *is*

$$v' = (1 - \frac{D}{NL}) (\frac{2S}{L+1}).$$

If $L = S$, the desired strategy β^* for the defender may not exist. For example, if $N = 3$, $S = 2$, $D = 5$, then $L = 2$ and $y^* = (1/9, 1/9, 7/9, 0, 0, 0,)$. However, the only strategy b for the defender which never puts more than two units in a single zone is the one which places two units in each of two zones and one unit in the third zone, with the resulting probability distribution $y(b) = (0,1/3,2/3, 0,0,0)$. We conjecture that the attacker always has a strategy α^*, and that the defender has a strategy β^* whenever $L = [\frac{2D}{N} + 1]$. The calculation of these strategies depends on arithmetic properties of D, S, and N, and we have been unable to prove the conjecture. However, in the range of parameters which would seem to be of greatest interest for the problem at hand, i.e., $D/2 \leq S \leq N$, the strategies α^* and β^* seem fairly easy to compute.

Thus, when the payoff to the attacker is the number of zones from which at least one submarine successfully fires, there exists a number $L = \min(S, [\frac{2D}{N} + 1])$, such that when (12) is satisfied, optimal behavior may be described as follows. The attacker should play so that with probability $2S/NL(L+1)$, any given zone will contain $1, 2, \ldots,$ or L submarines, and with the remaining probability, the zone will contain no submarines. The defender should play so that with probability $(1 - \frac{D}{NL})\ (\frac{2}{L+1})$ any given zone will contain $0, 1, \ldots,$ or $L-1$ submarines, and with the remaining probability the zone will contain L submarines. If $L = S$, it may not be possible for the defender to do this. We conjecture, however, that the attacker can always play according to this strategy, and that the defender can whenever $L = [\frac{2D}{N} + 1]$.

5. DISCUSSION OF THE RESULTS

With no detection system, the value of the game with the payoff $M(a,b)$ to the attacker is high unless D is much greater than S; and an effective defensive system would be relatively costly, since N defense units are required for complete defense against a single submarine. Furthermore, when $S \geq D/N$, the attacker may increase the size of his submarine force with no increase in his expected losses. In this sense, the game is highly favorable to the attacker. On the other hand, the optimal strategy a^* for the attacker is unique, while Theorem 2 states that anything the defender does which does not purposely waste defensive units by putting more than S in a single zone will be optimal. Thus if, instead of successful launches, we choose any other payoff function which does not require the defender to waste units, he can optimize with respect to this payoff function and still maximize the attacker's losses, while optimal play with respect to the new payoff function by the attacker will, in general, increase his expected losses. For example, if we use the payoff function $M'(a,b)$, then in the case when $S < N$, $2D < N$, we have $L = 1$ and a^S and b^* are optimal, while $M(a^S,b^*) = S(1 - D/N) \leq M(a,b)$ for all a,b, so that from the standpoint of the payoff function $M(a,b)$, a^S is the worst possible strategy for the attacker.

With even a simple detection system, it does not seem possible to obtain a general solution to the game. The analysis does show however, that while a detection system of the type postulated forces the attacker to deploy some single submarines as decoys to increase the defender's uncertainty concerning the location of his main group, he still must bunch the remainder in order to maximize his payoff. Even with a detection probability of one and no false alarms, there is still a point ($S = 2\sqrt{D}$) such that for fixed N and D, any further increase in submarine fleet size will not increase his expected losses.

The assumption that the detection probability is independent of the number of submarines present is clearly unrealistic whenever $p_1 < 1$.

This analysis ignores completely the fact that the submarine defense problem takes place over a period of time, and considers the problem as a static game. This seems a reasonable simplification—at least, in the game without detection—if the purpose of the system is to deter or defend against a single mass attack, because then the only time of interest is the time of attack. In the game with detection, the amount of information provided would presumably be a function of time. The model would then be affected accordingly.

REFERENCES

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