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LINEARIZATION BASED UPON DIFFERENTIAL APPROXIMATION AND GALERKIN'S METHOD

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PREFACE

As part of its Project RAND research program, RAND engages in basic supporting studies in mathematics. The present Memorandum investigates the application of a new linearization technique based on differential approximation to ordinary differential equations of deterministic type.

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SUMMARY

We present a new linearization technique based upon differential approximation which is considerably simpler and more flexible than those used in our previous work.

CONTENTS

LINEARIZATION BASED UPON DIFFERENTIAL APPROXIMATION AND GALERKIN'S METHOD

1. INTRODUCTION

In some previous publications (see [1] for further references), we have applied the technique of differential approximation to obtain linear functional equations for approximate solutions to linear and nonlinear equations. In some cases, we are trying to avoid nonlinearity; in some cases, to eliminate random effects; in other cases, to reduce the dimension of a system to manageable terms. In this note, we wish to discuss some further aspects of differential approximation and to compare it with Galerkin's method for treating nonlinearity. We shall restrict our attention to ordinary differential equations of deterministic type.

To illustrate the first approach, consider the nonlinear equation

$$u'' + g(u) = 0, \quad u(0) = c_1, \quad u'(0) = c_2. \tag{1.1}$$

We wish to obtain an approximate solution v via a linear equation of the form

$$v'' + av' + bv = 0. \tag{1.2}$$

One way to do this is to use an approximation of the form

$$g(u) \cong au + b, \tag{1.3}$$

or perhaps $g(u) \cong au + bu' + c$. Let us consider the

simpler form for the purposes of illustration. How do we determine the parameters a and b? If u(t) may be considered to be known, either analytically or numerically, a and b can be conveniently obtained by means of the minimization of the functional

$$(1.4) \qquad J(u) = \int_0^T (g(u) - au - b)^2 dt.$$

If u(t) may not be considered to be known, in the sense that it is the function whose approximate form we are seeking, there are, nevertheless, several feasible approaches to the determination of a and b. One is a self-consistent technique. Referring to (1.2), consider v, the solution, as a function of a, b, and t, $v = v(t,a,b)$. In place of minimizing J(u) over a and b, let us minimize the function

$$(1.5) \qquad K(a,b) = \int_0^T (g(v(t,a,b)) - av(t,a,b) - b)^2 dt.$$

This procedure often leads to a complicated set of simultaneous transcendental equations.

It is not necessary, however, to proceed along this direct route. An alternate, and occasionally more felicitous, path is the following. Let $u_0(t)$ denote an approximate solution to (1.1) obtained in some other fashion. Let a_0 and b_0 then be determined as the values minimizing

$$(1.6) \qquad J(u_0) = \int_0^T (g(u_0) - au_0 - b)^2 dt.$$

Let u_1 be determined as the solution of

$$(1.7) \qquad u_1'' + a_0 u_1' + b_0 = 0, \quad u_1(0) = c_1, \quad u_1' = c_2,$$

then (a_1,b_1) by the minimization of $J(u_1)$, and so on. We expect a_n,b_n and u_n to converge as $n \to \infty$ to the quantities determined by the direct self-consistent technique. There has, however, been little effort devoted to the investigation of these matters.

2. HIGHER ORDER APPROXIMATIONS

Although the foregoing procedure yields excellent results in a number of cases, there remains the challenging question of obtaining improved estimates. Furthermore, it is desirable to establish the existence of an algorithm which will yield a sequence of functions $\{u_n\}$, each satisfying a linear differential equation of order d_n (with $d_n \to \infty$ as $n \to \infty$), with the property that u_n converges to u as $n \to \infty$.

One approach, sketched previously, is the following. Suppose that $g(u) = u + u^3$. Then

$$(2.1) \qquad (u^3)' = 3u^2u',$$

$$(u^3)'' = 3u^2u'' + 6uu'^2$$

$$= -3u^3 + 6uu'^2 - 3u^5.$$

We obtain corresponding expressions for the variables u^2u', $u(u')^2$, $(u')^3$. Hence, if we introduce the notation

$$(2.2) \qquad w_1 = u^3,\ w_2 = u^2u',\ w_3 = u(u')^2,\ w_4 = (u')^3,$$

we see that we have a system of simultaneous equations

$$(2.3) \qquad \frac{dw_i}{dt} = \sum_{j=1}^{4} a_{ij}w_j + g_i(u,u'), \quad i = 1,2,3,4.$$

The original equation for u has the form

$$(2.4) \qquad u'' + u + w_1 = 0.$$

Write $g(u,u')$ for the vector with components $g_i(u,u')$, $i = 1,2,3,4$. Proceeding by analogy with the foregoing, we look for an approximation of the type

$$(2.5) \qquad g(u,u') \cong B\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ u \\ u' \end{pmatrix},$$

where B is a constant 6×4 matrix. The matrix B can, in principle, be determined using one of the procedures described above.

3. DISCUSSION

There are several drawbacks to the method described in the foregoing section. In the first place, the dimension of the approximating system increases at a

disagreeable rate as the order of the approximation increases. The number of parameters in the linear system increases roughly as the square of the dimension. This leads to serious computational difficulties if we try to apply these techniques to a system of equations or to a high order nonlinear differential equation.

Secondly, this approach depends upon the ability to differentiate $g(u)$ repeatedly. This would prevent us from treating functions such as $|u|^{3/2}$.

A third disadvantage will be mentioned below in Sec. 6.

4. AN ALTERNATE LINEARIZATION

With the foregoing comments in mind, let us introduce an alternate approach. Write

$$(4.1) \qquad g(u) \cong au + b + \int_0^t k(t-s)u(s)ds,$$

where the parameters a and b and the function $k(t)$ are now to be chosen in a convenient fashion. If t ranges over $(0,\infty)$, it is often possible to choose $k(t)$, dependent upon u of course, so that (4.1) is an equality. Using the Laplace transform,

$$(4.2) \qquad \frac{L(g(u) - au - b)}{L(u)} = L(k).$$

This encourages us in the belief that (4.1) is a useful approximation. In general, we are interested in

$0 \leq t \leq T < \infty$. Consequently, let us choose $k(t)$ to have some simple analytic form which facilitates the minimization of

$$(4.3) \qquad \int_0^T \Big(g(u) - au - b - \int_0^t k(t-s)u(s)ds\Big)^2 dt.$$

Suppose, for example, we choose

$$(4.4) \qquad k(t) = \sum_{i=1}^{N} c_i e^{\lambda_i t}.$$

An important advantage of this representation is that the equation

$$(4.5) \qquad u'' + au + b + \int_0^t k(t-s)u(s)ds = 0$$

is equivalent to a linear differential equation of degree $N + 2$. To avoid the usual difficulties of exponential approximation, we resort to differential approximation and take $k(t)$ to satisfy a linear differential equation

$$(4.6) \qquad k^{(N)} + a_1 k^{(N-1)} + \cdots + a_N k = 0,$$

$$k^{(i)}(0) = b_i, \quad i = 0,1,\ldots,N-1,$$

where a, b, a_i, and b_i are to be chosen so as to minimize

$$(4.7) \qquad \int_0^T \left(g(u) - au - b - \int_0^t k(t-s)u(s)ds\right)^2 dt.$$

We can expect this procedure to be quite effective [2].

5. EXISTENCE OF A MINIMUM

It is worth noting that it is not immediate that the minimum value of the expression in (4.7) exists. Consider, for example, the problem of minimizing

$$(5.1) \qquad \int_0^T (u-b)^2 dt$$

over a and c where $u' = au$, $u(0) = c$. If we fix $c > 0$ and take $b < 0$, then although the infinum is $\int_0^T b^2 dt$, no finite value of a yields this value. On the other hand, if we allow a and c to vary, then the minimum is attained. These considerations, of course, are important if we are interested in an actual determination of the minimum via computer.

In a separate paper, general existence and uniqueness results for problems of differential approximation will be presented.

6. AVOIDANCE OF ANHARMONICS

In Sec. 3, we noted that the method sketched in Sec. 2 had a third disadvantage associated. To describe this, let us return to the nonlinear oscillator, $u'' + u + \epsilon u^3 = 0$.

The approximation $u^3 \cong au$, with a determined as indicated previously, yields the usual first order perturbation expansion if $\epsilon \ll 1$. If we attempt to find a second order approximation, we encounter the annoying fact that the optimal B found via (2.5) yields a linear system with anharmonic frequencies.

To avoid this, we could impose an initial constraint on the set of B's under consideration, to the effect that we admit only those matrices for which the resultant linear systems have harmonic frequencies. As might be surmised, even in this simple case, the constraint cannot be simply enforced.

On the other hand, it is quite easy to require this for the system resulting from (4.5) combined with (4.6). In subsequent papers, we will discuss this point, as well as more general functional equations and stochastic effects.

7. GALERKIN'S METHOD

Let us now consider an alternate approach to linearization. In Sec. 1, we outlined a procedure for finding first the approximating linear equation, and then the approximating solution. We can, if we wish, reverse the process in the sense of first finding an approximate solution and then deducing an associated linear equation.

To illustrate this technique, return to the equation of the nonlinear oscillator $u'' + u + \epsilon u^3 = 0$, $u(0) = c$,

$u'(0) = 0$. Let us look for a solution of the form

(7.1) $$u = c \cos \omega t,$$

and determine ω by the condition that

(7.2) $$\lim_{T\to\infty} \frac{1}{T} \int_0^T (-c\omega^2 \cos \omega t + c \cos \omega t + \epsilon c^3 \cos^3 \omega t)^2 dt$$

is a minimum. This gives a correction in the frequency if $\epsilon \ll 1$.

On the other hand, if we start with the Van der Pol equation

(7.3) $$u'' + \epsilon(u^2 - 1)u' + u = 0,$$

with $0 \leq \epsilon \ll 1$, and ask for an approximate solution of the form $u \cong a \cos t$, the same minimization procedure yields the amplitude of the approximating periodic solution.

To apply this method successfully, it is clear that some a priori information concerning the nature of the approximate solution is required.

8. RIGOROUS ASPECTS

Starting with the equation

(8.1) $$u'' + g(u,u') = 0,$$

the foregoing method yields a new equation of the form

$$v'' = g(v,v') + \Delta(t), \tag{8.2}$$

where

$$\|\Delta\|^2 = \lim_{T\to\infty} \frac{1}{T} \int_0^T \Delta(t)^2 dt \tag{8.3}$$

is small. Actually, in the foregoing cases, and in general, only behavior over a finite t—interval is required since we are looking for periodic or almost—periodic solutions.

Is it true that $u \cong v$? This is a question with the sphere of classical stability theory, as we see upon setting $v = u + w$, $u(0) = v(0)$, $u'(0) = v'(0)$. The function w satisfies the equation

$$\begin{aligned} w'' &= g(u + w,u' + w') - u'' \\ &= g(u + w,u' + w') - g(u,u') + \Delta(t). \end{aligned} \tag{8.4}$$

Suppose that $0 \le t \le t_0$. It follows that if $\|\Delta\|$ is sufficiently small, then $|w|$ will be of the same order of magnitude over $[0,t_0]$. What is interesting about this is that it shows that a heuristic technique can be made rigorous if it is successful enough.

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