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A SIMPLIFIED MODEL OF AIRCRAFT SORTIE GENERATION CAPABILITY

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PREFACE

This Memorandum evaluates the direct effects that aircraft reliability-maintainability characteristics have on achieving maximum flying hour programs, by evaluating the sortie generation capability of an aircraft when it is subjected to a maximum-level daytime flying program. The model makes simplifying assumptions about the flying schedule and maintenance resources. It provides a meaningful point of departure in developing planning factors for the employment of a particular weapon system, and can be used as a starting point in the analysis of aircraft performance in more complex operations-maintenance environments. Finally, it provides information that can be used in designing more complex operations-maintenance models.

The Memorandum should be of interest to personnel concerned with performance measures of aircraft, with maintenance-operations interfaces, and with the modeling of operations-maintenance systems.
SUMMARY

First, the upper theoretical limit of mean flying hours per aircraft per flying day provides a meaningful point of departure in developing planning factors for the employment of a particular weapon system.

This Memorandum analyzes the steady-state performance of an aircraft subjected to a maximum daytime flying effort under the condition of unlimited maintenance resources. By this we mean that the aircraft shall be allowed to fly only during the day, not the night, and that it shall fly whenever possible during the day. The analysis is steady-state in the sense that the aircraft is presumed to have been flying this schedule for enough days to allow any initial transient effects to die out. Also, it is assumed that delays in maintenance due to lack of resources never occur.

This analytic model has several uses. First, daytime sortie generation capability is one measure (among many) of the performance of an aircraft. Under the assumption that maintenance resources never become binding, the daytime sortie generation capability of a squadron of a aircraft is routinely determined from that of a single aircraft. Second, the model determines the load on maintenance under the condition of maximum continued daytime stress; this requirement is useful for planning purposes. Third, since a particular run with the model takes only a few seconds, the model can readily and economically be used to determine the sensitivity of the performance of the aircraft to variations in several parameters. This information is useful, for example, to the designer of much more complex operations-maintenance
models, as it helps him decide what to include and what to exclude from such models.

The Memorandum also reports on the results of some 200 runs with the model. One group of runs measures the sensitivity of the performance of the aircraft to the description of the maintenance function. Another group of runs measures the sortie generation capability as a function of the length of the flying day, the length of the flight and the mean turnaround time.
ACKNOWLEDGMENT

L. W. Miller of The RAND Corporation contributed some of the ideas underlying the analysis in Sec. III.
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I. INTRODUCTION

The following is an analysis of the steady-state performance of a single aircraft subjected to a maximum daytime flying effort under the condition of unlimited maintenance resources. By this we mean that the aircraft shall only be allowed to fly during the day, not the night, and that it shall fly whenever possible during the day. The analysis is steady-state in the sense that the aircraft is presumed to have been flying this schedule for enough days to allow any initial transient effects to die out. Also, it is assumed that delays in maintenance due to lack of resources never occur.

Elementary probability theory yields a set of linear equations that describe the performance of the aircraft. A program to solve these equations has been coded in FORTRAN IV. The equations, the program, and the results of computer runs are furnished below. Enough information is provided to enable the interested user to make runs of his own with the program. The model is analytic, rather than using simulation.*

The model has several uses. First, the daytime sortie generation capability is one measure (among many) of the performance of an aircraft. Under the assumption that maintenance resources never become binding, the daytime sortie generation capability of a squadron of n aircraft is routinely determined from that of a single aircraft.

Second, the model determines the load on maintenance under the condition of maximum continued daytime stress; this requirement is useful for planning purposes. Third, since a particular run with the model takes only a few seconds, the model can readily and economically be used to determine the sensitivity of the performance of the aircraft to variations in several parameters. This sensitivity information is useful in itself and relevant to the design of much more complex operations-maintenance models. It helps the designer decide what to include in and what to exclude from the more complex models -- judgments that affect the complexity, data requirements, and running speed of these operation-maintenance models.

The results of some 200 runs with the model are contained in Sec. II. As well as illustrating the uses of the model, Sec. II deals with two issues. First, it considers the sensitivity of the system performance to the description of the maintenance function and concludes that simplified models of the maintenance function are justified in many cases. Second, it measures the sortie generation capability as a function of three variables, presenting the results in the form of two functions which approximate the sortie generation capability.

Section III contains the mathematics which underlies the model and can be omitted by those who do not wish to check the analysis. The computer program constitutes an appendix. The remaining paragraphs of this section describe the model.

During the day, the aircraft moves among three states: preflight inspection, flight, and repair. During the night, it may also be waiting. Although the way in which these states are used is somewhat
flexible, we shall adopt the following for the sake of being specific. The "preflight inspection" state includes uploading. The "flight" state includes time in the air and in any postflight inspection that occurs immediately upon landing. The "repair" state includes whatever repair activity results from flights or from preflight inspections.

The evolution of the aircraft through the states is perhaps best explained by reference to Fig. 1. The aircraft is in the "wait" state only if it does not require repair (i.e., if it is operational) and, furthermore, if another takeoff is not possible during the day.

![Diagram](image)

Fig. 1 -- Maximum daytime flying model

If it enters the wait state, it emerges at the beginning of the next flying day and immediately enters preflight inspection. Preflight
inspection ends either with a takeoff or with the beginning of a repair activity. Similarly, repair may or may not be required when the aircraft lands. If either a preflight inspection or a flight generates a requirement for maintenance, repair activity commences immediately and continues until the aircraft becomes operational. Operational aircraft immediately enter preflight inspection if there is time for another takeoff or wait until the beginning of the next day if there is not.

We note in passing that mathematical interest in the model stems primarily from the fact that operational aircraft wait at night, since otherwise the model is much more readily analyzed.

In order to replace integration by summation in the computations, we break the calendar day up into a number of time intervals of equal length and allow events to occur only at the beginning of each time interval. This discretization effects a substantial computational saving and does not materially affect the results, providing the time intervals are small enough.

We are now in a position to specify the data required to make a run with the model. The symbols introduced here are used in Sec. II for describing the runs we have made, in Sec. III for developing the equations that specify the model's performance, and in the computer program. Some of the required data are:

\[ N = \text{the number of time intervals in the 24-hour calendar day} \]
\[ M = \text{the number of time points at which preflight inspections can commence} \]
a = the length of the preflight inspection, measured in time intervals
b = the length of the flight, measured in time intervals
p = the probability that a takeoff occurs at the end of a preflight inspection
q = the probability that the aircraft is operational when it lands
J = a truncation constant.

For an example, suppose the calendar day is broken up into \( \frac{1}{2} \)-hour intervals, that the flying day is 14 hours, that flights are 1\( \frac{1}{2} \) hours long, that preflight activity (uploading) requires \( \frac{1}{4} \) hour, that malfunctions are never discovered during preflight, and that the aircraft always requires repair upon landing. Then one would have \( N = 96, M = 56 \) (or 4.14), \( a = 2, b = 6, p = 1, \) and \( q = 0. \)

The repair time distribution is now specified. If one wishes to use an empirical repair time distribution, he must enter

\[
r(n) = \text{the probability that repair takes more than } n - 1 \text{ but not more than } n \text{ time periods, for } n = 1, 2, \ldots, JN.
\]

The integer \( J \) is a truncation constant; repair time in excess of \( J \) days is impossible. Then, one has \( \sum_{n=1}^{JN} r(n) = 1. \) In lieu of entering an empirical repair time distribution, one might wish to use one of the standard functional forms for the repair time distribution. Provision has been incorporated for using the exponential distribution, the lognormal distribution, or the sum of two exponentials (with different means) for the repair time. Using these, one need specify only the
parameters of the distribution, namely

$$EZ = \text{the mean of the repair time distribution (in hours)}$$

and, for the latter two distributions,

$$RZ = \text{the ratio of the standard deviation of the repair time distribution to the mean repair time.}$$

The particular repair time distribution used in each run is determined by setting an indicator. A potential user can readily add any other distribution of his own choosing by incorporating a routine that calculates its cumulative distribution function and defining another setting for the indicator.

Several variants on the model make sense economically and can be analyzed using similar techniques. First, both the repair time distribution and the probability $(1 - q)$ that repair is necessary following a flight could depend on the time of day at which the flight lands. This would model the situation in which maintenance is deferred. Second, the repair time distribution following preflight inspection could differ from the repair time distribution following flight. Also, the length of the preflight inspection and of the flight might be random variables and might also depend on the time of day. Each of these situations could be modeled with a minor variation on the computer program.

With a major revision of the computer program, one could deal in an approximate manner with the case in which maintenance resources are limited and sometimes become binding.
II. RESULTS OF SOME COMPUTER RUNS

This section illustrates three uses of the model. For the first of these, Fig. 2 displays the information obtained from a particular run of the model. The second discussion depicts the influence of the maintenance function on system performance. The third part of this section examines the joint influence of flight length, mean turnaround time and length of flying day upon the sortie generation capability.

DATA FROM A SINGLE COMPUTER RUN

Fig. 2 contains the performance of the aircraft under the following conditions: the day is segmented into 15-minute intervals ($N = 96$), flights are 1.5 hours long ($b = 6$), the take-off interval is 14 hours long ($M = 56$), there are no preflight inspections ($a = 0$ and $p = 1$), the repair time distribution is truncated at 6 days ($J = 6$), and the repair time is lognormally distributed ($K_{ind} = 2$) with mean of 6 hours ($EZ = 6$) and variance of 36 hours ($RZ = 1$). This figure depicts a possible level of performance of certain fighter aircraft and indicates the type of information yielded by a single 10-second computer run.

For concreteness, we give a formula for the lognormal distribution. Suppose $Z$ is a random variable which is distributed according to the lognormal distribution with mean $EZ$ and variance $Var Z$. With $RZ$ denoting the ratio of the standard deviation of $Z$ to its mean, i.e., with $RZ = \sqrt{Var Z / EZ}$, Appendix 2 verifies that

$$P(Z < t) = \int_{-\infty}^{\frac{X - m}{\sigma \sqrt{2\pi}}} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(X - m)^2}{2\sigma^2}} dx$$
with

\[ \sigma^2 = \ln[1 + RZ^2] \]

(3) \[ m = \ln(EZ) - \sigma^2/2. \]

Time of day is measured on the abscissa of Fig. 2. Various probabilities are measured on the ordinate. A logarithmic scale is selected for the ordinate mainly as a matter of convenience; it encompasses three orders of magnitude on the same graph.

Reading from Fig. 2, the probability that the aircraft is in repair at the beginning of the day is .161. As time increases, this number varies sharply and settles down to about .801 at the beginning of the 7th hour. It falls off approximately exponentially from the 16th through the 24th hour. (In this range it looks linear on a logarithmic scale and is therefore exponential.) Similarly, the probability of takeoff is .839 at the beginning of the flying day and settles down to about .033 by the 7th hour. The four remaining curves represent the probability that a takeoff is the 1st, 2d, 3d, or 4th flight of the day.

The mean time between takeoffs is 7-5/8 hours -- 6 hours mean repair plus 1-1/2 hours in flight plus a 1/8-hour average wait between end of repair and beginning of the next time period. After the start-of-day effect has died out, the probability of takeoff at a particular time point can be expected to settle down to \( 1/(4 \times 7.625) = .0328 \). The system settles down to this steady-state behavior after only 7 hours, which is rather quickly.

As noted in Fig. 2, the expected number of takeoffs during the day is 2.444 and the variance in the number of takeoffs is .940. The
Fig. 2 -- Takeoff and repair probabilities

Flight Probabilities
EF = 2.444
VarF = .940
F(0) = .027
F(1) = .128
F(2) = .347
F(3) = .374
F(4) = .116
F(5) = .007
probability that exactly \( n \) takeoffs occur is \( F(n) \), also given in Fig. 2.

**SENSITIVITY OF PERFORMANCE TO THE MAINTENANCE MODEL**

Figures 3 and 4 contain results of 78 computer runs and constitute attempts to deal with the following general issue: to what extent does the system performance depend on the description of the maintenance function? For instance, does the sortie generation capability depend significantly on anything other than the mean turnaround time? This question is important for someone who is attempting to measure sortie generation capability of a given aircraft in a particular environment. The issue also bears on the effects of excluding detail from more complex operations-maintenance models.

We distinguish between turnaround time distribution and repair time distribution: the turnaround time distribution is the probability distribution governing the interval between when the aircraft lands and when it becomes operational. The turnaround time distribution will differ from the repair time distribution (defined in Sec. I) in the case when \( q > 0 \), i.e., when there is a positive probability that the aircraft will not require maintenance upon landing.

Toward dealing with the issue raised above, several computer runs were made with each of four different maintenance models. In model i), repair time is lognormally distributed and repair is always required. In model ii), repair time is again lognormally distributed (this time with \( RZ = 1 \)) but there is a positive probability that repair will not be required, i.e., \( q > 0 \). Model iii) is like model i), except that repair time is distributed as the sum of two exponential
random variables with different means. Model iv) is like model ii), except that repair time is exponential. Formulas for these four models are presented later in this subsection.

These four maintenance models generate widely different turnaround time distributions. The lognormal distribution has a bell-shaped density function, while the exponential and sum of exponentials have decreasing density functions. The turnaround distributions in models ii) and iv) have jumps at the origin.

Let EF denote the expectation of the number of sorties generated per day, let ET be the expected turnaround time, and let RT denote the ratio of the standard deviation in turnaround time to the expected turnaround time. RT is a dimensionless measure of the spread in the turnaround distribution.

The various maintenance models can be compared in many ways. Using only model i), one might fix the mean turnaround time ET and observe the fluctuation of EF as a function of RT. The three solid curves in Fig. 3 do this for mean turnaround times of 3, 6, and 10 hours, respectively. Two of these curves fluctuate considerably (20% and in opposite directions) as RT is increased from 0 to .5. Note that RT = 0 corresponds to a situation with no random elements and RT = .5 represents a turnaround distribution which is still very heavily concentrated about its mean. Then, this fluctuation roughly represents the effect of introducing any randomness at all; this effect is significant, and in most cases deterministic reductions of the maintenance function are inappropriate. The range of RT from .75 to 1.25 constitutes a fairly broad spectrum of repair time distributions, since RZ below
Fig. 3 -- Sensitivity of expected number of takeoffs, $E_{FR}$, to distribution shape for mean turnaround times of 3, 6, and 10 hours.
(above) this range corresponds to a highly concentrated (dispersed) distribution. Again following the solid curves, we see that as RT ranges from .75 to 1.25, the sortie generation capability EF varies 2.8%, 2.8% and 2.1% for ET = 3, 6, and 10 hours, respectively.

In many problem contexts, a considerable simplification results from assuming that the repair phenomenon obeys an exponential distribution; this allows one to exploit the memoryless property of that distribution. Comparing the exponential with the lognormal having the same variance (RT = 1.), the exponential results in an increase in EF of .6%, 1.3%, and 1.1% for ET = 3, 6, and 10 hours, respectively. These increases might seem surprisingly low to one who notes the wide difference between the bell-shaped lognormal density function and the decreasing (constant failure rate) exponential density function.

Cases iii) and iv) carry this analogy between the exponential and lognormal repair time models to the case $0 \leq q \leq .5$, i.e., up to 50% of the time the aircraft is operational upon landing. (The case $q = .5$ corresponds to RT = 1.732, as solving Eq. 7b for RT would verify.) Referring to Fig. 3, curves iii) and iv) lie close together throughout this range of q; maximum differences are 1.2%, 1.3% and 1.1% for ET = 3, 6, and 10 hours, respectively. Similarly, performance under the sum of two exponentials (curve iii)) very closely resembles performance under the lognormal throughout the range in which the former is defined, i.e., for $.75 \leq RT \leq 1.0$.

Turning to grosser approximations, one could choose to match only the mean turnaround time (not q or the variance) and use an exponential turnaround model. For RT in the range from .5 to 1.25, the maximum
errors are 3.5%, 4% and 2.5% for ET = 3, 6, and 10 hours, respectively. Increasing the upper bound on RT to 1.5 increases these maximum errors to 5.8%, 5.3% and 4.2% for ET = 3, 6, and 10 hours, respectively. Comparing these percentages with the large ranges of variance in the turnaround distribution which they reflect, one sees that sensitivity of performance to the variance in turnaround time is quite low.

Turning to Fig. 4, the same types of things are plotted, except that the ordinate, EF, of Fig. 3 is replaced by the ratio, RF, of the standard deviation in the number of takeoffs to the expected number of takeoffs. RF is then a (dimensionless) measure of the variation in the sortie generation capability. The curves in Fig. 4 are not so tightly bunched as in Fig. 3, i.e., RF depends more strongly than EF on the maintenance model. Also, RF increases more or less linearly with RT. If one is interested in measuring the variability of the performance level, he need be more careful in his description of the maintenance model.

Note that the lognormal curves lie significantly below the exponential curves. The stability of the sample mean of a stochastic simulation decreases as the variance increases. An individual interested in reliable estimates of average performance of simulations might be well advised to a) underestimate the process's variance (since we have just argued that performance is insensitive to variance) and b) prefer lognormal models to exponential models.

To complete this discussion, we present formulas for the parameters of the four maintenance models in terms of ET and RT. For case i), the turnaround distribution is obviously given by Eqs. (1)-(3),
Fig. 4 -- Sensitivity of the spread in the number of takeoffs, $\frac{RF}{\bar{T}}$, to the distribution shape for mean turnaround times of 3, 6, and 10 hours.
with ET and RT replacing EZ and RZ. For case iii), routine algebraic manipulations would verify that the turnaround distribution is given by

\[ P(T < t) = 1 - e^{-t/\mu} \mu/(\mu - \nu) - e^{-t/\nu} \nu/(\nu - \mu) \]

with

\[ \mu = \left[ 1 + \sqrt{2(RT)^2 - 1} \right] ET/2 \]

\[ \nu = \left[ 1 - \sqrt{2(RT)^2 - 1} \right] ET/2. \]

In case ii), the probability distribution of the turnaround time is given by

\[ P(T < t) = q + (1 - q)P(Z < t) \]

with \( P(Z < t) \) as defined by Eq. (1). Similarly, for case iv), the distribution of turnaround time is as given by Eq. (7), but with

\[ P(Z < t) = 1 - e^{-t/\lambda}. \]

In case ii) and in case iv), the probability \( q \) that no failure occurs upon landing is given in terms of RT by

\[ q = \frac{(RT)^2 - 1}{(RT)^2 + 1} \]

and the mean repair time EZ is determined by

\[ EZ = ET/(1 - q). \]
FACTORS AFFECTING SORTIE GENERATION CAPABILITY

The preceding subsection indicates that a simplified view of the maintenance function suffices in many cases for an approximate determination of the sortie generation capability. In this subsection, we assume a particular simple maintenance model and measure the effect of three factors on sortie generation capability. The maintenance model is model 1) of the previous subsection with RZ = 1; i.e., the turnaround time is lognormal with standard deviation equal to the mean. The three variables considered here are the mean turnaround time, the length of the flying day and the length of the flight.

The model was run with all combinations of 8 mean turnaround times (from 3 to 10 hours in one hour increments), 4 lengths of the flying day (9, 12, 15, and 18 hours) and 4 lengths of flight (1.5, 2, 2.5, and 3 hours), for a total of 128 cases. We could present the results of these runs in tabular form. Rather than doing this, we made least squares fits of several functions to the data and present two of these functions. Let $x_1$ denote the length of the flying day in hours, $x_2$ denote the mean turnaround time in hours, and $x_3$ denote the length of the flight in hours. Then one has $x_1 = 24M/N$, $x_2 = EZ$, and $x_3 = 24b/N$.

Fitting a linear function to the data, we obtain the estimate $f(x)$ of EF given by

$$f(x) = 2.758 + 0.1029x_1 - 0.2116x_2 - 0.2551x_3,$$

which explains 91.2% of the variance in the data. A somewhat better fit is obtained by the more complex function
\[ f(x) = 2.937 + .0601x_1 + .0519x_2 - .0931x_3 \]

\[ - .5581 \log x_1 - 1.575 \log x_2 - .3518 \log x_3, \]

explaining 93.9% of the variance. (Logarithms are taken to the base e in the above.)
III. ANALYSIS OF THE MODEL

This section develops the equations describing the performance of the aircraft. The development incorporates observations made by L. W. Miller of RAND's Logistics Department.

First, we outline the argument. Think of $Y(n)$ as the probability that a preflight inspection starts at time point $n$. Once $Y(n)$ is determined for each $n$, all the other quantities of interest can be calculated readily. If a preflight is to commence at time point $n$, the preceding preflight must have started sometime; this observation gives rise to a set of $M$ equations describing transition probabilities. One additional equation is necessary, since the preceding $M$ equations are dependent. For this, we exploit the fact that probabilities must sum to one.

It will be convenient to work with a network diagram equivalent to Fig. 1. If we denote a delay of $n$ time periods by $H_n$ and the repair time distribution by the symbol $R$, Fig. 1 has the equivalent representation shown in Fig. 5. Note that the aircraft moves unidirectionally,

![Network Diagram](image-url)

Fig. 5 -- Network diagram for Figure 1
though randomly, from node A to node F of Fig. 5. Then, by a simple algebraic manipulation we can calculate the probability distribution governing the transit time from node A to node F. Let

\[ R(n) = \text{probability that time in repair is } n \text{ time intervals or less}, \]
\[ = \sum_{i=1}^{n} r(i), \]

\[ P(n) = \text{probability that the transit time from node A to node F is } n \text{ time intervals or less}. \]

Then, referring to Fig. 5 and tracing out the three ways in which the transit time can be \( n \) time intervals or less, we obtain the formula

\[ P(n) = (1 - p)R(n - a) + pqH_k(n) + p(l - q)R(n - k) \]

where \( k = a + b \) and the Heaviside function \( H_i(n) \) is defined by

\[ H_i(n) = \begin{cases} 
0 & \text{for } n < i \\
1 & \text{for } n \geq i. 
\end{cases} \]

With \( P \) denoting the transition time, Fig. 5 reduces to Fig. 6.

---

**Fig. 6 -- Reduced network diagram for Figure 1**
ANALYSIS OF REDUCED MODEL

We shall analyze the model in Fig. 6 and then relate it to the original problem. Given that one is at node A (that is, one is starting a preflight inspection), the transit time to node F is of course governed by the probability distribution \( P(t) \) given in Eq. (8). If one gets to node F early enough in the day, one transits immediately back to node A; if not, one waits for the beginning of the next flying day. We shall, unimaginatively, call being at node A "state A" and being at node F "state F." Transitions occur from state A to state F and vice versa.

Throughout the remainder of the discussion "time point \( n \)" will refer to the beginning of the \( n^{\text{th}} \) time interval of the calendar day, with time point 1 being the first time at which preflight inspection can commence. Let

\[
q(i,j) = \text{the probability, given node A at time point } i, \text{ that the next occurrence of node A happens at time point } j \text{ of some day.}
\]

The number \( q(i,j) \) is a transition probability. Note that \( q(i,j) \) says nothing about the number of days that elapse between the two events involved -- state \( j \) could be attained the same day as state \( i \), the next day, two days later, etc. Then, \( \sum_{j=1}^{M} q(i,j) = 1. \)

Different expressions for \( q(i,j) \) result from the two cases \( j = 1 \) and \( j > 1 \). For \( j > 1 \), transition to state \( j \) could only happen if for some number \( m \) of days, the transition time is between \( j - i + mN - 1 \) and \( j - i + mN \) time periods, and the probability of this particular
event is \( P[j - i + mN] - P[j - i + mN - 1] \). Summing out over all values of \( m \) yields the probability of a transition from state \( i \) to state \( j \), namely

\[
q(i,j) = \sum_{m=0}^{\infty} \left( P[j - i + mN] - P[j - i + mN - 1] \right) \quad \text{for } 1 < j \leq M.
\]

Similarly, if \( j = 1 \), repair must have been completed later than time point \( M \) of the day at which state \( F \) is attained, since otherwise another takeoff would have occurred that day. Then, with \( m \) being the number of days involved, the transition time must be more than \( M + (m - 1)N - i \) time periods, but not more than \( 1 + mN - i \) time periods. The probability of this particular event is \( P[1 + mN - i] - P[M + (m - 1)N - i] \), and summing over all values of \( m \) yields

\[
q(i,1) = \sum_{m=0}^{\infty} \left( P[1 + mN - i] - P[M + (m - 1)N - i] \right).
\]

Both intuitive considerations and examination of Eq. (9) would reveal that \( q(i,j) = q(i + n,j + n) \) for \( j > 1 \) and every feasible \( i \) and \( n \). Then, one need store only the left-most two columns and the top row of the matrix \( \{q(i,j)\} \). The reduction in required computer memory would be significant if, for instance, \( M = 100 \).

In steady state, the probability that event \( A \) occurs at time point \( n \) will be independent of the day, which justifies the definition

\[
Y(n) = \text{the probability that event } A \text{ occurs at time point } n.
\]

Our immediate objective is to specify the probability \( Y(n) \), for \( n = 1, 2, \ldots, M \), in terms of the numbers \( q(i,j) \). When this is done, we shall calculate other relevant numbers from the \( Y(n) \)'s.
Since the probability of being in state $n$ is the sum of the probabilities of the $M$ different transitions to that state, one has

$$ Y(n) = \sum_{t=1}^{M} Y(t) \ q(t,n) \quad \text{for } 1 \leq n \leq M. \quad (11) $$

The above constitutes $M$ linear equations in $M$ unknowns. If these equations were independent, then they would uniquely specify $Y(n)$. Recall, however, that $\sum_{n=1}^{M} q(t,n) = 1$. Then, summing Eq. (11) over $n$ yields the identity $\sum_{n=1}^{M} Y(n) = \sum_{n=1}^{M} Y(n)$, which demonstrates that the equations are dependent. (Note also that $Y(n) = 0$ satisfies Eq. (11) and that twice a solution is also a solution.)

What is missing so far is the notion that probabilities must sum to one. Toward filling this gap, we temporarily introduce $M$ additional variables given by

$$ Z(n) = \text{Prob} \left\{ \text{state A occurs for the last time on a given day at time point } n \right\}. $$

Then, $Z(n)$ is the joint probability that state A occurs at time point $n$ today and state A does not occur again today. Hence,

$$ Z(n) = Y(n)[1 - P(M - n)]. \quad (12) $$

Denote by $\chi$ the probability that state A occurs at least once during the day. Since the last occurrence of state A must occur sometime during the day, one has

$$ \chi = \sum_{n=1}^{M} Z(n). \quad (13) $$
An independent expression is available for the probability, \(1 - \chi\), that state A does not occur today. Note that \(Y(n)[1 - P(M + mN - n)]\) is the joint probability that state A occurred at time point \(n\) exactly \(m\) days ago and that state A did not occur again until after time point \(M\) of today. Then, summing this expression over \(m \geq 1\) and all \(n\) yields the probability that state A does not occur today, i.e.,

\[
(14) \quad 1 - \chi = \sum_{m=1}^{\infty} \sum_{n=1}^{M} Y(n)[1 - P(M + mN - n)].
\]

Adding Eqs. (13) and (14) and substituting Eq. (12) for \(Z(n)\) yields

\[
(15) \quad 1 = \sum_{n=1}^{M} Y(n) a(n)
\]

with

\[
(16) \quad a(n) = \sum_{m=0}^{\infty} [1 - P(M + mN - n)].
\]

Equations (11) and (15) suffice to specify \(Y(n)\) uniquely, under the mild additional condition that \{\(q(i,j)\)\} is ergodic.* This condition holds for almost any repair-time distribution one might conceivably apply to this problem. Indeed, in most cases \(q(i,j) > 0\) for each \(i\) and \(j\), which suffices. We assume from here on that \{\(q(i,j)\)\} is ergodic.

Eq. (15) can be deduced from another argument. Note that the expression for \(a(n)\) in Eq. (16) can be interpreted as the expectation (in days) of the interval of time which elapses before a second occurrence of event A, given that event A occurs at time point \(n\) today.

Label this time interval between successive occurrences of event A the "cycle time." Since no time is spent in any state other than between successive occurrences of event A, one must have

\[ 1 = E[\text{cycle time (in days) per day}] \]

\[ = \sum_{n=1}^{M} E[\text{cycle time} \mid \text{event A at time point n}] \cdot Pr[\text{Event A at time point n}]. \]

\[ = \sum_{n=1}^{M} a(n) Y(n), \]

which obtains (15) by a second route.

Of course, Eqs. (11) and (15) can be solved by matrix inversion. Rather than doing this, the computer program uses a minor variant of the method of successive approximations given below. Let \( Y_i(n) \) be the \( i \)th approximate of \( Y(n) \), with \( Y_1(n) = 1/M \). For \( i \geq 1 \), we compute

\[
V_{i+1}(n) = \sum_{t=1}^{M} Y_i(t) q(t,n)
\]

\[
Y_{i+1}(n) = V_{i+1}(n) \left/ \left( \sum_{t=1}^{M} V_{i+1}(t) a(t) \right) \right.
\]

The program iterates until \( \max_n \left| \frac{Y_{i+1}(n) - Y_i(n)}{Y_i(n)} \right| < .00001 \), which usually happens within 5 iterations.

**SUPPLEMENTARY CALCULATIONS**

Having determined \( Y(n) \) for each \( n \), we can calculate a host of other numbers. Doing the easiest first, let:

\[ W(n) = \text{Prob}[\text{a sortie takes off at time point } n]; \]

\[ A(n) = \text{Prob}[\text{aircraft is aloft at time point } n]; \]
I(n) = \text{Prob}[\text{aircraft is in preflight inspection at time point } n].

M(n) = \text{Prob}[\text{aircraft is undergoing repair at time point } n].

Takeoffs are possible only at time points \( a + 1 \) through \( a + M \), and the probability of a takeoff is the probability of a successful preflight inspection. Hence,

\[
W(n) = \begin{cases} 
  p \ Y(n-a) & \text{for } 1 + a \leq n \leq M + a \\
  0 & \text{otherwise.}
\end{cases}
\]

Simply by adding up the possibilities, (and recalling that \( Y(t) = 0 \) for \( t > M \)), one has

\[
I(n) = \sum_{t = \max\{1, n-a+1\}}^{n} Y(t)
\]

\[
A(n) = \sum_{t = \max\{1, n-b+1\}}^{n} W(t)
\]

Actually, we have assumed for Eqs. (20) and (21) that \( M + k \leq N \), i.e., that sorties land the same day they take off.

At the first \( M \) time points, the aircraft is undergoing maintenance if it is neither aloft nor in preflight inspection. Hence,

\[
M(n) = 1 - A(n) - I(n) \quad \text{for } 1 \leq n \leq M.
\]

For \( n > M \), note first that \( Y(t)P(mN + n - t) \) is the joint probability that a preflight inspection started \( m \) days ago at time point \( t \) and that the aircraft did not become operational again until after time point \( n \) today. Summing out over the possibilities and subtracting the probability that the aircraft is in flight or in preflight inspection yields
(23) \[ M(n) = \sum_{t=1}^{M} Y(t) \left\{ \sum_{m=0}^{\infty} [1 - P(mN + n - t)] \right\} - A(n) - I(n) \text{ for } n > M. \]

The probability \( W(n) \) that a takeoff occurs at time point \( n \) is now broken up into the probability that this takeoff is the first of the day, the second, etc. Let

\[ X(i,n) = \text{the probability that the } i^{\text{th}} \text{ flight of the day takes off at time point } n. \]

In preparation for calculation \( X(i,n) \), we first determine the distribution of time between successive takeoffs for the case in which the later takeoff occurs the same day as the former one. With reference to Fig. 5, we shall compute the distribution of time between successive occurrences of node C under the assumption that the second occurrence occurs the same day as the first. For this purpose, we short-circuit the "wait (nights only)" box of Fig. 5 and define

\[ U(n) = \text{the probability that the time between successive takeoffs is no more than } n \text{ time periods, with the arc from F to A replaced by } H_0. \]

The distribution \( U(n) \) is determined by enumerating all the paths that lead from node C back to node C. Let \( L \) be the probability distribution governing a loop \( A \rightarrow B \rightarrow A \), with

\[ L = (1 - p)H_a \ast R. \]

The transition time from node A to node C is governed by the distribution of the transition \( A \rightarrow B \rightarrow C \) and is \( pH_k \). Since the loop can be
traversed any number of times (including zero), enumerating the paths from C back to C yields

\[ U = p_{H_k} \ast [q_{H_0} + (1 - q) \ast R] \ast [H_0 + L + L^2 + \ldots] \]

where \( L^n \) is the \( n \)-fold convolution of \( L \) with itself. Eq. (25) contains an infinite sum and, strictly speaking, does not specify a finite procedure for determining \( U(n) \). However, we shall only be interested in \( U(i) \) for \( i \leq M \), and the sequence \( \{L^n(M)\}_{n=1,2} \) decreases to zero, permitting a simple finite approximation.

Since a flight that takes off at time point 1 can only be the first flight of the day

\[ X(1,1) = W(1). \]

For \( n > 1 \), the joint probability that a flight takes off at time point \( n \) and that it is not the first flight of the day is \( \sum_{t=1}^{n-1} W(t) [U(n - t) - U(n - t - 1)] \), since a flight must have taken off at some time point \( t < n \) with inter-takeoff time between \( n - t - 1 \) and \( n - t \) periods. Hence

\[ X(1,n) = W(n) - \sum_{t=1}^{n-1} W(t) [U(n - t) - U(n - t - 1)] \quad \text{for } n > 1. \]

For \( i > 1 \), the \( i^{th} \) flight of the day takes off at time point \( n \) if the \( i - 1^{th} \) flight took off at some earlier time point \( t \) and the inter-takeoff time is between \( n - t - 1 \) and \( n - t \) periods. Hence

\[ X(i,n) = \sum_{t=1}^{n-1} X(i - 1,t) [U(n - t) - U(n - t - 1)] \quad \text{for } i > 1 \text{ and } n \geq i. \]
The probabilities, $X(i,n)$, are calculated from the $W(n)$'s using Eqs. (24)-(28). Having determined $X(i,n)$, we are prepared to calculate

$$F(i) = \text{probability that exactly } i \text{ takeoffs occur during the day}$$
$$EF = \text{expectation of the number of takeoffs during the day}$$
$$\text{Var } F = \text{the variance of the number of takeoffs during the day.}$$

Note that $\sum_{n=1}^{M} X(i,n)$ is the probability that at least $i$ takeoffs occur. Hence

$$F(0) = 1 - \sum_{n=1}^{M} X(1,n) \tag{29}$$

$$F(i) = \sum_{n=1}^{M} X(i,n) - \sum_{n=1}^{M} X(i+1,n) \quad \text{for } i \geq 1 \tag{30}$$

$$EF = \sum_{i=0}^{M} iF(i) \tag{31}$$

$$\text{Var } F = \sum_{i=0}^{M} i^2F(i) - (EF)^2. \tag{32}$$

The preceding equations specify a number of important aspects of the aircraft's performance. The computer program calculates and prints each of these numbers.
APPENDIX I. COMPUTER PROGRAM

The appendix contains a listing of the computer program and the instructions for its use. For each calculation, one must prepare a data card containing

N in columns 1 - 3 and format \* I3
M in columns 4 - 6 and format I3
a in columns 7 - 9 and format I3
p in columns 10 - 13 and format F4.3
b in columns 14 - 16 and format I3
q in columns 17 - 20 and format F4.3
J in columns 21 - 23 and format I3
Kind in columns 24 - 26 and format I3
EZ in columns 27 - 30 and format F4.3
RZ in columns 31 - 34 and format F4.3.

Except for "Kind," these terms are all defined in Sec. I. The indicator, Kind, determines which repair time distribution will be used, with

\[
\text{Kind} = \begin{cases} 
1 & \text{if empirical} \\
2 & \text{if lognormal} \\
3 & \text{if exponential} \\
4 & \text{if sum-of-2-exponentials.}
\end{cases}
\]

If Kind = 1, supplementary cards need to be entered. These contain

* The format is given in FORTRAN nomenclature.
the repair time probabilities $r(n)$, $n = 1, 2, \ldots, JN$, and are entered in format 12F5.3, requiring JN/12 additional cards.

Several calculations can be made in a single computer set-up; one stacks several data cards or sets of data cards behind the program. Debugging information can be obtained by including a special card with 999 in position 1 - 3; all calculations which follow this card are accompanied by debugging information. The normal output consists of

1) the information on the input card,
2) the number of iterations required for convergence of the algorithm.
3) the expected number of flights, $EF$, the variance in the number of flights, $Var F$, and the probability $F(i)$ that exactly $i$ takeoffs occur,
4) a table containing $M(n)$, $Y(n)$, $I(n)$, $A(n)$ and $X(i,n)$ for each time point $n$.

With certain exceptions, the notation used in the program closely resembles that of the Memorandum. Exceptions are $a \equiv LP$, $b \equiv LF$, $p \equiv PP$, $q \equiv PF$. As an aid in reading the program, comment cards which refer to the appropriate equation numbers are included in the program.
$JCBB 3C49,SORTIE,C283C,G2,1CC,3CC,C
$S1JGB MAP
$S1EFCT REF
  DIMENSION P(8CO),Q(1CC,1CO),GQ(1CC),Y(1CO),F(21),EM(1GC),
  XRANI(6CC),U(1CC),UQ(1CC),LB(1CC),UB(1CC),XX(20,1CC),YY(1CC),
  XPRF(1CC),PRF(1CC),
  XCUIT(4,48),A(3,3)
  FIRST=0.
  T=0.
  1 READ 7CC0,N,M,LP,PP,LF,PF,J,KINC,EZ,RZ
     IF (N.EQ.999) GO TO 997
7CC0 FCRMAT (3I3,F4.3,13,F4.3,2I3,2F4.2)
  ERR=C.
  K=LP+LF
  KPI = K+1
  NMI = N-1
  JN = J*N
  JNP1 = JN-1
  I = (M+K-1)/K
  IP1 = I+1
  CC 2 LJ = I, K
  2 P(LJ) = 0.
  P(JN) = 1.
  IF ((KINC.GT.4).OR.(KINC.LT.1)) GO TO 999
  GC TC (19C,2CC,3CC,4CC),KINC
997 T=1.
  GC TC 1
  GC TC C
EMPIRICAL PROBABILITY DENSITY FUNCTION
1SC JNPK=JN*K
   READ 7CC1,(P(LN),LN=KPI,JNP1)
7CC1 FCRMAT (12F5.3)
   CC 191 LN=KPI,JN
191 P(LN)=P(LN)+P(LN-1)
   IF (P(JN).GT.1.) GO TO 999
   P(JN) = 1.
   GC TC 4
C LCGNORMT L LN L RN R R R
2CC IF (FIRST.EQ.1.) GC TC 2CC
   FIRST=1.
C ROUTINE TO GENERATE NORMAL CDF.
C EQUATION 1
SUM = 0.
CC 21C LN = 1, 6CC
Z = FC LGN LN-1)/1CC.
RANI(LN) = (1.0/C)/(LZ/4.0/EXPTL(Z+C0.05)+EXPTL(Z+C1))
21C SUM = SUM +RANI(LN)
SUM = 2.0*SUM
RANI(1) = (RANI(1)/SUM)
CC 22C LN = 2, 6CC
22C RANI(LN) = RANI(LN-1)+(RANI(LN)/SUM)
C ROUTINE TO GENERATE LCGNORMT DISTRIBUTION
C EQUATIONS 2, 3
2CC SIG = ALCG(1.+RZ**2)
SIG = SCRT(SIG)
AVG = ALCG(EZ) - SIG**2/2.
Z = 24./FC LGN(N)
CC 223 LJ=KPI,JNP1
X1=ALCG(FLCAT(LJ-K)*Z) - AVG
IF (SIG.EQ.C.) GC TC 202
X1=X1/SIG
XS=-1.
IF (X1.GE.C.) XS=+1.
X1=(10C.)*ABS(X1)
JX=X1
XF=X1-FLOAT(JX)
IF (JX.GE.599) GC TG 221
P(LJ)=.5+XS*XF*RANI(JX+1)
IF (JX.GT.0) P(LJ)=P(LJ)+XS*(1.-XF)*RANI(JX)
223 CONTINUE
GC TC 4
221 P(LJ)=.5+XS*(.5)
GC TC 223
202 IF (X1.LT.0.) P(LJ)=C.
IF (X1.EQ.C.) P(LJ)=.5
IF (X1.GT.0.) P(LJ)=1.
GC TC 223
C EXPONENTIAL REPAIR TIME ENTERED WITH EZ.
300 Z = 24./(FLOAT(N)*EZ)
CC 301 LN=KP1,JNM1
301 P(LN)=1.-EXP(-FLCAT(LN-K)*Z)
GC TC 4
C REPAIR TIME=SUM OF 2 EXPONENTIALS ENTERED WITH EZ,RZ
C EQUATIONS 4, 5, 6
400 Z1=2.*(RZ**2)-1.
IF (((Z1.LT.0.),OR.Z1.GT.1.) )GO TO 999
Z1=SQRT(Z1)
ZA=(1.+Z1)*EZ/2.
ZB=(1.-Z1)*EZ/2.
Z1=24./(FLCAT(N)*ZA)
Z2=24./(FLCAT(N)*ZB)
ZR1=Z1/(Z1-ZB)
ZR2=ZB/(ZB-Z1)
CC 401 LN=KP1,JNM1
ZLN=LN-K
401 P(LN)=1.-ZR1*EXP(-ZLN*Z1)-ZR2*EXP(-ZLN*Z2)
GC TC 4
999 ERR=1.
GC TC 152
4 CC 199 LN=1,JNM1
199 IF (P(LN).GT.P(LN+1)) GC TG 999
IF (T.EQ.C.) GO TO 3000
CC 198 LN=1,JN
198 PRINT 7007,LN,P(LN)
3000 S=(1.-PF)*PP
C EQUATION 9
SS=PP*PF
CC 3 LN=1,N
U(LN)=S*P(LN)
3 IF (LN.GE.K) U(LN)=U(LN)+SS
PPPL=PP+CCCC01
IF (PPPL.GE.1.) GC TC 9
IF (T.EQ.C.) GC TC 3100
CC 31 LN=1,N
31 PRINT 7C07*,LN,U(LN)
31CC CC 51 LN=1,N
LT=MINC(LN+LF,JN)
LTL=MINC(LN+LF-1,JN)
C
EQUATIONS 24, 25
UB(LN)=(1,-PP)*(P(LT)-P(LTL))
IF (LN .GE. 1) UA(1)=U(1)
51 IF (LN.GT.1) UA(LN)=U(LN)-U(LN-1)
UC(1)=0.
6 TOTAL=C.
CC 7 LN=1,NM1
Z=0.
CC 52 LT=1,LN
IL=LN-LT+1
52 Z=Z+UA(LT)*UP(IL)
UC(LN+1)=Z
TOTAL=TOTAL+Z
7 LL(LN)=U(LN)+TOTAL
IF(TOTAL.LT..CCC1) GO TO 9
IF (T.EQ.0.) GC TO 71CC
CC 71 LN=1,NM1
71CC CC 81 LN=1,NM1
81 UA(LN)=UC(LN)
CC TC 6
9 SS=PP*PF
SF=PP*(1.-PF)
CC 91 LN=1,JNM1
LT=MINC(JN,LN+LF)
Z=SF*P(LN)+(1.-PP)*P(LT)
IF (LN .GE. K) Z=Z+SS
91 P(LN)=Z
IF (T.EQ.0.) GC TO 5
CC 92 LN=1,JN
92 PRINT 7C07*,LN,P(LN)
CC 93 LN=1,M
93 PRINT 7C07*,LN,U(LN)
C
EQUATIONS 9, 1C, 16
5 CC 2C LT=1,M
CC(LT)=0.
Y(LT)=(1.)/FLCAT(M)
CC 8 LM=0,J
IL=M-LT+LM*N
II=MINC(IL,JN)
CC(ILT)=CC(LT)+1.
8 IF (II.GT.CC) CC(LT)=CC(LT)-P(IL)
CC 2C LN=1,M
IF ((LN.LE.2).OR.(LT.EQ.1)) GC TO 11
CL(LT,LN)=CL(LT-1,LN-1)
GC TO 2C
11 CL(LT,LN)=0.
CC 2C LM=C,J
IL=LM*N+LN-LT
IF (II.LE.C) GC TO 2C
IF (LN.EQ.1) I2=(LM-1)*N+M-LT
IF (LN.GT.1) I2=I1-1
IF (I2.LE.C) GC TO 1C
I1=MINC(I1,JI)
I2=MINC(I2,JN)
G(LT,LN)=G(LT,LN)+P(I1)-P(I2)
GC TO 20
20 GC(LT,LN)=G(LT,LN)+P(I1)
2C CCONTINUE
ITER=1
LI=0
IF (T.EQ.C) GC TO 25
PRINT 7007,LI,(G(LN),LN=1,M)
CC 22 LN=1,M
22 PRINT 7007,LN,(G(LN,LI),LI=1,M)
25 CP=C.
ZT=0.
CC 30 LN=1,M
Z=0.
CC 28 LT=1,M
C EQUATION 17
28 Z=Z+Y(LT)*G(LT,LN)
IF (Z.EQ.C) ZT=AMAX1(ZT,(ABS(Z-Y(LN)))/Z)
IF (Z.EQ.C) ZT=AMAX1(ZT,ABS(Z-Y(LN)))
Y(LN)=Z
C EQUATION 15, MGRF OR LESS
3C CP=CP+Z*GQ(LN)
ZT=AMAX1(ZT,ABS(1.-CP))
IF (T.EQ.C) GC TO 3001
PRINT 7007,ITFR,CP
3001 Z=1./CP
CC 35 LN=1,M
IF (T.EQ.C) GC TO 35
PRINT 7007,LN,Y(LN)
C EQUATION 18
35 Y(LN)=Y(LN)+Z
IF (ZT.LT.CCCG1) GC TO 1GC
IF (ITER.GE.3C) GC TO 998
ITER=ITER+1
GC TO 25
998 ERR=2.
C EQUATION 19 WITH W=YY, 26
1GC CC 1C1 LN=1,M
1C1 YY(LN)=PP*Y(LN)
XX(1,1)=YY(1)
CC 1C3 LN=2,M
Z=YY(LN)-U(1)*YY(LN-1)
IF (LN.EQ.2) GC TO 1C3
LNM2=LN-2
CC 1C2 LJ=1,LNM2
I1=LN-LJ
I2=I1-1
C EQUATIONS 27, 28
1C2 Z=Z-YY(LJ)*U(I1)-U(I2))
1C3 XX(1,LN)=Z
IF (I.EQ.1) GO TO 1C6
CC 1C4 LI=2,I
CC 1C4 LN=LI,N
Z=XX(LI-1,LN-1)*U(1)
IF (LN.EQ.2) GO TO 1C4
LN=M-LN-2
CC 1C5 LT=1,LM2
II=LN-LT
I2=II-1
105 Z=Z+XX(LI-1,LT)*(U(I1)-U(I2))
1C4 XX(LI,LN)=Z
1C6 F(I)=1.
C
EQUATIONS 29-32
CC 11C LI=1,I
F(LI+1)=0.
CC 11C LN=1,N
110 F(LI+1)=F(LI+1)+XX(LI,LN)
CC 112 LI=1,I
112 F(LI)=F(LI)-F(LI+1)
121 EF=0.
VARF=0.
CC 13C LI=1,IP1
FIM1=LI-1
13C EF=EF+FIM1*F(LI)
CC 131 LI=1,IP1
FIM1=LI-1
131 VARF=VARF+(FIM1-EF)**2*F(LI)
CC 144 LN=1,N
PRP(LN)=0.
PRF(LN)=0.
J1=MAXC(LN-K+1,1)
J11=MINC(LN-LP,M)
J2=MAXO(LN-LP+1,1)
J12=MINO(LN,M)
IF (J1.GT.J11) GO TO 142
CC 141 LJ=J1,J11
141 PRP(LN)=PRP(LN)+YY(LJ)
142 IF (J2.GT.J12) GO TO 144
CC 143 LJ=J2,J12
143 PRP(LN)=PRP(LN)+Y(LJ)
144 CONTINUE
CC 145 LN=1,N
145 EM(LN)=1.-PRP(LN)-PRF(LN)
MP1=M+1
JP1=J+1
CC 15C LN=MP1,N
EM(LN)=PRP(LN)-PRF(LN)
CC 15C LT=1,M
Z=Y(LT)
CC 150 LM=L,JP1
150 EM(LN)=EM(LN)+Z*(1.-P(II))
152 PRINT 7CC8
7CC8 FORMAT (1IC)
IF (ERR.NE.2.) GO TO 151
PRINT 7201

7201 FORMAT (19HFAILED TO CONVERGE.)
151 PRINT 7004

7004 FORMAT (55H KIND N M LP LF J PP PF EZ RZ ITER)
PRINT 7013,KIND,N,M,LP,LF,J,PP,PF,EZ,RZ,ITER

7C13 FORMAT (1X,(4,5I3,2F7.3,2F7.2,I6))
IF (ERR.EQ.1.) GC TC 169
PRINT 7006

7C06 FORMAT (1HC,43H EF VARG F(C) F(1) F(2) ....)
PRINT 7C03,EF,VARG,(F(LI),LI=1,IP1)

7CC3 FORMAT (2X,13(2X,F5.3))
PRINT 7111

7111 FORMAT (1HC,24H TABLE OF PROBABILITIES)
PRINT 7C05

7CC5 FORMAT (53H N M(N) Y(N) I(N) W(N) A(N) X1(N) X2(N))
IF (LP.EQ.C) GC TC 165
CC 155 LN=1,LP

155 PRINT 7C02, LN,EM(LN),Y(LN),PRP(LN)

7CC2 FORMAT (1X,(13,18(F7.4))
165 I1=LP+1
   I2=LP+M
   CC 166 LN=I1,I2
   Z1=Y(LN)
   Z2=PRP(LN)
   LJ=LN-LP
   IF (LN.LE.M) GC TC 166
   Z1=0.
   Z2=0.

166 PRINT 7002, LN, EM(LN), Z1, Z2, YY(LJ), PRP(LN), (XX(LI,LJ),LI=1,I)
   I1=I2+1
   I2=M+K-1
   Z=0.
   IF (LF.LE.1) GC TC 157
   CC 158 LN=I1,I2

158 PRINT 7C02, LN, EM(LN), Z, Z, PRP(LN)
157 I1=I2+1
   IF (I1.GT.N) GC TC 160
   CC 159 LN=I1,N

159 PRINT 7C02, LN, EM(LN)
160 IF (ERR.EQ.C) GC TC 1
169 PRINT 7200

7C00 FORMAT (1HC,39HABOVE RUN IS WRONG. OUTPUT P(LN),U(LN).)
170 CC 171 LN=1,N+1
171 PRINT 7007, LN,P(LN),U(LN)
CC 172 LN=N,JN

172 PRINT 7C07, LN,P(LN)
7C07 FORMAT (1X,(13,18(F7.4))
CC TC 1
END

$IBFIC EXPTL
FUNCTION EXPTL(X)
   EXPTL = EXP(-X**2/2.)
RETURN
END
APPENDIX 2. DERIVATION OF LOGNORMAL DISTRIBUTION

For completeness, we include a derivation of certain facts concerning the lognormal distribution. Let $X$ be a normally distributed random variable with mean $m$ and variance $\sigma^2$. The density function $f_X(t)$ of $X$ is given by

$$f_X(t) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(t-m)^2}{2\sigma^2}}.$$  

Let $Z = e^X$. The random variable $Z$ is said to have the lognormal distribution, since the logarithm of $Z$ is normally distributed. For $Z$, we shall calculate the density function $f_Z(t)$, the expected value $EZ$ and the variance $Var Z$, the last two in terms of $m$ and $\sigma$.

For the density function of $Z$, note that $P(Z < t) = P(e^X < t) = P(X < \ln t)$; hence, by the chain rule

$$f_Z(t) = \frac{d}{dt} P(Z < t) = \frac{1}{t} f_X(\ln t)$$

$$= \frac{1}{\sqrt{2\pi} \sigma t} e^{-\frac{(\ln t-m)^2}{2\sigma^2}}.$$  

For the expectation of $Z$, and of $Z^2$, one has

$$EZ = \int_0^\infty tf_Z(t)dt = \int_0^\infty f_X(\ln t)dt = e^{m+\sigma^2/2}$$

$$EZ^2 = \int_0^\infty t^2 f_Z(t)dt = e^{2m+2\sigma^2},$$
where the last expression on each line is obtained by change of variables. One then has

\[ \text{Var } Z = E(Z^2) - (EZ)^2 = e^{2m + \sigma^2} \left[ e^{\sigma^2} - 1 \right]. \]

The ratio RZ of the standard deviation of Z to its expectation is then given by

\[ RZ^2 = \left[ e^{\sigma^2} - 1 \right]. \]

(34)

Note that RZ is independent of m. One could solve Eq. (34) for \( \sigma \), yielding Eq. (2) and then, using this value of \( \sigma \), solve Eq. (33) for m, yielding Eq. (3).