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SOME LINEAR PROGRAMMING APPLICATIONS TO STOCKAGE PROBLEMS

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APPLICATIONS TO STOCKAGE PROBLEMS
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This Memorandum is part of RAND's continuing research on the problems of designing optimal stockage policies. Although the study is theoretical, it is hoped that the ideas are susceptible to practical application. For a full understanding of the text the reader should have some acquaintance with both linear programming and the Air Force supply system.
SUMMARY

In this Memorandum we formulate a range of Air Force stockage problems as linear programming problems. We consider analysis of such problems as a two-part process. Part one is evolving ways to identify decision variables and to compute cost-effectiveness measures connected with a proposed policy. Part two is evolving methods for choosing optimal policies. Our interest here is with part two. We do not describe computational techniques other than to say that the linear programs may be solved by the simplex method using multipliers. The stockage problems described are:

A. Design of mobility kits (M-kits) of reparable items.
B. Reparable item stockage of a multi-weapon-system base.
C. Base stockage of recoverable items and WRM supplements thereto.
D. Base stockage of EOQ items.
E. Single-echelon multibase stockage and procurement of recoverable items.
F. Single-echelon multibase stockage of recoverable items without additional procurement.
G. Multi-echelon stockage of recoverable items with additional procurement.
H. Multi-echelon stockage of recoverable items without additional procurement.
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I. INTRODUCTION

The analysis of an Air Force inventory problem may be thought of as a two-part process. Part one is cataloging the available policies (i.e., identifying the "decision variables") and the important characteristics of each policy (i.e., choosing cost and effectiveness measures), and devising methods for computing the characteristics of particular policies. Part two is devising methods for choosing policies with optimal characteristics. In this Memorandum we consider problems whose first part has been accomplished, and concentrate on part two—optimization. Specifically, we consider the following problems:

A. **Design of mobility kits (M-kits) of reparable items.** The decision variables are the quantities of each item in the kit. The policy characteristics are: cost, weight, volume, and performance of the kit.

B. **Reparable item stockage of a multi-weapon-system base.** The decision variables are the item levels. The policy characteristics are the investment cost and the support given to each weapon system as measured, e.g., by fill rates.

C. **Base stockage of recoverable items and WRM supplements thereunto.** The decision variables are peacetime item levels and the item quantities held on the base for war reserve. The policy characteristics are the total dollar investment in inventory, the ability of the peacetime assets to support the peacetime mission, and the ability of peacetime assets plus war reserve to meet the war mission.

D. **Base stockage of EOQ items.** The decision variables are the items' economic-order-quantities and reorder points. The policy
characteristics are average investment in inventory, average number of orders per unit time, and average shortages per unit time.

E. **Single-echelon multibase stockage and procurement of recoverable items.** The decision variables are the amount of each item that should be added to existing Air Force assets, and the bases' stock levels. The policy characteristics are additional procurement costs, support levels provided to the bases, and the amounts by which procurement plus existing assets fall short of the sum of the base stock levels for each item.

F. **Single-echelon multibase stockage of recoverable items, without additional procurement.** The decision variables are the bases' stock levels. The policy characteristics are the amount of support provided at the bases, and the amount by which assets fall short of the sum of base stock levels for each item.

G. **Multi-echelon stockage of recoverable items with additional procurement.** The decision variables are depot stock levels for each item and base stock levels for each item and each base, and procurement for each item. The policy characteristics are total additional procurement cost, the amount of support given the bases, and the amount for each item by which procurement, plus assets, falls short of the sum of base and depot levels.

H. **Multi-echelon stockage of recoverable items without additional procurement.** The decision variables are depot stock levels for each item, and base stock levels for each item and each base. The policy characteristics are the amount of support given to the bases, and the amount, for each item, by which assets fall short of the sum of base and depot levels.
In each of the above problems there is a set $Q$ of policies and a set of real valued functions $a^i : Q \to \mathbb{R}$, $i = 1, \ldots, m + 1$, of policy characteristics. By using the negative of a characteristic if necessary we may assume that low values of a characteristic are good whereas high values are bad. An optimization problem then takes the following form:

Subject to

$$a^i(q) \leq b^i \quad (i = 1, \ldots, m),$$

(1)

minimize

$$a^{m+1}(q) \quad (q \in Q).$$

Finding an exact solution to (1) for any of the problems A-H above is a formidable problem. Instead of attempting exact solutions, we use the procedure suggested in [11] to find approximate solutions. Let $R^Q$ be the set of all real valued functions on $Q$ that assume the value zero at all but a finite number of points. We will assume that $Q$ -- though possibly huge -- is finite. This assumption is a mild one for any of the problems listed above, since we may always put large upper bounds on kit quantities, stock levels, and order quantities. The support of a function $p \in R^Q$ is the set of policies $q \in Q$ for which $p(q) \neq 0$. Consider the following problem:

*The notation $a^i : Q \to \mathbb{R}$ means that $a^i$ is a function whose argument comes from the set $Q$ of policies, and whose value lies in the set $\mathbb{R}$ of real numbers.*
Subject to

\[ p(q) \geq 0 \quad (q \in Q), \]

\[ \sum_q p(q) = 1, \]

(2)

\[ \sum_q p(q) a_i^i(q) \leq b_i^i \quad (i=1, \ldots, m), \]

minimize

\[ \sum_q p(q) a_{m+1}(q) \quad (q \in Q). \]

The above problem is a linear programming problem in the "variables" \( p(q), q \in Q. \)

The set of \( p \in R^Q \) that satisfy the constraints in (2) is compact, and the minimand is a continuous function of \( p. \) Therefore if there is a \( p \in R^Q \) that satisfies the constraints, (2) will have a solution. If (1) has a solution \( q^* \in Q, \) then the function \( p^* \in R^Q \) defined by

\[ p^*(q) = \begin{cases} 1 & \text{if } q = q^*, \\ 0 & \text{if } q \neq q^*, \end{cases} \]

satisfies the constraints in (2). Thus if (1) has a solution, then so does (2). The procedure suggested in [1] is to solve (2) for a \( p^* \in R^Q \) and then use the support of \( p^* \) as a set of approximate solutions to (1). For the problems considered in this Memorandum, the approximations will be good to the extent that the number of different decision variables is large in relation to the number of different policy characteristics. The procedure is also "fail-safe" in the
sense that if the policies in the support of a solution to (2) are good approximations to one another, then any of them provides a good approximation to a solution of (1). These two statements are made more precise and proven in Appendix A.

In this Memorandum we do not give a detailed description of the techniques for solving problems in the form of (2). A few remarks on computational technique might, however, be helpful. For all the problems we consider, the set $Q$ is huge. For example, in problem A, $Q$ consists of all different possible mobility kits. If there are 500 candidate items for inclusion in the $M$-kit, and each of these items could be put in the kit in any quantity from 0 through 9, then $Q$ will consist of $10^{500}$ different kits. Now the number of variables $p(q), q \in Q$, in (2) is equal to the number of elements in $Q$. This means that a straightforward application of the standard simplex technique to (2) is, for our problems, practically impossible, since this method would require explicit enumeration and storage of all the vectors $(a^1(q), \ldots, a^{m+1}(q))$ for $q \in Q$. It does seem feasible, however, to solve (2) by the simplex method using multipliers. This method requires the explicit recording of only $m+2$ of the vectors $(a^1(q), \ldots, a^{m+1}(q)), q \in Q$. Two types of problems are then solved alternately. The first takes the form

Subject to

$$\xi_j \geq 0 \quad (j=1, \ldots, m+2),$$

$$\sum_j \xi_j = 1,$$
\[ \sum_j z_j a_i^j(q_j) \leq b_i \quad (i=1, \ldots, m), \]

minimize

\[ \sum_j z_j a_{m+1}^j(q_j) \quad (\xi_j \in \mathbb{R}; \ j=1, \ldots, m+2). \]

This problem is referred to as the "master problem"; it is to be solved for the variables \( \xi_j, j=1, \ldots, m+2 \); the policies \( q_j \in Q, j=1, \ldots, m+2 \) are fixed. When the master problem is solved there will be (at least) one \( k \) for which \( \xi_k = 0 \). The corresponding \( q_k \) is then removed from the list \( q_1, \ldots, q_{m+2} \); and a replacement for it is found by solving the "side problem:"

Minimize

\[ \gamma + \sum_{i=1}^m a_i^j(q) + a_{m+1}^j(q) \quad (q \in Q), \]

where \(-\gamma\) is the dual variable associated with the "convexity constraint" \( \sum_j \xi_j = 1 \) in the master problem, and \(-\alpha^j \geq 0\) is the dual variable associated with the constraint \( \sum_j \xi_j a_i^j(q_j) \leq a_i^j \). A solution \( q \) to the subproblem is then used to replace the \( q_k \) thrown out from the previous solution to the master problem, and the master problem is solved anew. The process is terminated when there is no \( q \in Q \) for which \( \gamma + \sum_{i=1}^m a_i(q) + a_{m+1}^j(q) < 0 \). When this point is reached we may define \( p \in \mathbb{R}^Q \) by \( p(q_j) = \xi_j \) and \( p(q) = 0 \) when \( q \not\in \{q_1, \ldots, q_{m+2}\}. \)

* Solving the master problem, solving the side problem, and the termination rule are the analogues to the standard simplex method of deciding what column to throw out of the basis and pivoting, deciding what column to introduce into the basis, and finding a column that "prices out" negative.
The above description is a very brief sketch of the simplex method using multipliers. For a complete description the reader may refer to
[2] and [3]. Before leaving the topic, however, a few more remarks may be helpful. The dual variables \( \gamma, \alpha^1, \ldots, \alpha^m \) are sometimes referred to as "Lagrange multipliers" or "prices." The technique for solving the master problem can be the same for all of the problems in this Memorandum. The technique for solving the side problem, however, will vary from one problem to the next depending upon the structure of \( Q \) and the functions \( a^i: Q \to R, i = 1, \ldots, m + 1 \). In attempting to solve the side problem, it is not always necessary to solve it completely; in fact it suffices to find a \( q \in Q \) for which
\[
\gamma + \sum_i \alpha^i a_i(q) + a^{m+1}(q) < 0.
\] This fact allows us to use quick and dirty methods on the side problem at first, and save more time consuming methods until the process is near completion.

In Sec. II we consider the single base problems A through D. In Sec. III we consider the multiple base problems E through H.
II. SINGLE-BASE PROBLEMS

A. M-kits of Reparable Items

A TAC M-kit is supposed to contain enough units of recoverable items to support a deployed unit for thirty days. In addition, the kits are supposed to be mobile, which means that they should be light and compact. They should also be low-cost.

Let \( \gamma_j(x) \) be the probability that no more than \( x \) units of item \( j \) are demanded in the deployed situation. Assuming that the number of demands for item \( j \) is independent of the number for item \( k, j \neq k \), the probability of meeting all demands during the thirty-day mission is

\[
\prod_j \gamma_j(q_j),
\]

where \( q_j \) is the quantity of item \( j \) in the kit, \( j = 1, \ldots, n \), and \( n \) is the number of candidates for inclusion in the kit. The weight of a kit containing \( q_j \) of item \( j \), \( j = 1, \ldots, n \), is

\[
\sum_j w_j q_j,
\]

where \( w_j \) is the weight of item \( j \), and the volume of such a kit is

\[
\sum_j v_j q_j,
\]

where \( v_j \) is the volume of the \( j \)th item. Finally, the cost of a kit is

\[
\sum_j c_j q_j,
\]

where \( c_j \) is the cost of the \( j \)th item.
The optimization problem is to maximize (3) subject to constraints on (4), (5) and (6), or minimize one of (4), (5) and (6) subject to constraints on the other functions (3) through (6). We will consider the problem of maximizing performance (as measured by (3)) subject to constraints on weight, volume, and cost; the other optimization problems are similar.

Maximizing (3) is equivalent to minimizing the negative of its logarithm. Hence we may state our problem as

Subject to

\[ \sum_j w_j q_j \leq W, \]
\[ \sum_j v_j q_j \leq V, \]
\[ \sum_j c_j q_j \leq C, \]

(7)

minimize

\[ \sum_j f_j(q_j) (q \in Q), \]

where \( f_j(q_j) = - \ln \bar{v}_j(q_j) \) and \( Q \) is the set of n-tuples \( q = (q_1, \ldots, q_n) \) of nonnegative integers \( q_j \leq \) a large number \( L \). We replace (7) by the linear programming problem:

Subject to

\[ \sum_{q \in Q} p(q) \sum_j w_j q_j \leq W, \]
\[ \sum_{q \in Q} p(q) \sum_j v_j q_j \leq V, \]
\[ \sum_{q \in Q} p(q) \sum_j c_j q_j \leq C, \]
\[ \sum_{q \in Q} p(q) = 1, \]
\[ p(q) \geq 0 \quad (q \in Q), \]
minimize
\[ \sum_{q \in Q} p(q) f_j(q_j) \quad (p \in R^Q). \]

This problem may be solved by the "simplex method using multipliers" [2], [3]. In applying this method one obtains a master problem with four constraints (weight, volume, cost, and convexity) and a side problem of the form

Minimize
\[ \sum f_j(q_j) + \alpha \Sigma_j w_j q_j + \beta \Sigma_j v_j q_j + \gamma \Sigma_j c_j q_j \quad (q \in Q), \]

where \( \alpha, \beta, \gamma \) are the "prices" associated with the weight, volume, and cost constraints in (8). (In order that our dual variables be non-negative we are reversing the usual dual variable sign conventions throughout this Memorandum.)

The side problem (9) separates into \( n \) subproblems:

Minimize
\[ f_j(q_j) + \alpha w_j q_j + \beta v_j q_j + \gamma c_j q_j \quad (j = 1, \ldots, n; q_j = 0, \ldots, L). \]

A program for computing M-kits using the optimization technique outlined here has been written; the computation of 27 different M-kits involving 408 different items \( (n = 408) \) required approximately eleven minutes on the 7044.

B. Reparable Item Stockage of a Multi-Weapon-System Base

One of the problems that has arisen during the implementation of the RAND base stockage model [4] is the desire of the Air Force to specify specific supply-support targets for individual weapon systems
on a base, rather than aggregate the measure of supply support across all weapon systems. Thus, for example, it may be desired to keep the nonfill rate of F4C demands below ten percent, rather than specify the nonfill rate across all F4s. If there are no common items among the various weapon systems, then the base stockage model can be applied to each weapon system in turn. If there are common items, however, this procedure might cause an overstockage of common items.

Assume that all demands are treated on a first-come, first-served basis. Let \( f_j(q_j) \) be the expected number of nonfills in a unit of time (or back orders if you prefer) on item \( j \) when its base stock level is \( q_j \); the functions \( f_j \), \( j = 1, \ldots, n \), number of items, may be computed exactly as in the base stockage model. Let \( t^i_j \) be the proportion of total base demands against item \( j \) that are generated by weapon system \( i \). Then \( t^i_j f_j(q_j) \) is the number of nonfills per unit time on item \( j \) and weapon-system \( i \), when \( q_j \) is the base stock level for item \( j \). (This claim is justified in Appendix B.) Suppose that we wish to find the cheapest stockage policy that keeps the number of nonfills per unit time on demands levied by weapon system \( i \) below \( a^i \), for each \( i = 1, \ldots, m \), \( m \), number of weapon systems. Then the problem may be written as

Subject to

\[
\sum_j t^i_j f_j(q_j) \leq a^i \quad (i = 1, \ldots, m),
\]

(11)

minimize

\[
\sum_j c_j q_j \quad (q \in Q),
\]
where $c_j$ is the cost of item $j$; $q_j$ is the base stock level for item $j$; $t_j^i$ and $f_j^i$ are defined as above; and $Q$ is the set of all $n$-tuples $q = (q_1, \ldots, q_n)$, $q_j$ an nonnegative integer no more than a large number $L$.

The linear-programming approximation to (11) is

Subject to

\[ p(q) \geq 0, \]
\[ \sum_{q \in Q} p(q) = 1, \]
\[ \sum_{q \in Q} p(q) t_j^i f_j^i(q_j) \leq a_i \quad (i = 1, \ldots, m), \]

minimize

\[ \sum_{q \in Q} p(q) c_j q_j \quad (q \in Q). \]

This problem may also be solved by the simplex method using multipliers. The master problem will have $m + 1$ constraints -- the first of which is the convexity constraint, and the remaining $m$ of which are the performance constraints for supporting each of the $m$ weapon systems. The side problem takes the form

Minimize

\[ \sum_{j} c_j q_j + \sum_{i} \alpha_i \sum_{j} t_j^i f_j^i(q_j) \quad (q \in Q), \]

where $\alpha^i$ is the dual variable associated with the $i$th weapon system constraint. (Again, we have reversed the usual dual variable sign conventions, so as to make the $\alpha^i \geq 0$.)
Again, this side problem separates into n problems:

Minimize

\[ c_j q_j + \sum_i \alpha_i \beta_j(i) \frac{q_j}{j} \]  \( j = 1, \ldots, n; q_j = 0, \ldots, L \).  

We have had no computational experience with the multi-weapon-system problem; however, it is so mathematically similar to the M-kit problem that we should encounter no surprises in programming it for the computer.

C. Base Stockage of Recoverable Items and the WRM Supplemental Thereto

This problem arose out of conversations with ADC. Spare parts stocked at a base are normally broken into two parts. The first consists of normal base stockage to be used in day-to-day operations; the second consists of WRM (war reserve materiel). The WRM portion may be designed to be mobile, in which case it is an M-kit (problem A), or it may be designed for use at the base in the event of war. It is the latter type of WRM that we are considering here.

Current Air Force policy is that WRM is not to be used in day-to-day operations (except, perhaps, to alleviate or prevent a NORS condition). The normal base stock levels and the WRM-kits are designed in isolation from each other. In the problem formulation to follow we still assume that the WRM-kit is inviolate under peacetime conditions, but we take into account the role of the normal base stock levels in supplementing WRM under wartime conditions.

*Actually there are two types of M-kits. One kind is prepositioned, in which case weight and cube are not so important; the other kind is deployed with the aircraft. Problem A considered this second kind.
Let \( f_j(q_j) \) be the nonfills per unit time on item \( j \) under normal operations when \( q_j \) is item \( j \)'s stock level. \( f_j(q_j) \) may be computed as in the base stockage model. Let \( \psi_j(k, q_j) \) be the stationary probability of having exactly \( k \) units of item \( j \) on hand, under normal operations, if \( q_j \) is its stock level. Finally let \( \varphi_j(k) \) be the probability of having \( k \) demands or less for item \( j \) under wartime conditions (we assume that all resupply and parts repair is suspended during the war). We will use nonfills under normal conditions as a measure of peacetime performance, and probability of meeting all demands as a wartime measure of performance. Then our peacetime performance criterion is

\[
\Sigma_j f_j(q_j),
\]

where \( q_j \) is the stock level for item \( j \). Under wartime conditions, the probability of meeting all demands for item \( j \) will be

\[
\sum_{k=0}^{\infty} \psi_j(k, q_j) \varphi_j(k + r_j),
\]

where \( q_j \) is the stock level for item \( j \) and \( r_j \) is the quantity of item \( j \) in the WRM-kit. Assuming that demands are independent from item to item yields the following measure of wartime performance:

\[
\prod_j \sum_{k=0}^{\infty} \psi_j(k, q_j) \varphi_j(k + r_j).
\]

We are thus led to the following minimization problem:
Subject to

\[ \sum_{j} f_j(q_j) \leq a, \]

(15) \[ \prod_{j} \mathbb{R}^+ k \geq 0, k, q_j \varphi_j(k + r_j) \geq b', \]

minimize

\[ \sum_{j} c_j (r_j + q_j) \quad (q, r) \in Q, \]

where \( Q \) is now the set of all \( 2n \)-tuples \( (q, r) = (q_1, \ldots, q_n, r_1, \ldots, r_n) \) of nonnegative integers no more than a large integer \( L \), \( c_j \) is the cost of item \( j \), \( f_j, \varphi_j \) are as defined above, \( a \) is the performance target for peacetime operations, and \( b' \) is the performance target for wartime operations.

Let

\[ h_j(q_j, r_j) = -\ln \sum_{k} \varphi_j(k, q_j) \varphi_j(k + r_j), \]

and \[ b = -\ln b'. \]

An equivalent problem to (15) is then

Subject to

\[ \sum_{j} f_j(q_j) \leq a, \]

(16) \[ \sum_{j} h_j(q_j, r_j) \leq b, \]

minimize

\[ \sum_{j} c_j (q_j + r_j) \quad ((q, r) \in Q). \]
The linear programming approximation to (16) is

Subject to

\[ p(q, r) \geq 0 \quad ((q, r) \in Q), \]

\[ \sum_{(q, r) \in Q} p(q, r) = 1, \]

\[ \sum_{(q, r) \in Q} p(q, r) \sum_j f_j(q_j) \leq a, \]

\[ \sum_{(q, r) \in Q} p(q, r) \sum_j h_j(q_j, r_j) \leq b, \]

minimize

\[ \sum_{(q, r) \in Q} p(q, r) \sum_j c_j(q_j + r_j) \quad (p \in Q^Q). \]

After applying the simplex method using multipliers, we obtain a master problem with three constraints: a convexity constraint, a peacetime performance constraint, and a wartime performance constraint. The side problem takes the form

Minimize

\[ \sum_j c_j(q_j + r_j) + \alpha \sum_j f_j(q_j) + \beta \sum_j h_j(q_j, r_j) \quad ((q, r) \in Q), \]

where \( \alpha \geq 0 \) is the dual variable associated with the peacetime constraint, and \( \beta \) is associated with the wartime constraint. This separates, yielding \( n \) problems:
Minimize

\[ c_j(q_j + r_j) + \alpha f_j(q_j) + \beta h_j(q_j, r_j) \]

\[ (j = 1, \ldots, n; q_j = 0, 1, \ldots, L; r_j = 0, 1, \ldots, L). \]

D. Base Stockage of EOQ Items

In the problems thus far we have been considering reparable items where a reorder quantity of one is best. For cheaper, throw-away items, the cost incurred in the requisitioning process may be significant relative to the investment cost in inventory. For such items we wish to establish not only reorder points, but also reorder quantities. An EOQ policy is characterized by the average investment in stock that it implies, the average number of shortages per unit time, and the average number of requisitions that must be made. Let \( q_j \) be \( 1 + \) the reorder point for item \( j \), and \( r_j \) be the order quantity. An optimization problem involving these three policy characteristics takes the following form:

Subject to

\[ \Sigma_j s_j(q_j, r_j) \leq a, \]

\[ \Sigma_j w_j(r_j) \leq b, \]

minimize

\[ \Sigma_j c_j(q_j, r_j) \quad ((q, r) \in Q), \]

* This problem, based on the work of John Lu, will be described in a forthcoming Memorandum.
where \( s_j(q_j, r_j), w_j(r_j), \) and \( c_j(q_j, r_j) \) measure shortages per unit time, orders per unit time, and average investment, respectively, for item \( j \), and \( Q \) is the set of all \( 2n \)-tuples \( (q_1, \ldots, q_n, r_1, \ldots, r_n) \) of nonnegative integers \( q_j, r_j \leq \) a large number.

We may approximate (20) by a linear programming problem:

Subject to

\[
P(q, r) \geq 0 \quad ((q, r) \in Q),
\]

\[
\sum_{(q, r) \in Q} p(q, r) = 1,
\]

\[
\sum_{(q, r) \in Q} p(q, r) \sum_j s_j(q_j, r_j) \leq a,
\]

\[
\sum_{(q, r) \in Q} p(q, r) \sum_j w_j(r_j) \leq b,
\]

minimize

\[
\sum_{(q, r) \in Q} p(q, r) c_j(q_j, r_j) \quad (p \in R^Q).
\]

When we solve (21) by the simplex method using multipliers we have a master problem with three constraints (convexity, shortages, and orders), and a side problem that separates into \( n \) side problems -- one for each item \( j = 1, \ldots, n \):

Minimize

\[
\alpha s_j(q_j, r_j) + \beta w_j(q_j, r_j) + c_j(q_j, r_j)
\]

(22)

\( (0 \leq q_j, r_j \leq \text{large number}) \).
This optimization procedure is now imbedded in John Lu's EOQ model. Running time on the IBM 7044 has varied between one and three minutes to compute an EOQ policy for about 18,000 items grouped into about 800 categories. Although this running time seems perfectly satisfactory, it is considerably more than the running times for the M-kits. This is due largely to the fact that minimizing a function of two integer variables (22) is considerably more time-consuming than minimizing a function of one integer variable (10).
III. MULTIBASE PROBLEMS

E. Single-echelon Multibase Stockage
   and Procurement of Recoverable Items

   This problem has arisen during implementation of the RAND base
   stockage model (BSM). In using the BSM for computing base stockage
   requirements it might well happen that for some items the existing
   Air Force assets greatly exceed the computed base stockage requirement.
   In these cases it is desirable to have a method to distribute some
   of these assets to the bases and thereby obtain target base supply-
   support effectiveness with a smaller additional procurement of other
   assets. In short, we would like to have a method for substituting
   assets in long supply for assets in short supply when computing base
   stockage requirements.

   Let $f_j^i(q_j^i)$ be the expected number of nonfills per unit time on
   item $j$ at base $i$ when base $i$ has a stock level of $q_j^i$ for item $j$,
   $i = 1, \ldots, m = \text{number of bases}$, $j = 1, \ldots, n = \text{number of items}$. Let
   $a_i$ be the target expected number of nonfills per unit time at base $i$,
   $b_j$ be the supply of item $j$ that is currently available for base distri-
   bution, and $c_j$ be the market cost of item $j$. If $r_j$ of item $j$ is
   procured and $q_j^i$ is the stock level for item $j$ at base $i$, then $\Sigma c_j r_j$
   will be the cost of the policy $\{r_j\}, \{q_j^i\}$, $\Sigma f_j^i(q_j^i)$ will be the
   effectiveness of the policy at base $i$, and $\Sigma q_j^i - r_j - b_j$ will be
   the amount by which the requirement for item $j$ exceeds available
   assets plus procurement. The optimization problem is
Subject to

\[ \sum_{j} f_{j}^{i}(q_{j}^{i}) \leq a^{i} \quad (i = 1, \ldots, m), \]

(23)

\[ \sum_{i} q_{j}^{i} - r_{j} \leq b_{j} \quad (j = 1, \ldots, n), \]

minimize

\[ \sum_{j} c_{j} r_{j} \quad ((r, q) \in Q), \]

where \( Q \) is the set of all pairs \((r, q) = ([r_{j}], [q_{j}^{i}]), [r_{j}]\) is an \( n \)-tuple of nonnegative integers no more than a large number \( L \), and \([q_{j}^{i}]\) is an \( m \times n \) matrix of nonnegative integers no more than \( L \).

The linear programming approximation to (23) is

Subject to

\[ p(r, q) \geq 0 \quad ((r, q) \in Q), \]

\[ (r, q) \in Q \quad p(r, q) = 1, \]

(24)

\[ (r, q) \in Q \quad p(r, q) \sum_{j} f_{j}^{i}(q_{j}^{i}) \leq a^{i} \quad (i = 1, \ldots, m), \]

\[ (r, q) \in Q \quad p(r, q) \left( \sum_{i} q_{j}^{i} - r_{j} \right) \leq b_{j} \quad (j = 1, \ldots, n), \]

minimize

\[ \sum_{(r, q) \in Q} p(r, q) \sum_{j} c_{j} r_{j} \quad (p \in R^{Q}). \]
A straightforward application of the simplex method using multipliers to (24) is practically infeasible. Although \( m \), the number of bases, is unlikely to exceed 200 (a manageable number), \( n \), the number of items, could be as large as 90,000 and will almost always be several thousands. Thus the number of constraints in (24) is huge. By a little manipulation, however, we can reformulate (24) so as to arrive at a linear program that has only \( m + 1 \) constraints. When we apply the simplex method using multipliers to this new problem we will obtain a manageable master problem with \( m + 1 \) constraints, and a side problem that separates into \( n \) problems, each of which is a manageable 2-constraint linear programming problem that, in turn, can be solved by the simplex method using multipliers. To do this, let \( P \subset R^Q \) be the set of all \( p \in R^Q \) satisfying

\[
p(r, q) \geq 0 \quad ((r, q) \in Q),
\]

\[
(r, q) \in Q, p(r, q) = 1,
\]

\[
(r, q) \in Q, \Sigma_{i} q_i (r, q) - r_j \leq b_j \quad (j = 1, \ldots, n).
\]

Then (24) is equivalent to

Subject to

\[
\xi(p) \geq 0 \quad (p \in R^Q),
\]

\[
\Sigma_{p} \xi(p) = 1,
\]

(25)

\[
\Sigma_{p} \xi(p) (r, q) \in Q, p(r, q) \Sigma_{j} f_{j}^{i}(q_{j}) \leq a_{i} \quad (i = 1, \ldots, m),
\]

minimize
\[ \sum_{p \in P} \xi(p) (r, q) \in Q \ p(r, q)^{\sum_{j} c_j r_j} \quad (\xi \in \mathbb{R}^P). \]

To see that (24) and (25) are equivalent, suppose first that \( p^* \) is a solution to (24). Then \( p^* \in P \), so we may define \( \xi \in \mathbb{R}^P \) by

\[ \xi^*(p) = \begin{cases} 
1, & \text{if } p = p^* \\
0, & \text{if } p \neq p^* 
\end{cases}. \]

Then \( \xi^* \) is easily seen to be a solution to (25). Conversely, suppose that \( \xi^* \in \mathbb{R}^P \) is a solution to (25). Then we may define \( p^* \in \mathbb{R}^Q \) by

\[ p^*(r, q) = \sum_{p \in P} \xi^*(p) p(r, q), \]

and \( p^* \) will be a solution to (24).

Applying the simplex method using multipliers to (25) we obtain a master problem with \( m + 1 \) constraints -- the first of which is a convexity constraint, and the remaining \( m \) of which are performance constraints for the \( m \) bases. The side problem is

Minimize

\[ \sum_{(r, q) \in Q} p(r, q) (\sum_{j} c_j r_j + \sum_{i} c_i f_i(q_j)) \quad (p \in P), \]

which, from the definition of \( P \), may be rewritten as

Subject to

\[ p(r, q) \geq 0 \quad ((r, q) \in Q), \]

\[ \sum_{(r, q) \in Q} p(r, q) = 1, \]
(26) \( (\tilde{r}_j, \tilde{q}_j) \in Q \ p(r, q) \ (\Sigma_{i} q_{j}^{i} - r_{j}) \leq b_{j} \quad (j=1, \ldots, n), \)

minimize

\[ \Sigma_{j} (\tilde{r}_j, \tilde{q}_j) \in Q \ p(r, q) \Sigma_{j} (c_{j} \tilde{r}_{j} + \Sigma_{i} a_{i} \tilde{r}_{j}^{i} (q_{j}^{i})) \quad (p \in R_{Q}). \]

Now let \( Q_{j} \) be the set of all \((m+1)\)-tuples \((r_{j}, q_{j}) = (r_{j}, q_{j}^{1}, \ldots, q_{j}^{m})\).

Then (26) is equivalent to

Subject to

\[ p_{j}(r_{j}, q_{j}) \geq 0 \quad (j=1, \ldots, n; p_{j} \in R_{Q_{j}}), \]

(27) \( (r_{j}, q_{j}) \in Q_{j} \ p_{j}(r_{j}, q_{j}) = 1 \quad (j=1, \ldots, n), \)

minimize

\[ \Sigma_{j} (r_{j}, q_{j}) \in Q_{j} \ p_{j}(r_{j}, q_{j}) \ (\Sigma_{i} q_{j}^{i} - r_{j}) \leq b_{j} \quad (j=1, \ldots, n), \]

\[ \Sigma_{j} (r_{j}, q_{j}) \in Q_{j} \ p_{j}(r_{j}, q_{j}) \ (c_{j} r_{j} + \Sigma_{i} a_{i} r_{j}^{i} (q_{j}^{i})) \quad (p_{j} \in R_{Q_{j}}, j=1, \ldots, n). \]

To see that (26) and (27) are equivalent, suppose first that \( p^{*} \in R_{Q} \) is a solution to (26). Define \( p_{j}^{*} \in R_{Q_{j}}, j=1, \ldots, n, \) by

\[ p_{j}^{*}(r_{j}, q_{j}) = \Sigma p^{*}(r, q) \]

\[ \{(r, q) \in Q \mid r_{j} = \tilde{r}_{j}, \ q_{j}^{i} = q_{j}^{i}, \ i=1, \ldots, m\} \]

for \( j = 1, \ldots, n. \) Then \( (p_{1}^{*}, \ldots, p_{n}^{*}) \) is easily seen to be a solution to (27). Conversely, suppose that \( (p_{1}^{*}, \ldots, p_{n}^{*}) \) is a solution to (27).
Define \( p^* \in \mathbb{R}^Q \) by

\[
p^*(r, q) = \prod_j p_j^*(r_j, q_j, \ldots, q_j).
\]

Then \( p^* \) becomes a solution to (26).

Now note that (27) separates into \( n \) two-constraint problems, \( j = 1, \ldots, n \).

Subject to

\[
p_j(r_j, q_j) \geq 0 \quad ((r_j, q_j) \in Q_j),
\]

\[
(r_j, q_j) \in Q_j p_j(r_j, q_j) = 1, 
\]

\[
(r_j, q_j) \in Q_j p_j(r_j, q_j) (\Sigma_j q_j^i - r_j) \leq b_j, 
\]

minimize

\[
(r_j, q_j) \in Q_j p_j(r_j, q_j) (c_j q_j + \Sigma_i r_j^i (q_j^i)) 
\]

\[
(p_j \in \mathbb{R}^Q). 
\]

Applying the simplex method using multipliers to (28) we obtain a two-constraint master problem and a side problem of the form

Minimize

\[
(c_j - \beta_j) r_j + \Sigma_i \alpha_j r_j^i (q_j^i) + \beta_j q_j^i 
\]

\[
((r_j, q_j) \in Q_j). 
\]

This last problem separates into \( m + 1 \) problems:

Minimize
\[(c_j - \beta_j)r_j \quad (r_j = 0, 1, \ldots, L).\]

(30)

Minimize

\[\alpha_i f_j^i(q_j^i) + \beta_j q_j^i \quad (i = 1, \ldots, m; \ q_j^i = 0, \ldots, L).\]

F. Single-echelon Multibase Stockage of Recoverable Items Without Additional Procurement

In the previous problem, we assumed that Air Force repairable assets could be augmented, at a cost, by additional procurement. This assumption is valid in computing base stockage requirements. In setting base stockage levels for distribution purposes, however, it is desirable to have a method that will set levels for items that are consistent with current assets without additional procurement. For this problem, there is no investment cost to be minimized, and there is no opportunity to meet performance targets via additional investment. One of many ways of formulating the problem is as follows:

Subject to

\[\sum_j f_j^i(q_j^i) \leq a_i \quad (i = 1, \ldots, m),\]

(31)

\[\sum_j q_j^i \leq b_j \quad (j = 1, \ldots, n),\]

minimize

\[\sum_i \sum_j f_j^i(q_j^i),\]

where \(q_j^i\) is the stock level of item \(j\) at base \(i\), \(f_j^i(x)\) is the nonfills per unit time on item \(j\) at base \(i\) with stock level of \(x\), \(a_i\) is the nonfill target for base \(i\), and \(b_j\) is the amount of asset \(j\) available for base distribution.
Approximate (31) by a linear programming problem. As in the previous section, this may be expressed as an \( m + 1 \) constraint problem:

Subject to

\[ \xi(p) \geq 0 \quad (p \in P), \]

\[ \sum_{p \in P} \xi(p) = 1, \]

\[ \sum_{p \in P} \xi(p) \sum_{q \in Q} p(q) \Sigma f^i_j(q^i_j) \leq a^i, \]  \hspace{1cm} (32)

minimize

\[ \sum_{p \in P} \xi(p) \sum_{q \in Q} p(q) \Sigma f^i_j(q^i_j) \quad (\xi \in \mathbb{R}^P), \]

where \( Q \) is the set of \( m \times n \) matrices \( q = [q^i_j] \) of nonnegative integers \( q^i_j \leq \) a large number, and \( P \subset \mathbb{R}^Q \) is the set of all \( p \in \mathbb{R}^P \) satisfying

\[ p(q) \geq 0 \quad (q \in Q), \]

\[ \sum_{q \in Q} p(q) = 1, \]

\[ \sum_{q \in Q} p(q) \Sigma q^i_j \leq b^i \quad (j=1, \ldots, n). \]

When we solve (32) by the simplex method using multipliers, we obtain a master problem involving \( m + 1 \) constraints and a side problem that separates into \( n \) two constraint problems:

Subject to

\[ p^i_j(q^i_j) \geq 0 \quad (q_j \in Q_j), \]
\[ \sum_{q_j \in Q_j} p_j(q_j) = 1, \]

(33)

\[ \sum_{q_j \in Q_j} p_j(q_j) \sum_i q_{ij}^i \leq b_j, \]

minimize

\[ \sum_{q_j \in Q_j} (\alpha_i^i + 1) \xi_j^i(q_j^i) \quad (p_j \in R^{Q_j}), \]

where \( j = 1, ..., n \), \( Q_j \) is the set of \( m \)-tuples \((q_j^1, ..., q_j^m)\) of non-negative integers \( \leq \) large number, and \( \alpha_i^i \) is the dual variable associated with the performance constraint on the \( i^{th} \) base.

When we solve each problem in (33) by the simplex method using multipliers, we obtain a master problem with two constraints, and \( m \) side problems:

Minimize

\[ (\alpha_i^i + 1) \xi_j^i(q_j^i) + \beta_j q_j^i \quad (0 \leq q_j^i \leq \text{large number}), \]

where \( i = 1, ..., m \).

G. Base and Depot Stockage of Recoverable Items With Additional Procurement

In addition to establishing individual base levels on each item, we also wish to establish depot levels and generate requirements for each item. We wish to minimize total procurement cost subject to performance (which we will measure in back orders on base supply) constraints on the bases. Sherbrooke [5] has shown that under suitable
assumptions * the back orders on base i for item j depend only on the level of item j at base i and the level of item j at the depot. Let $f_j^i(q^i_j, d_j)$ be the back orders at base i on item j when base i has a level of $q^i_j$ on item j, and the depot has a level of $d_j$ on item j.

Let $c_j$ be the procurement cost of one unit of item j. We wish then to solve the following problem:

Subject to

$$\sum_j f_j^i(q^i_j, d_j) \leq a^i \quad (i=1, \ldots, m),$$

(35)

$$\sum_i q^i_j + d_j - r_j \leq b_j \quad (j=1, \ldots, n),$$

minimize

$$\sum_j c_j r_j \quad \text{for } (q, d, r) \in Q,$$

where Q is the set of all triples $(q, d, r)$, $q = [q^i_j]$ is an $m \times n$ matrix of nonnegative integers ≤ large number, and $d = [d_j]$ and $r = [r_j]$ are n-tuples of such integers.

We will present two linear programming approximations to (35). For the first, let $Q' \subset Q$ be all triples $(q, d, r)$ satisfying

$$\sum_i q^i_j + d_j - r_j \leq b_j \quad (j=1, \ldots, n).$$

Then we approximate (35) by

*For example, no lateral resupply among bases.
Subject to

\[ p(q,d,r) \geq 0 \quad ((q,d,r) \in Q'), \]

\[ \Sigma_{(q,d,r) \in Q'} p(q,d,r) = 1, \]

\[ \Sigma_{(q,d,r) \in Q'} p(q,d,r) \Sigma_j f^i_j(q^i_j,d_j) \leq a^i \quad (i=1, \ldots, m), \]

minimize

\[ \Sigma_{(q,d,r) \in Q'} p(q,d,r) \Sigma_j c_j r_j \quad (p \in R^{Q'}). \]

This formulation leads to a master problem with \( m + 1 \) constraints and a side problem that separates into \( n \) problems -- one for each item \( j = 1, \ldots, n \):

Subject to

\[ \Sigma_i q^i_j + d_j - r_j \leq b_j, \]

minimize

\[ \Sigma_i \alpha^i \Sigma_j f^i_j(q^i_j,d_j) + c_j r_j \quad ((q_j,d_j,r_j) \in Q_j), \]

where \( Q_j \) is the set of all triples \( (q_j,d_j,r_j) \) wherein \( q_j = (q^1_j, \ldots, q^m_j) \) is an \( m \)-tuple of nonnegative integers \( \leq \) large number, and \( r_j \) and \( d_j \) are the same sort of integers.

A procedure similar to the one in [5] may be used to solve (37). Since however, (37) must be solved for a number of different \( \alpha^i \), \( i=1, \ldots, m \),
and all values of \( j = 1, \ldots, n \), the amount of computation involved might be prohibitive. A procedure that might be less time consuming is as follows. Approximate (35) by

Subject to

\[
\xi(p) \geq 0 \quad (p \in P),
\]

\[
\sum_{p \in P} \xi(p) = 1,
\]

(38) \[
\sum_{p \in P} \xi(p) \Sigma(q,d,r) \in Q \quad p(q,d,r) \Sigma_j c_j d_j \leq a_i
\]

\[
(i = 1, \ldots, m),
\]

minimize

\[
\sum_{p \in P} \xi(p) \Sigma(q,d,r) \in Q \quad p(q,d,r) \Sigma_j c_j d_j
\]

\[
(\xi \in R^p),
\]

where \( P \) is the set of all \( p \in R^Q \) satisfying

\[
p(q,d,r) \geq 0 \quad ((q,d,r) \in Q),
\]

\[
\Sigma(q,d,r) \in Q \quad p(q,d,r) = 1,
\]

\[
\Sigma(q,d,r) \in Q \quad p(q,d,r)(\Sigma_j q_j^i + d_j - r_j) \leq b_j \quad (j = 1, \ldots, n).
\]

This formulation leads (as in problem E) to a master problem involving \( m + 1 \) constraints, and a side problem that separates into \( n \) problems, one for each item \( j = 1, \ldots, n \):
Subject to

\[ p_j(q_j, d_j, r_j) = 0 \quad \left((q_j, d_j, r_j) \in Q_j, \right) \]

\[ \Sigma (q_j, d_j, r_j) \in Q_j, p_j(q_j, d_j, r_j) = 1, \]

\[ \Sigma (q_j, d_j, r_j) \in Q_j, p_j(q_j, d_j, r_j) \left(\Sigma_i q_j^i + d_j - r_j\right) \leq b_j, \]

minimize

\[ \Sigma (q_j, d_j, r_j) \in Q_j, p_j(q_j, d_j, r_j) \left(\Sigma_i a_j^i f_j(q_j^i, d_j) + c_j r_j\right) \quad (p_j \in R^{Q_j}). \]

Each of the n problems (39) may be solved by the simplex method using multipliers. In doing this one has (for each \( j = 1, \ldots, n \)) a master problem with two constraints, and a side problem of the form

Minimize

\[ \beta_j \left(\Sigma_i q_j^i + d_j - r_j\right) + \Sigma_i a_j^i f_j(q_j^i, d_j) + c_j r_j \quad \left((q_j, r_j, d_j) \in Q_j, \right) \]

where \( \beta_j \) is the dual variable associated with the last constraint in (39). This separates for each \( j = 1, \ldots, n \) into two minimization problems:

Minimize

\[ (c_j - \beta_j) r_j \quad (0 \leq r_j \leq \text{large number}), \]
\[(42) \quad \$_j (\Sigma_i q^i_j + d_j) + \Sigma_i \alpha^i f^i_j(q^i_j, d_j) \quad (0 \leq r_j, q^i_j \leq \text{large number}).\]

Problem (41) is easy (in economic terms, buy only when shadow cost exceeds market cost). Problem (42) may present more difficulties, since it involves a function of two variables, \(f^i_j(q^i_j, r_j)\). For fixed \(d_j\), however, it separates into \(m\) easy minimizations (\(i = 1, \ldots, m\)). One might therefore solve (42) for fixed \(d_j\), and only vary \(d_j\) when the solution for a new set of dual variables is no better than the last solution (using the new duals to make the comparison).

H. Base and Depot Stockage of Recoverable Items
   Without Additional Procurement

This problem bears the same relation to \(G\) as \(F\) did to \(E\). We wish to distribute existing assets among bases and depots in such a way as to minimize total base-level backorders, subject to backorder constraints at each of the bases. The original problem is:

Subject to

\[\Sigma_j f^i_j(q^i_j, d_j) \leq a^i \quad (i = 1, \ldots, n),\]

\[\Sigma_i q^i_j + d_j \leq b_j,\]

minimize

\[\Sigma_i \Sigma_j f^i_j(q^i_j, d_j) \quad ((q, r) \in Q),\]

where \(Q\) is now the set of all pairs \((q, r)\) wherein \(q = \{q^i_j\}\) is an \(m \times m\) matrix of nonnegative integers \(\leq \) a large number and \(d = \{d_j\}\) is an
n-tuple of such integers. As in the previous problem, this problem has two possible linear programming approximations. The formulation of these approximations is probably best left, at this point, to the reader.
APPENDIX A

We first show that if the policies in the support of a solution to (2) are good approximations to one another, then any one of them is a good approximation to a solution to (1).

PROPOSITION 1. Suppose (1) has a solution \( q^* \in Q \) and (2) has a solution \( p^* \in \mathbb{R}^Q \). Suppose also that for some positive numbers \( \epsilon^1, \ldots, \epsilon^{m+1} \) we have

\[
(i) \quad | a^i(q) - a^i(q') | \leq \epsilon^i \quad (i = 1, \ldots, m+1)
\]

whenever \( p^*(q) > 0 \) and \( p^*(q') > 0 \). Then

\[
(ii) \quad a^i(q) \leq b^i + \epsilon^i \quad (i = 1, \ldots, m)
\]

and

\[
(iii) \quad a^{m+1}(q) \leq a^{m+1}(q^*) + \epsilon^{m+1}
\]

whenever \( p^*(q) > 0 \).

Proof. Since \( p^* \) is a solution to (2), \( p^*(q') \geq 0 \) for all \( q \in Q \).

Thus (i) implies that

\[
\sum_{q'} \in Q p^*(q)(a^i(q) - a^i(q')) \leq \sum_{q'} \in Q p^*(q') \epsilon^i \quad (i = 1, \ldots, m+1).
\]

Since \( p^* \) is a solution to (2), \( \sum_{q} \in Q p^*(q) = \sum_{q'} \in Q p^*(q') = 1 \), so the above inequality yields

\[
(iv) \quad a^i(q) - \sum_{q'} \in Q p^*(q') a^i(q') \leq \epsilon^i \quad (i = 1, \ldots, m+1);
\]

but \( p^* \) is a solution to (2) and therefore satisfies
\[ \sum_{q'} \in Q \; p^*(q') a^i(q') \leq b^i \quad (i = 1, \ldots, m). \]

When we add our last two inequalities for \( i = 1, \ldots, m \) we obtain (ii).

Now define \( p \in R^Q \) by

\[
p(q) = \begin{cases} 
1, & \text{if } q = q^* \\
0, & \text{if } q \neq q^*.
\end{cases}
\]

Then, since \( q^* \) satisfies the constraints in (1), it follows that \( p \) satisfies the constraints in (2). Thus, since \( p^* \) solves (2), we have

\[ \sum_{q} \in Q \; p^*(q) a^{m+1}(q) \leq \sum_{q} \in Q \; p(q) a^{m+1}(q). \]

But \( \sum_{q} \in Q \; p(q) a^{m+1}(q) = a^{m+1}(q^*) \) so the last inequality becomes

\[ \sum_{q} \in Q \; p^*(q) a^{m+1}(q) \leq a^{m+1}(q^*). \]

This inequality may be added to (iv), for \( i = m + 1 \), to yield (iii).

Q.E.D.

We next show that for the problems considered in this Memorandum, (2) is likely to have a solution whose support contains policies that are good approximations to one another and therefore (in view of Proposition 1), good approximations to a solution to (1). All the problems in this paper are separable in the following sense: The set \( Q \) of policies may be expressed as the cartesian product \( Q = Q_1 \times \ldots \times Q_n \) of a number of sets, where \( Q_j \) is the set of policies that may be followed on the \( j \)th item, and the functions \( a^i, i = 1, \ldots, m + 1 \) may be written in the form
\[ a^i(q_1, \ldots, q_n) = \sum_{j=1}^{m+1} a^i_j(q_j) \quad (i = 1, \ldots, m + 1), \]

where \( a^i_j : Q_j \to \mathbb{R} \) is a real valued function on \( Q_j \) for each \( i = 1, \ldots, m + 1 \) and each \( j = 1, \ldots, n \). The sets \( Q_j \) will be called cells. Two policies \( (q_1, \ldots, q_n), (q'_1, \ldots, q'_n) \in Q \) agree on the \( j \)th cell if \( q_j = q'_j \).

We may rewrite (2) as

Subject to

\[ p(q) \geq 0 \quad (q \in Q), \]

\[ \sum_{q \in Q} p(q) = 1, \]

(43)

\[ \sum_{q \in Q} p(q) \sum_{j=1}^{m+1} a^i_j(q_j) \leq b^i \quad (i = 1, \ldots, m), \]

minimize

\[ \sum_{q \in Q} p(q) \sum_{j=1}^{m+1} a^i_j(q_j) \quad (p \in \mathbb{R}^Q). \]

PROPOSITION 2. Suppose (43) has a solution and \( n > m \). Then it has a solution \( p^* \) with the property that there are at least \( n-m \) cells \( Q_{j_1}, \ldots, Q_{j_{n-m}} \), such that for any two \( q, q' \) in the support of \( p^* \), \( q \) and \( q' \) agree on \( Q_{j_1}, \ldots, Q_{j_{n-m}} \).

Proof. Consider the following problem.

Subject to

\[ p_j(q_j) \geq 0 \quad (j = 1, \ldots, n; \ q_j \in Q_j), \]

\[ \sum_{j=1}^{n} p_j(q_j) = 1 \quad (j = 1, \ldots, n), \]

(44)

\[ \sum_{j=1}^{n} p_j(q_j) \sum_{j=1}^{m+1} a^i_j(q_j) \leq b^i \quad (i = 1, \ldots, m), \]
minimize
\[ \sum_{j} p_j(q_j) \sum_{q_j \in Q_j} a_{j, q_j}^{m+1}(q_j) \quad (j = 1, \ldots, n; \ p_j \in \mathbb{R}^{Q_j}). \]

Suppose that \( p^* \in \mathbb{R}^Q \) is a solution to (43). Define functions
\[ p_j^*(q_j') = \sum_{q_j \in Q_j} p_j^*(q_j) \quad (q_j' \in Q_j), \]
where the summation is extended over all n-tuples \( (q_1, \ldots, q_n) \in Q \) such that \( q_j = q_j' \). Then it is easily seen that \( p_1^*, \ldots, p_n^* \) form a solution to (44). Conversely, if \( p_1^*, \ldots, p_n^* \) form a solution to (44), then the function \( p^* \in \mathbb{R}^Q \) defined by
\[ p^*(q_1, \ldots, q_n) = \prod_{j=1}^{n} p_j^*(q_j) \]
is a solution to (43).

Suppose now that (43) has a solution. Then, since there are \( m + n \) constraints in (44), it follows that (44) has a solution \( p_1^*, \ldots, p_n^* \) with the property that \( p_j^*(q_j) > 0 \) for at most \( m + n \) values of \( j \) and \( q_j \). Now in view of the convexity constraints in (44), for each \( j \) there is at least one \( q_j \in Q_j \) such that \( p_j^*(q_j) > 0 \). Thus there are at most \( m \) values of \( j \) for which \( p_j^*(q_j) > 0 \) for more than one \( q_j \in Q_j \), i.e., there are \( j_1, \ldots, j_{n-m} \), such that the support of \( p_j^* \) has only one element,

\[ k = 1, \ldots, n-m. \]

Let \( p^* \) be the solution to (43) obtained from \( p_1^*, \ldots, p_n^* \) by means of (ii). Then any two \( q, q' \) in the support of \( p^* \) agree on the cells \( Q_{j_1}, \ldots, Q_{j_{n-m}} \). Q.E.D.
Now for all of the problems considered in this Memorandum, the number \( n \) of items is large in relation to the number \( m + 1 \) of policy characteristics, \( a^1, \ldots, a^{m+1} \), so Proposition 2 says that if (43) has a solution, it has a solution \( p^* \) such that any two policies in the support of \( p^* \) agree on all but a few \( (m) \) items. Thus if the policy chosen for any one item has little effect on the overall policy characteristics (this is the case for the problems we are considering), then the policies in the support of \( p^* \) will be good approximations to one another and hence, by Proposition 1, good approximations to a solution to (1).
APPENDIX B

Assume that demands levied by weapon system $i$ on base supply for a particular item are generated by a compound Poisson process with customer arrival rate $\lambda_i$ and compounding distribution $g$. Assume also that the compounding distribution $g$ is the same from one weapon system to another. Finally assume that base supply issues on a first-come, first-served basis. Let $E_i(t)$ be the steady state expected number of nonfills on demands levied by weapon system $i$ during a period of length $t$, and let $E(t) = \sum_i E_i(t)$ be the expected number of nonfills across all weapon systems during the same period. We claim that

\begin{equation}
E_i(t) = \left(\frac{\lambda_i}{\lambda}\right)E(t),
\end{equation}

where $\lambda = \sum_i \lambda_i$. To see this let $P_k$ be the steady state probability of $k$ items on the shelf. Then the probability of weapon system $i$ levying exactly $r$ demands in excess of available assets during a period of time $h$ is

$$
\lambda_i h \sum_{k=0}^{\infty} P_k g(k+r) + R(h) \quad (r > 0),
$$

where

$$
\lim_{h \to 0} \frac{R(h)}{h} = 0.
$$

Thus, since $E_i(t+h) = E_i(t) + E_i(h)$, we have

$$
E_i(t+h) - E_i(t) = \lambda_i h \sum_{r=1}^{\infty} \sum_{k=0}^{\infty} P_k g(k+r) + R^*(h),
$$

where $\lim_{h \to 0} R^*(h)/h = 0$. Dividing both sides of the last equation by $h$ and letting $h \to 0$ we have
\[ E_{i}'(t) = \lambda_i \sum_{r=1}^{\infty} r \sum_{k=0}^{\infty} P_k g(k+r). \]

Clearly \( E_{i}(0) = 0 \), so when we integrate both sides of the last equation over the interval from 0 to \( t \) we have

\[ E_{i}(t) = \lambda_i t \sum_{r=1}^{\infty} r \sum_{k=0}^{\infty} P_k g(k+r). \]

Similar reasoning shows

\[ E(t) = \lambda t \sum_{r=1}^{\infty} r \sum_{k=0}^{\infty} P_k g(k+r). \]

So dividing the first of these last two equations by the second we have

\[ \frac{E_{i}(t)}{E(t)} = \frac{\lambda_i}{\lambda}, \]

which is equivalent to (45).

If we assume \( \lambda \) to be random variable, but assume the ratios \( \frac{\lambda_i}{\lambda} \) to be known, (45) is still valid.
REFERENCES


3. -----, Chapter 22.

