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AN APPLICATION OF THE WIENER-HERMITE EXPANSION TO THE DIFFUSION OF A PASSIVE SCALAR IN A HOMOGENEOUS TURBULENT FLOW

P. G. Saffman

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The RAND Corporation
SANTA MONICA • CALIFORNIA

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PREFACE

A difficult and recurrent problem in many applications of fluid mechanics is the description of the convection of a passive scalar quantity in a turbulent flow field. Examples of the diverse areas in which this problem arises are in reentry physics, when electron concentration is effected by wake turbulence, or in pollution studies, when the passive scalar, pollutant concentration, is transported by the turbulent flow field of the surrounding medium.

A theoretical investigation of this problem is presented in this Memorandum, using a systematic approximation based on an expansion in Weiner-Hermite functionals. This approximation leads to analytic solutions to two crucial problems in the theory of turbulent diffusion: the diffusion of a passive scalar from a point source, and the spectrum of a statistically homogeneous scalar -- both in a homogeneous turbulent flow.

The research described here is part of RAND's work on reentry aerodynamics for the Advanced Research Projects Agency.

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SUMMARY

The velocity and concentration fields in a homogeneous turbulent flow convecting a passive scalar are expanded in series of Wiener-Hermite functionals. The equations of motion lead to integro-differential equations for the coefficients of the expansion. The expansion and the equations are truncated at the first stage that gives nontrivial results, and analytical solutions are obtained for the concentration field with the velocity field given and with zero molecular conductivity.

The diffusion from a point source is considered first. It is shown that the mean concentration satisfies an integro-differential equation with diffusive and wave-like properties, which reduces for large time-values to the diffusion equation in accordance with the classical theory of G. I. Taylor. The effective diffusivity is determined in terms of Eulerian velocity correlations.

The spectrum of a statistically homogeneous distribution of the scalar is considered second. It is shown that if the spectrum is initially concentrated around the wave number of the energy-containing eddies, an equilibrium is set up for larger wave numbers, with

$$\Gamma(k) = \frac{\bar{\theta}^2}{3u^2} E(k)$$

where $\Gamma(k)$ and $E(k)$ are the concentration and energy spectra, respectively, and $\bar{\theta}^2$ and u^2 are the mean-square concentration and the mean-square velocity component. Combined with a modification, to take account of small-scale intermittency, of Batchelor's theory of the spectrum for very large wave numbers, this equation is used to predict the decay rate of the concentration intensity.

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I. INTRODUCTION

The diffusion and mixing of convected quantities in a turbulent flow is a subject of great importance and has attracted much attention. For a passive scalar in an incompressible fluid with uniform properties, the concentration $\theta(\underline{x}, t)$ satisfies the equation

$$\frac{\partial \theta}{\partial t} + \underline{u} \cdot \nabla\theta = \kappa \nabla^2 \theta \qquad (1)$$

where $\underline{u}(x, t)$ is the velocity field and κ is the molecular diffusivity. Given the velocity field $\underline{u}$, Eq. (1) is linear, unlike the nonlinear Navier-Stokes equations for the velocity $\underline{u}$ itself. But this advantage is apparent rather than real, and a linear equation with random coefficients seems to present as hard a problem as a nonlinear one.

The problem has two distinct aspects. First, there is the question of how an inhomogeneous distribution is mixed and transported by the fluid motion. This can be subdivided further into diffusion from a point source (sometimes called one-particle analysis), first studied by Taylor,[1] and the spreading of a cloud (two-particle analysis), first investigated by Richardson.[2] A survey of this work is to be found in an article by Batchelor and Townsend.[3] There is also the difficult matter of deciding to what extent the mixing due to turbulence can be described by an eddy diffusivity, and how the eddy diffusivity (if it is a reasonable approximation) is related to the statistics of the velocity field. The second aspect concerns the intensity decay and spectrum of a homogeneous distribution in an approximately homogeneous flow. The pioneering work on this subject was done by Obukhov and Corrsin, and a unified treatment has been given by Batchelor.[4]

Although intensive effort on these problems has uncovered many details of the processes and provided not too unsatisfactory empirical laws for practical applications in certain cases, a basic understanding of the fundamental questions is still seriously incomplete. Part of the difficulty, but by no means all, lies in ignorance about the velocity field itself.

For diffusion from a point source in an isotropic flow, Taylor[1] showed in effect that for large values of time the mean concentration is described by the heat diffusion equation

$$\frac{\partial \bar{\theta}}{\partial t} + \underline{U} \cdot \nabla \bar{\theta} = \varkappa_e \nabla^2 \bar{\theta} \qquad (2)$$

where $\underline{U}$ is the constant mean velocity and $\varkappa_e$ is an effective diffusivity, related to the Lagrangian auto-correlation of the velocity following a fluid particle. The definition and relation are

$$u^2 R_L(\tau) = \langle u(t)u(t+\tau) \rangle, \qquad \varkappa_e = u^2 \int_0^\infty R_L(\tau)\, d\tau \qquad (3)$$

where u(t) is the component of velocity fluctuation of a fluid particle and u is the root mean square. One of the unsolved problems of turbulence is the relation between the Lagrangian auto-correlation and the more readily measured Eulerian correlations, although crude theoretical estimates have been made in profusion.[5,6]

All previous work on the spectrum and decay of a homogeneous distribution has been based on the hypothesis of a cascade and the independence of large and small wave numbers, and has been closely related to the Kolmogorov universal-equilibrium theory. However, the Kolmogorov theory itself has been both modified[7] and questioned[8] in recent years, and it seems that the evidence in its favor is by no means as strong or as convincing as was once supposed. Thus it is by no means certain that the applications and extensions of the Kolmogorov hypothesis to turbulent diffusion are firmly based.

The main purpose of the present Memorandum is to examine the results of applying to these two problems -- the diffusion from a point source and the spectrum and decay of homogeneous fluctuations -- a truncated Wiener-Hermite expansion of the velocity and concentration fields. This method has recently been applied by Meecham and Jeng[9] to the decay of isotropic turbulence with results that are not too bad. However, there are doubts about the convergence and completeness of the Wiener-Hermite expansion, and it has recently been pointed out[10] that in

homogeneous turbulence the truncation prevents energy transfer to high wave numbers if the energy spectrum has a cutoff at the initial instant. However, the present writer believes that the objections to the method arise because the critics take literally the assertion by the method's developers that turbulence can be regarded as a small perturbation about a normally distributed velocity field. This interpretation is clearly untenable, both on general theoretical grounds and because numerical work shows that terms initially small do not remain small, and is rightly criticized. On the other hand, the method can be interpreted in the spirit of the integral methods used in kinetic theory, boundary-layer theory, etc. That is, a finite truncation of the Wiener-Hermite expansion can be regarded as an approximation to the velocity field, the approximation being determined by the requirement that it satisfy certain integrals of the Navier-Stokes equations, in a manner similar, for example, to the Mott-Smith or Karman-Pohlhausen approximations.* It goes without saying that the results of such an analysis must be inconclusive, since in the complete absence of any reliable theory there is nothing with which they can be convincingly compared. But it is precisely because the prospects of a completely sound theory are so distant (and the experimental situation is far from satisfactory) that the method is worth pursuing. In principle, at least, the accuracy can be improved by taking more terms; a virtue that some of the other crude approximations do not possess. Furthermore, the method is "honest" in the sense that preconceived physical ideas are not built into or hidden in the mathematical approximations.

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*A general name for such approximations, which we henceforth adopt, is "Galerkin methods."

II. THE WIENER-HERMITE EXPANSION

The basic idea of the method is due to Wiener,[11] and was applied to turbulence-type problems by Meecham and Siegel[12] and others. The technique is founded on the white noise or ideal random function $a(x)$, which is the derivative of a continuous random walk and has the property that $a(x)$ is a normally distributed random function with zero mean and

$$\langle a(x)a(x') \rangle = \delta(x - x') \tag{4}$$

One then considers the stationary random function defined by

$$f(x) = K^{(0)}H^{(0)} + \int K^{(1)}(x - x')H^{(1)}(x')\, dx'$$

$$+ \iint K^{(2)}(x - x', x - x'')H^{(2)}(x', x'')\, dx'\, dx'' + \ldots \tag{5}$$

where the Wiener-Hermite functions are statistically orthonormal combinations of $a(x)$ with

$$H^{(0)} = 1, \quad H^{(1)}(x) = a(x), \quad H^{(2)}(x, x') = a(x)a(x') - \delta(x - x'), \text{ etc.} \tag{6}$$

The kernel functions K are nonrandom ordinary functions. It has been shown[13] that the expansion Eq. (5) is complete, but the question of how one determines in practice the kernel functions, given the probability distribution of f, seems unanswered, except when f is itself normally distributed and the problem is trivial. If f satisfies an equation of motion, substitution of Eq. (5) and equating coefficients of the $H^{(n)}$ gives equations for the nonrandom $K^{(m)}$. Since in general there is an infinite number of coupled equations, the usual difficulty of turbulence theory is not avoided, but the hope is that approximate solutions will give useful results.

For the problem of turbulent diffusion, we shall take the ideal random function to be a random vector function of both position vector

$\underline{x}$ and time t. We then consider the normally distributed vector function $a_i(\underline{x}, t)$ with the property

$$\left\langle a_i(\underline{x}, t)a_j(\underline{x}, t) \right\rangle = \delta_{ij}\delta(\underline{x} - \underline{x}')\delta(t - t') \tag{7}$$

We now expand the random concentration $\theta(\underline{x}, t)$ as a series of statistically orthonormal Wiener-Hermite functions obtained by generalizing Eq. (6):

$$\theta(\underline{x}, t) = \Theta(\underline{x}, t) + \int C_k(\underline{x}, \underline{\xi}, t, \tau)a_k(\underline{\xi}, \tau)\, d\underline{\xi}d\tau$$

$$+ \int D_{k\ell}(\underline{x}, \underline{\xi}, \underline{\xi}', t, \tau, \tau')$$

$$\{a_k(\underline{\xi}, \tau)a_\ell(\underline{\xi}', \tau') - \delta_{k\ell}\delta(\underline{\xi} - \underline{\xi}')\delta(\tau - \tau')\}d\underline{\xi}d\underline{\xi}'d\tau d\tau' + \ldots \tag{8}$$

The scalar, vector, and tensor kernels Θ, C_k, $D_{k\ell}$, etc., are ordinary nonrandom functions of their arguments. So far no assumptions have been made about statistical stationarity. Mean values should be interpreted as ensemble averages. Clearly

$$\left\langle \theta(\underline{x}, t) \right\rangle = \Theta(\underline{x}, t) \tag{9}$$

However, there are no analogous relations for mean values of products of θ; for instance, $\left\langle \theta^2 \right\rangle$ involves integrals of kernels of all orders. In this respect the Wiener-Hermite expansion is clumsy. Apparently it has not been proved for random functions of more than one variable that the expansion Eq. (8) exists and is complete, but the proof[13] for functions of one variable should be readily generalizable.

The main applications of the method so far reported[9,12] have not used the expansion Eq. (8), but an alternative form in which the ideal random function is independent of t and a function of $\underline{x}$ only. Now for the problem of decaying turbulence, in which the statistical

properties are given at t = 0 and one tries to calculate how mean values change with time, it would appear that an expansion in the restricted Wiener-Hermite functionals based on $a_i(\underline{x})$ is equally appropriate and mathematically complete, and that the time dependence is adequately described by the variation of the kernels with t. The method thus suffers in this case from a lack of uniqueness. Indeed, for decaying turbulence where the time dependence is imposed by the equations of motion and not by the statistics of the ensemble, it would appear that we are free to choose ideal random functions of $\underline{x}$ with an arbitrary dependence on t. Thus an infinite number of different expansions may be possible, for each of which the kernels would have a different t dependence. The relationship between different expansions has not been examined analytically. It is reasonable to suppose that all expansions will give the same final answers if the infinite number of equations is solved exactly. However, truncations of the expansions will give different results. (Similar remarks can be made for the application of the method to inhomogeneous turbulent flows which are statistically stationary in time.) Clearly, we would like to choose an expansion for which the Wiener-Hermite expansion converges as rapidly as possible, in order that the error in the Galerkin method is small. It is not known how such a choice should be made, but it appears that choosing $a_i(\underline{x}, t)$ to be a stationary random function of t is less restrictive and probably "safer" than taking it to be independent of t; since the latter choice ignores the fact that any realization will drift away from its initial state, thus requiring more and more higher-order kernels to describe it in terms of the initial statistics as t increases. In the present Memorandum we shall use the expansion Eq. (8) with ideal random functions that satisfy Eq. (7), and leave for further study the question of whether the results are better or worse than those that might be obtained with a different time dependence of the ideal random function.

III. DIFFUSION FROM A POINT SOURCE

We apply the method first to the one-particle diffusion problem. We examine the solutions of Eq. (1) with $\underline{u}(\underline{x}, t)$ a given stationary random function of $\underline{x}$ and t, and the initial condition that at $t = 0$, $\theta = \delta(\underline{x})$; i.e., a blob is released at the origin at time zero. In order to restrict the algebra, we shall retain in the Galerkin method the smallest number of terms that gives nontrivial results. Thus we shall assume that the velocity field has a Gaussian distribution with zero mean; i.e.,

$$u_i(\underline{x}, t) = \int K_{ij}(\underline{x} - \underline{\xi}, t - \tau) a_j(\underline{\xi}, \tau)\, d\underline{\xi}\, d\tau \tag{10}$$

where K_{ij} is a given function. For the concentration, we take as the approximation to its distribution the first two terms,

$$\theta(\underline{x}, t) = \Theta(\underline{x}, t) + \int C_k(\underline{x}, \underline{\xi}, t, \tau) a_k(\underline{\xi}, \tau)\, d\underline{\xi}\, d\tau \tag{11}$$

where Θ and C_k are to be found. The kernel K_{ij} in Eq. (10) is a function of $\underline{x} - \underline{\xi}$ and $t - \tau$ because the velocity field is statistically stationary in both $\underline{x}$ and t; the concentration does not have this property.

Following the spirit of the Galerkin method, we substitute Eqs. (10) and (11) into the equation of motion, Eq. (1), and obtain equations for Θ and C_k by imposing the condition that the result of the substitution should have zero mean and should be statistically orthogonal to $a_j(\underline{x}, t)$. After some algebra, we obtain

$$\frac{\partial \Theta}{\partial t}(\underline{x}, t) + \int K_{ij}(\underline{x} - \underline{\xi}, t - \tau) \frac{\partial C_j}{\partial x_i}(\underline{x}, \underline{\xi}, t, \tau)\, d\underline{\xi}\, d\tau = \nu \frac{\partial^2 \Theta}{\partial x_i^2}(\underline{x}, t) \tag{12}$$

and

$$\frac{\partial C_j}{\partial t}(\underline{x}, \underline{\xi}, t, \tau) + K_{ij}(\underline{x} - \underline{\xi}, t - \tau) \frac{\partial}{\partial x_i} \Theta(\underline{x}, t) = \varkappa \frac{\partial^2}{\partial x_i^2} C_j(\underline{x}, \underline{\xi}, t, \tau) \tag{13}$$

The initial conditions for these equations are, at $t = 0$,

$$\Theta = \delta(\underline{x}), \qquad C_j = 0 \tag{14}$$

For the sake of simplicity we now put $\varkappa = 0$, and thus neglect the effect of molecular diffusivity. For diffusion from a point source, the molecular transport is generally negligible. Integrating Eq. (13) with respect to t and differentiating with respect to $\underline{x}$, we obtain

$$\frac{\partial C_j}{\partial x_i}(\underline{x}, \underline{\xi}, t, \tau) = - \int_o^t \frac{\partial}{\partial x_i} \left\{ K_{kj}(\underline{x} - \underline{\xi}, t' - \tau) \frac{\partial}{\partial x_k} \Theta(\underline{x}, t') \right\} dt' \tag{15}$$

Further,

$$\frac{\partial K_{ij}}{\partial x_i}(\underline{x} - \underline{\xi}, t - \tau) = 0 \tag{16}$$

because div $\underline{u} = 0$. Substitution of Eq. (15) into Eq. (12) now gives (with $\varkappa = 0$)

$$\frac{\partial}{\partial t} \Theta(\underline{x}, t)$$

$$= \frac{\partial}{\partial x_i} \int_o^t dt' \frac{\partial \Theta}{\partial x_k}(\underline{x}, t') \int K_{ij}(\underline{x} - \underline{\xi}, t - \tau) K_{kj}(\underline{x} - \underline{\xi}, t' - \tau) \, d\underline{\xi} \, d\tau \tag{17}$$

Now From Eq. (10), we have

$R_{ik}(t - t')$

$$\equiv \left\langle u_i(\underline{x}, t)u_k(\underline{x}, t') \right\rangle = \int K_{ij}(\underline{x} - \underline{\xi}, t - \tau)K_{kj}(\underline{x} - \underline{\xi}, t' - \tau)\, d\underline{\xi}\, d\tau \tag{18}$$

where R_{ik} is the Eulerian one-point, two-time velocity covariance tensor (the velocities are measured in a frame with zero mean velocity). Substituting Eq. (18) into Eq. (17), we obtain

$$\frac{\partial \Theta}{\partial t}(\underline{x}, t) = \frac{\partial^2}{\partial x_i \partial x_k} \int_o^t \Theta(\underline{x}, t')R_{ik}(t - t')\, dt' \tag{19}$$

as an integro-differential equation for the mean concentration Θ.

When Θ is found from Eq. (19), C_j follows from Eq. (15) and the structure of the concentration field θ is given. Thus the Galerkin method applied to the Wiener-Hermite expansion yields relatively simple equations without too much labor, at least if molecular transport is neglected. The question remains whether the results are reasonable.

Let T_E be the time scale of the Eulerian covariance. It could be defined, for instance, by

$$T_E = \int_o^\infty R_{ii}(t)\, dt/R_{ii}(0) \tag{20}$$

Then for $t \gg T_E$ it is almost obvious that Eq. (19) reduces to

$$\frac{\partial \Theta}{\partial t} = D_{ik} \frac{\partial^2}{\partial x_i \partial x_k} \Theta \tag{21}$$

where

$$D_{ik} = \int_o^\infty R_{ik}(t)\, dt \tag{22}$$

Thus for large time-values the diffusion of mean concentration is given by a heat-diffusion equation whose effective diffusivity is given by Eq. (22). This result is consistent with that obtained by Taylor[1] [see Eq. (3)] as generalized to three-dimensional diffusion, and says that the integrated Lagrangian correlation coefficient is given by the integrated Eulerian correlation at a fixed point (i.e., at a point moving with the mean velocity). This conclusion is plausible and reinforces the belief that the Wiener-Hermite expansion can give reasonable results.

On the other hand, for small t we can replace $R_{ik}(t - t')$ by $R_{ik}(0) = \langle u_i u_k \rangle$, and on differentiating Eq. (19) we obtain

$$\frac{\partial^2 \Theta}{\partial t^2} = \langle u_1 u_k \rangle \frac{\partial^2 \Theta}{\partial x_i \partial x_k} \tag{23}$$

i.e., for small time-values Θ satisfies a wave equation. For the sake of illustration, it is useful to consider isotropic turbulence for which

$$R_{ik}(t) = u^2 \delta_{ik} R_E(t) \tag{24}$$

Then Eq. (23) becomes

$$\frac{\partial^2 \Theta}{\partial t^2} = u^2 \nabla^2 \Theta \tag{25}$$

which has the solution

$$\Theta = \frac{-1}{4\pi r} \delta'(r - ut) \tag{26}$$

where $r = |\underline{x}|$ and δ' is the derivative of the (one-dimensional) Dirac δ-function. However, the solution [Eq. (26)] is physically unreasonable, because it predicts that the mean concentration lies only on a spherical surface of radius ut, and also gives both positive and negative

values. Thus for small time-values the Wiener-Hermite truncation is apparently unreliable.*

It is worth examining this difficulty a little further to show that it is not a consequence of the approximation that led to the wave equation [Eq. (23)]. We take Laplace transforms (denoted by a tilde)

$$\tilde{\Theta}(\underline{x},\ p) \ = \ \int_0^\infty e^{-pt}\Theta(\underline{x},\ t)\ dt \tag{27}$$

etc. Then Eq. (19) becomes (for isotropic turbulence)

$$u^2\tilde{R}_E(p)\nabla^2\tilde{\Theta}(\underline{x},\ p) - p\tilde{\Theta}(\underline{x},\ p) \ = \ -\delta(\underline{x}) \tag{28}$$

with the solution

$$\tilde{\Theta}(\underline{x},\ p) \ = \ \frac{1}{4\pi u^2\tilde{R}_E(p)}\,\frac{e^{-kr}}{r}, \qquad k \ = \ \left[\frac{p}{u^2\tilde{R}_E(p)}\right]^{1/2} \tag{29}$$

As $p \to \infty$, $\tilde{R}_E(p) \to p^{-1}$, and it follows from the properties of Laplace transforms that Θ is discontinuous at $r = ut$. Thus the wave property of Eq. (19) persists for all time, but if $t^{1/2} \gg T_E^{1/2}$ the concentration is spread out over a distance $u(tT_E)^{1/2}$, which is small compared with the cutoff radius, and the nonphysical cutoff is not significant. (But strangely enough, casual observation shows that the edge of a cloud of diffusing material is sharp.)

If we take, for the sake of illustration,

$$R_E(t) \ = \ e^{-t/T_E}, \qquad \tilde{R}_E(p) \ = \ \frac{1}{p + T_E^{-1}} \tag{30}$$

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*This difficulty is not due to the use of an ideal random function that fluctuates in time. The wave equation is also obtained if one uses a time-independent, ideal random function of position.

we find after some analysis that

$$\Theta(\underline{x},\ t) = \frac{e^{-t/2T_E}}{4\pi u^2 r}\left\{\delta'(t - \frac{r}{u}) + \delta(t - \frac{r}{u})\left\{\frac{T_E + r/u}{8T_E^2}\right\}\right.$$

$$\left. + H(t - \frac{r}{u})\left[I_1\left(\frac{\sqrt{t^2 - \frac{r^2}{u^2}}}{2T_E}\right) + \frac{t}{\sqrt{t^2 - \frac{r^2}{u^2}}} I_2\left(\frac{\sqrt{t^2 - \frac{r^2}{u^2}}}{2T_E}\right)\right]\frac{r}{4uT_E^2}\right\} \tag{31}$$

where H is the Heaviside function and I_υ is the modified Bessel function of order υ. This expression agrees in the limits $t \to \infty$ and $t \to 0$ with the values obtained by approximating the equation, and confirms the conclusions reached above.

However, the inadequacy of the solution for small t may not result from a general failure of the method, but rather from the fact that the method is unsuitable for an initial concentration in which all the material is concentrated at one point, since two terms of the expansion are insufficient to describe the distribution when it is highly concentrated. The Galerkin method may work better when the material is initially more diffuse or is closer to a concentration of a type that exists some time after the initial conditions. In other words, although the solution of Eq. (19) with initial condition Eq. (14) may not describe adequately for small t the real diffusion of a point source, the error may cancel out when this solution is used as the Green's function to calculate the spread of an initially diffuse cloud.

To demonstrate this argument, consider the <u>exact</u> solution for small t:

$$\theta(\underline{x},\ t) = \theta\left(\underline{x} - \underline{u}^{(0)}t - \left\{\frac{\partial \underline{u}^{(0)}}{\partial t} - \underline{u}^{(0)} \cdot \nabla \underline{u}^{(0)}\right\}\frac{t^2}{2},\quad 0\right) + O(t^3) \tag{32}$$

where $\underline{u}^{(0)} = \underline{u}(\underline{x}, 0)$ and $\theta(\underline{x}, 0) = \theta^{(0)}$ is the initial concentration. Suppose also that the initial concentration is statistically uncorrelated with the velocity field. Expanding Eq. (32) and taking mean values, we obtain

$$\Theta(\underline{x}, t) = \Theta^{(0)} + \frac{t^2}{2} \left\langle u_i^{(0)} u_j^{(0)} \right\rangle \frac{\partial^2 \Theta^{(0)}}{\partial x_i \partial x_j} + \ldots$$

$$= \Theta^{(0)} + \frac{u^2 t^2}{2} \nabla^2 \Theta^{(0)} + O(t^3) \tag{33}$$

for isotropic turbulence. Now we solve the problem using the Wiener-Hermite truncation; i.e., we solve Eqs. (12) and (13) with the initial condition $\theta = \Theta^{(0)}$. For small time-values, Eq. (26) gives the fundamental solution (since the equations for θ are linear), and

$$\Theta(\underline{x}, t) = - \int \frac{1}{4\pi r} \delta'(r - ut) \Theta(\underline{\xi}, 0) \, d\underline{\xi} \tag{34}$$

where $\underline{r} = \underline{x} - \underline{\xi}$. Expanding $\Theta(\underline{\xi}, 0)$ as a power series about $\underline{x}$ and integrating, we find that

$$\Theta(\underline{x}, t) = \frac{-1}{4\pi} \int \frac{\delta'(r - ut)}{r} \left[\Theta^{(0)} + r_i \frac{\partial \Theta^{(0)}}{\partial x_i} + \frac{1}{2} r_i r_j \frac{\partial^2 \Theta^{(0)}}{\partial x_i \partial x_j} + \ldots \right] d\underline{r}$$

$$= \Theta^{(0)} + \frac{1}{2} u^2 t^2 \nabla^2 \epsilon^{(0)} + \ldots \tag{35}$$

Thus for small time-values, the Wiener-Hermite expansion and the exact solution agree to order t^2 when the initial distribution is continuous rather than concentrated at a point.

There is therefore reason to believe that the method is useful when applied with care and in appropriate circumstances. Since it is only an approximation, it is unreasonable to expect more, or that it should always give good results even if misused.

We can use the results of our analysis to find expressions for the Lagrangian auto-correlation coefficient. It has been shown[6] that

$$u^2 R_L(t) = \int \left\langle u_1(\underline{x}, t) u_1(0, 0) \theta(\underline{x}, t) \right\rangle d\underline{x} \tag{36}$$

where θ is the solution of the diffusion equation with $\theta = \delta(\underline{x})$ at $t = 0$; for simplicity we suppose the turbulence is isotropic. Substituting the expansion Eqs. (10) and (11) into Eq. (36), we obtain

$$u^2 R_L(t) = \int \left\langle u_1(\underline{x}, t) u_1(0, 0) \right\rangle \Theta(\underline{x}, t)\, d\underline{x} \tag{37}$$

where Θ is the solution of Eq. (19) with initial condition Eq. (14). Thus, given the two-point, two-time Eulerian correlation, Eq. (37) yields the Lagrangian auto-correlation. For consistency, it should be possible to show directly from Eq. (37) that

$$\int_0^\infty R_L(t)\, dt = \int_0^\infty R_E(t)\, dt \tag{38}$$

but this problem has so far defied attack.

The formula [Eq. (37)] has the same structure as a hypothesis sometimes called "Corrsin's conjecture."[6] There is an important difference, however, since in Corrsin's conjecture the exact mean value should be substituted, whereas in the present analysis we should substitute the approximate value of Θ given by the analysis.

For the problem of diffusion from a point source, it is easy to interpret the Wiener-Hermite truncation in terms of approximations made directly to the convection equation. Thus, if we write

$$\theta(\underline{x}, t) = \Theta(x, t) + \phi(\underline{x}, t) \tag{39}$$

Eqs. (12) and (13) are obtained on substituting the complete, nontruncated, Wiener-Hermite expansion of θ together with the velocity given by Eq. (10) in the pair of equations

$$\frac{\partial \Theta}{\partial t} + \left\langle u_i \frac{\partial \phi}{\partial x_i} \right\rangle = \kappa \nabla^2 \Theta \tag{40}$$

and

$$\frac{\partial \phi}{\partial t} + u_i \frac{\partial \Theta}{\partial x_i} = \kappa \nabla^2 \phi \tag{41}$$

Now Eq. (40) is exact, and is obtained by taking the mean of Eq. (1). Then subtracting Eq. (40) from Eq. (1) gives

$$\frac{\partial \phi}{\partial t} + u_i \frac{\partial \Theta}{\partial x_i} - \kappa \nabla^2 \phi = u_i \frac{\partial \phi}{\partial x_i} - \left\langle u_i \frac{\partial \phi}{\partial x_i} \right\rangle \tag{42}$$

which demonstrates the error in Eq. (41). In other words, use of the Wiener-Hermite truncation is equivalent to neglecting the fluctuations in the convection of the random part of the field, and essentially replaces $u_i \, \partial\phi/\partial x_i$ by its mean value. For $\kappa = 0$, Eq. (41) can be integrated with respect to t, and substitution into Eq. (40) gives Eq. (19) directly.* It is curious that Eq. (41), which seems a natural approximation, does not appear to have been used before in the present context.

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*I am grateful to Dr. S. C. Crow for pointing out this alternative, and indeed more general, method of obtaining Eq. (19).

IV. DECAY OF HOMOGENEOUS FLUCTUATIONS

We now apply the method to the decay of statistically homogeneous fluctuations of concentration in a statistically homogeneous and stationary velocity field. Again we retain only the minimum number of terms necessary to give nontrivial results. The velocity field is given by Eq. (10). For the concentration field, we now use

$$\theta(\underline{x}, t) = \int C_k(\underline{x} - \underline{\xi}, t, \tau) a_k(\underline{\xi}, \tau)\, d\underline{\xi}\, d\tau + \int D_{k\ell}(\underline{x} - \underline{\xi}, \underline{x} - \underline{\eta}, t, \tau, \tau') H^{(2)}_{k\ell}(\underline{\xi}, \underline{\eta}, \tau, \tau')\, d\underline{\xi}\, d\underline{\eta} d\tau d\tau' \tag{43}$$

where

$$H^{(2)}_{k\ell}(\underline{\xi}, \underline{\eta}, \tau, \tau') = a_k(\underline{\xi}, \tau) a_\ell(\underline{\eta}, \tau') - \delta_{k\ell}\delta(\underline{\xi} - \underline{\eta})\delta(\tau - \tau') \tag{44}$$

Since we wish mean properties to be independent of position, we suppose that the mean concentration is zero and that the kernels are functions of $\underline{x} - \underline{\xi}$ and $\underline{x} - \underline{\eta}$.

We substitute Eqs. (10) and (43) into the diffusion equation and require that the resulting expression be statistically orthogonal to the Wiener-Hermite functionals of the first and second order. After some reduction, and use of the relationship

$$H^{(2)}_{k\ell}(\underline{\xi}, \underline{\eta}, \tau, \tau') a_i(\underline{\delta}, \tau'') = H^{(3)}_{k\ell i} + a_k(\underline{\xi}, \tau)\delta_{i\ell}\delta(\underline{\eta} - \underline{\zeta})\delta(\tau' - \tau'') + a_\ell(\underline{\eta}, \tau')\delta_{ik}\delta(\underline{\xi} - \underline{\zeta})\delta(\tau - \tau'') \tag{45}$$

we find that

$$\frac{\partial}{\partial t} C_k(\underline{x} - \underline{\xi}, t, \tau)$$

$$= - \int K_{ij}(\underline{x} - \underline{\eta}, t - \tau') \frac{\partial}{\partial x_i} D_{kj}(\underline{x} - \underline{\xi}, \underline{x} - \underline{\eta}, t, \tau, \tau') \, d\underline{\eta} \, d\tau'$$

$$- \int K_{ij}(\underline{x} - \underline{\eta}, t - \tau') \frac{\partial}{\partial x_i} D_{jk}(\underline{x} - \underline{\eta}, \underline{x} - \underline{\xi}, t, \tau', \tau) \, d\underline{\eta} \, d\tau'$$

$$+ \nu \frac{\partial^2}{\partial x_i^2} C_k(\underline{x} - \underline{\xi}, t, \tau) \tag{46}$$

$$\frac{\partial}{\partial t} D_{kj}(\underline{x} - \underline{\xi}, \underline{x} - \underline{\eta}, t, \tau, \tau') = -K_{ij}(\underline{x} - \underline{\eta}, t - \tau') \frac{\partial}{\partial x_i} C_k(\underline{x} - \underline{\xi}, t, \tau)$$

$$+ \nu \frac{\partial^2}{\partial x_i^2} D_{kj}(\underline{x} - \underline{\xi}, \underline{x} - \underline{\eta}, t, \tau, \tau') \tag{47}$$

[There is some arbitrariness in the tensor D_{kj}, since, as is clear from Eq. (43), an arbitrary antisymmetrical tensor may be added to D_{kj} without changing θ. It appears to be more convenient not to impose the requirement of symmetry and to define D_{kj} by Eq. (47).]

These equations for the unknown kernel functions C_k and $D_{k\ell}$ are not simple. To evaluate the present approach, we shall assume for simplicity that $\nu = 0$ and that the variables are statistically isotropic. Now, by change of variable and use of the equation of continuity, we can rewrite Eqs. (46) and (47) as

$$\frac{\partial}{\partial t} C_i(\underline{x}, t, \tau)$$

$$= - \frac{\partial}{\partial x_k} \int K_{kj}(\underline{\xi}, t - \tau') \left\{ D_{ij}(\underline{x}, \underline{\xi}, t, \tau, \tau') + D_{ji}(\underline{\xi}, \underline{x}, t, \tau', \tau) \right\} d\underline{\xi} \, d\tau' \tag{48}$$

and

$$\frac{\partial}{\partial t} D_{ij}(\underline{x}, \underline{\xi}, t, \tau, \tau') = -K_{\ell j}(\underline{\xi}, t - \tau') \frac{\partial}{\partial x_\ell} C_i(\underline{x}, t, \tau) \quad (49)$$

respectively.

Let us consider now the distribution at time t, given that at the initial instant t = 0 the non-Gaussian part of the concentration was zero. Then Eq. (49) can be integrated with respect to t and the result substituted in Eq. (48). After some reduction we obtain

$$\frac{\partial}{\partial t} C_i(\underline{x}, t, \tau) = \nabla^2 \int_0^t R_E(t - t') C_i(\underline{x}, t', \tau)\, dt' \quad (50)$$

where

$$\delta_{ij} R_E(t - t') = \left\langle u_i(\underline{x}, t) u_j(\underline{x}, t') \right\rangle \quad (51)$$

is the Eulerian one-point, two-time covariance, and we have used Eq. (16) and the principle of isotropy, which implies that

$$\int K_{kj}(\underline{\xi}, t - \tau') \frac{\partial}{\partial \xi_\ell} C_j(\underline{\xi}, t', \tau')\, d\underline{\xi}\, d\tau' = \delta_{k\ell} F(t, t')$$

Notice that the second time-variable τ, which relates the concentration to the representations of the ideal random functions at different times, only enters parametrically into the equation. After Eq. (50) has been solved for C_i, Eq. (49) determines the D_{ij}. It is not surprising that the Gaussian part of the concentration field satisfies the same equation as the mean concentration in the diffusion from a point-source problem. The remarks made in Section III about the equation still apply, and we expect the results of the present analysis to have some physical significance.

We now turn to the correlation and spectrum functions of $\theta(\underline{x}, t)$. After reduction, it is found from Eq. (43) that

$$\left\langle \theta(\underline{x})\theta(\underline{x}+\underline{r}) \right\rangle = S(\underline{r}, t)$$

$$= \int C_j(\underline{\xi}, t, \tau)C_j(\underline{\xi}+\underline{r}, t, \tau)\, d\underline{\xi}\, d\tau$$

$$+ \int D_{j\ell}(\underline{\xi}, \underline{\eta}, t, \tau, \tau')D_{j\ell}(\underline{\xi}+\underline{r}, \underline{\eta}+\underline{r}, t, \tau, \tau')d\underline{\xi}d\underline{\eta}d\tau d\tau'$$

$$+ \int D_{j\ell}(\underline{\xi}, \eta, t, \tau, \tau')D_{\ell j}(\underline{\eta}+\underline{r}, \underline{\xi}+r, t, \tau', \tau)d\underline{\xi}d\underline{\eta}d\tau d\tau' \tag{52}$$

It can be verified from Eqs. (48) and (49) that

$$\frac{\partial}{\partial t} S(0, t) = \frac{\partial}{\partial t} \left\langle \theta^2 \right\rangle = 0 \tag{53}$$

Thus the truncated equations preserve mean-square concentrations as they should, since θ has been chosen to be statistically orthogonal to $D\theta/Dt$.

Expressing the right-hand side of Eq. (52) in terms of the Fourier transforms, we find that

$$S(\underline{r}, t) = (2\pi)^3 \int \tilde{C}_j(\underline{k}, t, \tau)\tilde{C}_j(-\underline{k}, t, \tau)e^{i\underline{k}\cdot r}\, d\underline{k}\, d\tau$$

$$+ (2\pi)^6 \int \tilde{D}_{j\ell}(\underline{k}, \underline{k}', t, \tau, \tau')\tilde{D}_{j\ell}(-\underline{k}, -\underline{k}', t, \tau, \tau')e^{i(\underline{k}+\underline{k}')\cdot\underline{r}}d\underline{k}d\underline{k}'d\tau d\tau'$$

$$+ (2\pi)^6 \int \tilde{D}_{j\ell}(\underline{k}, \underline{k}', t, \tau, \tau')\tilde{D}_{\ell j}(-\underline{k}', -\underline{k}, t, \tau', \tau)e^{i(\underline{k}+\underline{k}')\cdot\underline{r}}d\underline{k}d\underline{k}'d\tau d\tau' \tag{54}$$

where

$$\tilde{C}_j(\underline{k}, t, \tau = \frac{1}{(2\pi)^3} \int C_j(\underline{x}, t, \tau)e^{-i\underline{k}\cdot\underline{x}}\, d\underline{x} \tag{55}$$

$$\tilde{D}_{j\ell}(\underline{k}, \underline{k}', t, \tau, \tau') = \frac{1}{(2\pi)^6} \int D_{j\ell}(\underline{x}, \underline{\xi}, t, \tau, \tau')e^{-i\underline{k}\cdot\underline{x}-i\underline{k}'\cdot\underline{\xi}}\, d\underline{x}d\underline{\xi} \tag{56}$$

The equations for the Fourier transforms are

$$\frac{\partial}{\partial t}\tilde{C}_j(\underline{k}, t, \tau) = -ik_p(2\pi)^3 \int \tilde{K}_{p\ell}(-\underline{k}', t - \tau') \left\{\tilde{D}_{j\ell}(\underline{k}, \underline{k}', t, \tau, \tau') + D_{\ell j}(\underline{k}', \underline{k}, t, \tau', \tau)\right\} d\underline{k}' d\underline{\tau}' \tag{57}$$

$$\frac{\partial}{\partial t}\tilde{D}_{j\ell}(\underline{k}, \underline{k}', t, \tau, \tau') = -ik_p\tilde{K}_{p\ell}(\underline{k}', t - \tau')\tilde{C}_j(\underline{k}, t, \tau) \tag{58}$$

from which we deduce that

$$\frac{\partial}{\partial t}\tilde{C}_j(\underline{k}, t, \tau) = -k^2 \int_0^t R_E(t - t')\tilde{C}_j(k, t', \tau)\, dt' \tag{59}$$

If we define a spectrum function for the concentration distribution by

$$\Delta(\underline{k}, t) = \frac{1}{(2\pi)^3} \int S(\underline{r}, t)e^{-i\underline{k}\cdot\underline{r}}\, d\underline{r} \tag{60}$$

it follows from Eq. (54) that

$$\Delta(\underline{k}, t) = (2\pi)^3 \int \tilde{C}_j(\underline{k}, t, \tau)\tilde{C}_j(-\underline{k}, t, \tau)\, d\tau$$

$$+ (2\pi)^6 \int \tilde{D}_{j\ell}(\underline{k}', \underline{k} - \underline{k}', t, \tau, \tau')\tilde{D}_{j\ell}(-\underline{k}', -\underline{k} + \underline{k}', t, \tau, \tau')\, d\underline{k}' d\tau d\tau'$$

$$+ (2\pi)^6 \int \tilde{D}_{j\ell}(\underline{k}', \underline{k} - \underline{k}', t, \tau, \tau')\tilde{D}_{\ell j}(-\underline{k} + \underline{k}', -\underline{k}', t, \tau', \tau)\, d\underline{k}' d\tau d\tau' \tag{61}$$

For an isotropic distribution in which S and Δ are functions of the magnitudes r and k alone, the spectrum function with respect to wave-number magnitude is

$$\Gamma(k, t) = 4\pi k^2 \Delta(k, t) = \frac{2k}{\pi} \int_0^\infty S(r, t) r \sin kr \, dr \qquad (62)$$

By using Laplace transforms, we can now write the solution of Eq. (59) in closed form. It is easily shown that $\tilde{C}_j$ is proportional to e^{pt}, where p is the root of

$$p + k^2 \hat{R}_E(p) = 0 \qquad (63)$$

and $\hat{R}_E(p)$ is the Laplace transform of $R_E(t)$. If T_E is the Eulerian time scale, then it is clear that

$$\tilde{C}_j(\underline{k}, t, \tau) = \tilde{C}_j(\underline{k}, 0, \tau) e^{-k^2 u^2 T_E t}, \quad \text{for} \quad k \ll (uT_E)^{-1} \qquad (64)$$

and

$$\tilde{C}_j(\underline{k}, t, \tau) = \tilde{C}_j(\underline{k}, 0, \tau) \cos kut, \quad \text{for} \quad k \gg (uT_E)^{-1} \qquad (65)$$

The wave number $(uT_E)^{-1} = k_e$, say, is characteristic of the energy-containing eddies of the velocity field. This wave number separates the Gaussian contribution to the concentration spectrum into two parts. For wave numbers less than k_e, the interaction causes an exponential decay with time; for $k \gg k_e$, the interaction gives a rapidly oscillating fine-scale structure to the spectrum. The oscillatory behavior is associated with the wave-like features of Eq. (50), and may be a spurious effect due to the truncation. However, the quantity $\tilde{C}_j$ always appears squared in relevant physical quantities, and for large kut it seems appropriate to neglect the fine-scale oscillatory structure and replace $\cos^2$ kut by its mean value of one-half. Then if $\Gamma_G(k, t)$ is the contribution to the spectrum function from the Gaussian part of the concentration field,

$$\Gamma_G(\underline{k}, t) \doteq \Gamma_G(\underline{k}, 0)e^{-2k^2u^2T_E t} \quad \text{for} \quad k \ll k_e \tag{66a}$$

$$\doteq \frac{1}{2}\Gamma_G(k, 0) \quad \text{for} \quad k \gg k_e,\ kut \gg 1 \tag{66b}$$

The exact forms depend on the unspecified Eulerian correlation, but for the sake of discussion we can take these expressions as holding respectively for $k < k_e$ and $k > k_e$.

We consider now the contributions from the non-Gaussian part of the concentration field. Quite generally, we can write

$$\tilde{C}_j(\underline{k}, t, \tau) = \tilde{C}_j(\underline{k}, 0, \tau)C(k, t) \tag{67}$$

Then from Eq. (58),

$$\tilde{D}_{j\ell}(\underline{k}, \underline{k}', t, \tau, \tau') = -ik_p\tilde{C}_j(\underline{k}, 0, \tau)\int_o^t \tilde{K}_{p\ell}(k', t' - \tau')C(k, t')\,dt' \tag{68}$$

On substituting into the second and third terms of Eq. (54), we find after some reduction (using the isotropy and incompressibility of the velocity field) a contribution $S_{NG}(\underline{r}, t)$, say;

$$S_{NG}(\underline{r}, t)$$

$$= \int\int_o^t\int_o^t k_p k_q \Delta(\underline{k}, 0)e^{i\underline{k}\cdot\underline{r}}R_{pq}(\underline{r}, t' - t'')C(k, t')C(k, t'')dt'dt''d\underline{k} \tag{69}$$

where

$$R_{pq}(\underline{r}, t' - t'') = \left\langle u_p(\underline{x}, t')u_q(\underline{x} + \underline{r}, t'')\right\rangle \tag{70}$$

is now the two-point, two-time Eulerian velocity covariance.

The expression Eq. (69) can be given a simple interpretation if we use the approximation

$$\int_o^t\int_o^t R_{pq}(\underline{r},\ t' - t'')C(k,\ t')C(k,\ t'')\ dt'dt'' \ = \ \frac{1}{2k^2u^2}\,R_{pq}(\underline{r},\ 0) \qquad (71)$$

The approximation Eq. (71) is in fact asymptotically correct for large k if we use Eq. (65) and replace $\cos^2 kut$ by one-half. It is also correct for small k and large t if the characteristic time scale of $R_{pq}(\underline{r},\ t)$ is the same as that of $R_E(t)$, which is reasonable for small r. Then Eq. (69) can be written approximately as

$$S_{NG}(\underline{r},\ t) \ = \ \frac{1}{2u^2}\left\langle u_p(\underline{x},\ t)u_q(\underline{x} + \underline{r},\ t)\right\rangle \frac{\partial^2}{\partial r_p \partial r_q}\sum(r) \qquad (72)$$

where

$$\nabla^2\sum(r) \ = \ S(\underline{r},\ 0) \qquad (73)$$

In other words, the non-Gaussian contribution to the concentration covariance is one-half the Eulerian two-point, one-time velocity correlation, weighted by a function that depends on the initial concentration covariance. Note that the Gaussian contribution is given by

$$S_G(\underline{r},\ t) \ = \ \int [C(k,\ t)]^2\Delta(\underline{k},\ 0)e^{i\underline{k}\cdot\underline{r}}\ d\underline{k} \qquad (74)$$

and it follows immediately from Eqs. (69), (74) and (59) that S(0, t) is constant.

Let us now suppose that at the initial instant the concentration spectrum was peaked around k_e and was negligibly small at large wave numbers. Then for small r we have approximately

$$S(r,\ 0) \ = \ \bar{\theta}^2\left[1 - \frac{r^2}{2\lambda_o^2}\right] \qquad (75)$$

where $\lambda_o k_e \sim 1$. It now follows that for $r \ll k_e^{-1}$ and $ut \gg r$,

$$S(r, t)$$

$$= \frac{1}{2} S(r, 0) + \frac{1}{2u^2} \left\langle u_p(\underline{x}, t) u_q(\underline{x} + \underline{r}, t) \right\rangle \bar{\theta}^2 \left(\frac{1}{3} \delta_{pq} - \frac{r_p r_q}{5\lambda_o^2} - \frac{r^2 \delta_{pq}}{10\lambda_o^2} \right) \tag{76}$$

We can rewrite this result as

$$\left\langle (\theta - \theta')^2 \right\rangle = \frac{\bar{\theta}^2}{6u^2} \left\langle (\underline{u} - \underline{u}')^2 \right\rangle = \bar{\theta}^2 \left[1 - f(r) - \frac{r}{3} f'(r) \right] \tag{77}$$

where $f(r)$ is the longitudinal velocity correlation function of isotropic turbulence. In terms of the concentration spectrum function $\Gamma(k)$, we can say that if Γ is initially Gaussian and peaked on the scale of the energy-containing eddies, then the interactions predicted by a Wiener-Hermite truncation produce at larger wave numbers a spectrum due entirely to the non-Gaussian contributions

$$\Gamma(k) = \frac{\bar{\theta}^2}{3u^2} E(k) \qquad \text{for} \qquad k \gg k_e \tag{78}$$

where $E(k)$ is the energy spectrum function

$$\frac{1}{2} \left\langle \underline{u}^2 \right\rangle = \frac{3}{2} u^2 = \int_o^\infty E(k)\, dk \tag{79}$$

According to the analysis, the transfer of spectral density of $\bar{\theta}^2$ ceases when Γ attains the value given by Eq. (78). This can be verified directly by calculating the transfer function $\left\langle \theta(\underline{x}) \theta(\underline{x} + \underline{r}) u_i(\underline{x}) \right\rangle$ and showing that it has an average value of zero as kut becomes large. The probability distribution does not attain an equilibrium because the Gaussian contribution continues to decay with a slower time scale.

If the spectrum of concentration for all wave numbers were exactly proportional to the spectrum of energy, we would have

$$\Gamma(k) = \frac{2\bar{\theta}^2}{3u^2} E(k)$$

Thus the analysis has predicted that for $k \gg k_e$ the spectrum of concentration is proportional to the energy spectrum, but is only half as large as if proportionality existed for all wave numbers.

In the next section we shall discuss the results of our analysis and compare them with those of other theories.

V. THE SPECTRUM OF A CONVECTED SCALAR

We now use the analytical results of the previous section to build up a heuristic theory for the spectrum of a convected scalar when the Prandtl number is not small compared to unity. First, it is necessary to determine the reliability of the Wiener-Hermite truncation. Although we do not expect the results to be uniformly valid over all length scales, it is reasonable to believe that they may describe to a good approximation some of the properties of the distribution. We shall therefore make the hypothesis (which must ultimately be tested by experiment) that the interaction between the velocity field and the concentration field is well described by the analysis for scales which are larger than the Kolmogorov length $\ell = (\nu^3/\epsilon)^{1/4}$, but not for smaller scales. This hypothesis is equivalent to saying that the error introduced by the truncation is not important for wave numbers for which the energy spectrum is not exponentially small, and therefore avoids the criticism of Crow and Canavan.[10] The neglect of the molecular diffusivity $\varkappa$ is also justified if the Prandtl number $\nu/\varkappa$ is not small compared with unity.

PRANDTL NUMBER OF ORDER UNITY

First we consider the case where the Prandtl number is of order unity. The velocity fluctuations are damped out on a scale that is smaller than ℓ by the viscous stresses, and we expect similarly that the concentration fluctuations on the same scale are likewise damped by the conductivity. Thus, $\Gamma(k)$ should decay exponentially for $k > \ell^{-1}$. The rate at which $\bar{\theta}^2$ decays is given by

$$\chi = -\frac{d\bar{\theta}^2}{dt} = 2\varkappa\left\langle\left(\frac{\partial\theta}{\partial x_i}\right)^2\right\rangle = 2\varkappa\int_0^\infty k^2\Gamma(k)\,dk \qquad (80)$$

Then using Eq. (78), we have a decay rate

$$\chi = \frac{2\kappa\bar{\theta}^2}{3u^2}\int_0^\infty k^2 E(k)\, dk = \frac{\kappa\bar{\theta}^2\epsilon}{3\nu u^2} \tag{81}$$

since

$$\epsilon = 2\nu\int_0^\infty k^2 E(k)\, dk \tag{82}$$

The idea here is that when the Prandtl number is of order unity, concentration fluctuations are destroyed by molecular conduction on the scale ℓ. These fluctuations are produced by the nonlinear interaction of the velocity and concentration fields by which fluctuations with a scale L (that of the energy-containing eddies) are transformed into a smaller scale. We assume that this process (which can be called a cascade, although we do not imply or in fact believe that fluctuations with different length scales are statistically independent) is described by the Wiener-Hermite truncation analysis with molecular conduction neglected, according to which the spectrum at large wave numbers assumes in time ℓ/u (which is small) the form (78); and that the molecular conduction then acts to reduce the values of this spectrum. As conduction destroys or lowers the spectral density, the nonlinear interactions replace the spectral density to retain the form (78) for the spectrum, where $\bar{\theta}^2$ is now a function of time and has a time scale large compared with ℓ/u. In other words, Eq. (78) gives the spectrum, assuming the existence of a quasi-equilibrium.

Using Eq. (81), we can write Eq. (78) as

$$\Gamma(k) = \frac{\chi}{\epsilon}\frac{\nu}{\kappa}E(k) \tag{83}$$

for $k \gg L^{-1}$ and ν/κ of order unity.

In the inertial subrange $L^{-1} \ll k \ll \ell^{-1}$, the unmodified Kolmogorov theory predicts that

$$E(k) = K\epsilon^{2/3}k^{-5/3} \tag{84}$$

where K is a universal constant. Then Eq. (83) gives

$$\Gamma(k) = \frac{\upsilon\chi}{\nu} K\varepsilon^{-1/3} k^{-5/3} \tag{85}$$

The Obukhov-Corrsin application of the Kolmogorov theory (Ref. 4) gives

$$\Gamma(k) = K_\theta \chi\varepsilon^{-1/3} k^{-5/3} \tag{86}$$

The result of Eq. (85) implies that

$$K_\theta \doteq K \tag{87}$$

since υ/ν is supposed to be of order unity. Thus we have in this case confirmed the predictions of the universal equilibrium theory.

The experiments reported in the literature[14,15] were carried out for Prandtl numbers large compared with one, and cannot be used to check Eq. (87).

PRANDTL NUMBER LARGE COMPARED WITH UNITY

When $\upsilon/\nu \gg 1$, we can apply the theory of Batchelor[4] to derive the spectrum for $k \gg \ell^{-1}$ or the correlation function for $r \ll \ell$. Batchelor showed on the basis of a physical model (which makes the result far more convincing than if the argument relied solely on assumptions about interactions in wave-number space) that for $r \ll \ell$, the correlation function $S(r, t)$ satisfies the equation

$$\chi = -\gamma r \frac{\partial S}{\partial r} + 2\nu \left(\frac{\partial^2 S}{\partial r^2} + \frac{2}{r}\frac{\partial S}{\partial r}\right) \tag{88}$$

where γ (< 0) is an effective value of the least principal rate of strain [see also Saffman[16]]. Batchelor assumed that the straining field responsible for the fine-scale structure was uniformly distributed in space and had an order of magnitude given by the root-mean-square

vorticity $(\varepsilon/\nu)^{1/2}$; a plausible value of γ is then $-1/2(\varepsilon/\nu)^{1/2}$. However, there are strong reasons for believing that the vorticity is concentrated into small regions in which its order of magnitude is larger than $(\varepsilon/\nu)^{1/2}$. The small-scale fluctuations of concentration will then also be concentrated into these small regions, so that γ should be replaced by a quantity somewhat larger than in Batchelor's exposition. If σ is the intermittency factor, i.e., the fraction of volume in which the vorticity fluctuations are large, then

$$\sigma\gamma^2 \approx \varepsilon/\nu \tag{89}$$

since the major contributions to mean-square vorticity come from the regions of intense shear. Thus it is appropriate to write in Eq. (88)

$$\gamma = -\frac{1}{\sigma^{1/2}}\left(\frac{\varepsilon}{\nu}\right)^{1/2} \tag{90}$$

The value of σ is open to question. Saffman[8] has argued that

$$\sigma \sim \mathrm{Re}^{-1/2} = (\nu/uL)^{1/2} \tag{91}$$

and in the following discussion we shall use this value; but it should be kept in mind that further work on the structure of turbulence may suggest a different value. Equation (88) can be integrated[4] and gives, for $r \ll \ell$,

$$S(r, t) = \bar{\theta}^2 - \frac{\chi}{\gamma}\int_o^r \left[-\frac{1}{r} + \frac{1}{r^2}\, e^{\gamma r^2/4\kappa} \int_o^r e^{-\gamma r^2/4\kappa}\, dr\right] dr \tag{92}$$

The corresponding spectrum function for $k \gg \ell^{-1}$ is[4]

$$\Gamma(k, t) = -\frac{\chi}{\gamma k}\, e^{\kappa k^2/\gamma} \tag{93}$$

Thus the intermittency which increases the value of γ decreases the amplitude of the spectrum function in the "viscous-convective" subrange and also decreases the size of the conduction cutoff

$$\ell_\theta = (-\kappa/\gamma)^{1/2} = \ell(\kappa/\nu)^{1/2}\,\sigma^{1/4} \tag{94}$$

Batchelor pointed out that Eq. (93) and the Obukhov-Corrsin result (86) are continuous at $k = \ell^{-1}$ if σ and K_θ are both of order unity; i.e., if there is no intermittency. If there is intermittency, there would seem to be a serious conflict with the Kolmogorov hypotheses as applied to the convection problem.

In the present work, we suggest that the intermittent fine-scale structure should be retained, but that we drop the Obukhov-Corrsin result when $k \ll \ell^{-1}$, replacing it by the results of the truncated Wiener-Hermite expansion. The justification is similar to that employed above when ν/κ was of order unity; namely, that the truncation does not describe the fine-scale structure, but for $r \gg \ell$ the error is small and we can regard the analysis as giving a quasi-steady structure for $L \gg r \gg \ell$, the slow decrease in $\bar{\theta}^2$ being due to the generation of fine structure. Further, the small-scale fluctuations which are eventually dissipated by molecular conduction could in principle be analyzed by retaining many terms in the Wiener-Hermite expansion, but as an approximation are given by Batchelor's theory (modified because of intermittency). Thus we use Eq. (92) or Eq. (93) for $r \ll \ell$ or $k \gg \ell^{-1}$, respectively, in the so-called viscous-convective range; and Eq. (77) or (78) for $r \gg \ell$ or $k \ll \ell^{-1}$, respectively, in the so-called inertial-convective range.

The value of χ (which is not assumed here a priori to be independent of the Reynolds number, as it is in the Obukhov-Corrsin theory) can be found by matching the expressions for the viscous-convective and inertial-convective ranges at $r = \ell$ or $k = \ell^{-1}$. However, we do run into a troublesome difficulty which has not been resolved: matching in physical space and wave-number space gives different results.

In wave-number space, we use the form (84) for $E(k)$, since there is strong experimental support for this expression. For Eqs.

(93) and (78) to be comparable when $k = \ell^{-1}$, we must have

$$\chi \approx \frac{\bar{\theta}^2 K\epsilon}{3u^2\sigma^{1/2}} \tag{95}$$

It follows that in the inertial convective range,

$$\Gamma(k) \approx \chi\sigma^{1/2}\,\epsilon^{-1/3}\,k^{-5/3} \tag{96}$$

i.e.,

$$K_\theta \propto (Re)^{-1/4} \tag{97}$$

if we use Eq. (91).

These results are now different from those of the Obukhov-Corrsin theory. Firstly, χ, the rate of decay of scalar concentration, is dependent on the Reynolds number, behaving like $(Re)^{1/4}$ if K and ϵ are independent of the Reynolds number. Secondly, the coefficient of $\chi k^{-5/3}$ decreases with increasing Re, and is not constant as predicted by the arguments based on the hypotheses of the universal equilibrium theory. Insofar as the present results are based on a clear mathematical analysis rather than on vague assertions about the statistical independence of Fourier components, they deserve to be taken seriously. The relatively slow dependence of K_θ on Re makes it hard to check Eq. (97) against the available data,[14] especially since there is considerable uncertainty about the evaluation of χ.

If we match in physical space, we use the form of Eq. (92) for $r \gg (-\nu/\gamma)^{1/2}$, which is[4]

$$S(r, t) \sim \bar{\theta}^2 + \frac{\chi}{2\gamma} \log\left(-\frac{\gamma r^2}{\nu}\right) \tag{98}$$

The expression for $r \gg \ell$ is in physical space

$$S(r,\ t) \ = \ \bar{\theta}^2 \ - \ 1.2 \ \frac{\bar{\theta}^2}{3u^2} \ K\varepsilon^{2/3} \ r^{2/3} \tag{99}$$

Equating Eqs. (98) and (99) when $r = \ell$ gives

$$\chi \ \approx \ \frac{2.4 \ \bar{\theta}^2 \nu\varepsilon}{3u^2\sigma^{1/2}} \left[\log \ (v/\nu\sigma^{1/2})\right]^{-1} \tag{100}$$

This does not agree with Eq. (95), the difference being due to the logarithmic term. If we use Eq. (100), we would have in the inertial convective range

$$\Gamma(k) \ \propto \ \chi\sigma^{1/2} \ \varepsilon^{-1/3} \ k^{-5/3} \ \log \ (v/\nu\sigma^{1/2}) \tag{101}$$

and the spectrum function does not match at $k = \ell^{-1}$.

The difficulty is present in Batchelor's original analysis, and is not just a feature of the current approach. Until it is resolved, a question mark hangs over the assumption that we can determine χ by matching the two expressions for the inertial-convective and viscous-convective ranges in either physical or wave-number space. In any real situation the difference is not significant, since the logarithmic term is not large.

A crude physical justification of Eq. (95) can be given to demonstrate that it is consistent with intuitive ideas about the fine-scale structure.[8] We believe that the dissipation is due to fluctuations of order $(\bar{\theta}^2)^{1/2}$ having length scale ℓ_θ in the region of large vorticity. Then

$$\chi \ \approx \ \frac{\nu\bar{\theta}^2}{\ell_\theta^2} \ \sigma$$

which becomes, with the use of Eq. (94),

$$\chi \approx \frac{\nu\bar{\theta}^2}{\ell^2}\sigma^{1/2}$$

and on using the hypothesis that $\epsilon = \nu u^2\sigma/\ell^2$, we arrive at

$$\chi \approx \frac{\bar{\theta}^2\epsilon}{u^2\sigma^{1/2}} \tag{102}$$

Thus the Wiener-Hermite truncation combined with Batchelor's analysis has given results different from those previously obtained for the spectrum of a convected scalar with large Prandtl number. It is hoped that experimental results may eventually be available to test the theory. In any event, it is clear that progress will remain difficult without a better knowledge of the fine-scale structure of the velocity field in turbulent motion.

In this last connection, it is perhaps appropriate to mention here that the modification of Batchelor's theory to take account of intermittency does not affect the application of the theory to the vorticity field of turbulence by Saffman,[16] who assumed that vorticity correlations satisfied an equation like Eq. (88), with γ of order $(\epsilon/\nu)^{1/2}$ and with an extra term to take account of vortex line stretching. If there is intermittency, γ should be replaced by a value characteristic of the straining field distorting the vorticity. For reasons explained in Saffman,[8] it is believed that this straining field is not that of the vorticity itself, which is $O(u/\ell)$, but rather that of the straining motion whose length scale is the Taylor energy-dissipation parameter $\lambda = (15\nu\, u^2/\epsilon)^{1/2}$ and in which the characteristic rate of strain is $u/\lambda \propto (\epsilon/\nu)^{1/2}$. Thus, changing the fundamental assumptions does not in this case change the formal result.

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