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# MULTICOMMODITY SUPPLY AND TRANSPORTATION NETWORKS WITH RESOURCE CONSTRAINTS: The Generalized Multicommodity Flow Problem

Richard D. Wollmer

PREPARED FOR:
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The RAND Corporation
SANTA MONICA • CALIFORNIA

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The RAND Corporation

1700 MAIN ST. • SANTA MONICA • CALIFORNIA • 90406

# PREFACE

One problem that often arises in practice is that of finding generalized minimum cost multicommodity network flows. This problem has not previously been treated in the technical literature. Its importance is clearly demonstrated by its many applications. Among these are the military applications of interdiction, Military Airlift Command (MAC) scheduling, Logair*, and inter- and intra-theater transportation.

Previous work on multicommodity flows has treated networks in which individual arcs have flow capacities. This paper generalizes that notion to include capacities on linear combinations of flow on several arcs, thereby greatly increasing the scope of the problems that can be treated. Some of the multicommodity situations treated here which have not been handled previously are the allocation of directional flow in undirected arcs; node capacities; multimode networks with limited vehicle supplies; and the efficient use of scarce resources. In addition, this paper proposes a new solution technique which should be more efficient than ones given previously.

While this paper is directed towards a technical audience, the results will be of value to those working in the applied areas of interdiction, MAC scheduling, Logair, inter- and intra-theater transportation, and logistics system analysis.

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*Long term contract airlift service within continental U.S. for movement of cargo in support of military services' logistics systems.

## SUMMARY

This Memorandum presents an algorithm for finding a generalized minimum cost multicommodity network flow. Previous work on multicommodity flows has treated more limited situations in which flows on individual arcs are capacitated, or upper bounded. No problems have been treated in which limited resources are shared among several arcs, instead of only one. However, many actual network flow situations require that the latter be dealt with. Some general examples of this are operation of a transportation system with one or more vehicle types in limited supply, operation of a network under a limited budget constraint, or operation of a network where undirected arcs (which must be represented by two directed arcs) are capacitated. Specific military examples are the allocation of aircraft to routes for Logair* or the Military Airlift Command (MAC), where air corridors have virtually unlimited capacity, but the quantity of aircraft and crews are a limiting factor. Another military use is for interdiction where users of a lines-of-communication network are limited by their vehicle supply, but not by actual network throughput capacity. This problem is overcome in this paper by the introduction of generalized constraints. Specifically, upper bounds are placed on flows through nonnegative linear combinations of arcs rather than only through individual arcs.

This study is structured as follows:

- Section I discusses the generalized minimum cost multicommodity flow problem,
- Section II formulates it in terms of its arc subset-chain incidence matrix,

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*Long term contract airlift service within continental U.S. for movement of cargo in support of military services' logistics systems.

- Section III discusses examples of network problems involving joint capacity constraints that cannot be handled when dealing only with arc capacities,
- Section IV presents a method of solution based on an extension of Ford and Fulkerson's [1] algorithm for multicommodity flows with arc capacities only,
- Section V presents an alternative method of solution, which should prove to be more efficient than that cited in Sec. IV,
- Section VI formulates the problem in terms of its node-arc incidence matrices, and shows that when the Dantzig-Wolfe decomposition algorithm is applied, one arrives at methods of solution identical to those of Secs. IV and V.

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# I. INTRODUCTION

Problems that frequently arise in practice are those of finding generalized minimum cost multicommodity network flows. These have not previously been treated in the technical literature.

Multicommodity flow problems in which individual arcs are subject to flow (capacity) constraints have been dealt with quite extensively. Ford and Fulkerson [1], Jewell [2], Sakarovich [3], and Saigal [4] have treated the problem of maximizing the sum of all commodity flows subject to capacity constraints on the arcs and Tomlin [5] has treated the problem of finding minimum cost multicommodity flows under the same constraints. However, this formulation is often inadequate. In many actual network flow situations, the physical capacities of individual arcs are not limiting, but resources shared by units of flow on several arcs are. For example, when one wishes to assign flights to air routes, the capacities of the air corridors themselves are virtually unlimited; however, the number of flight assignments that could be made would be constrained by the number of planes and the number of pilots available. Examples of assigning flights to routes are Logair [6], MAC scheduling [7], and commercial airline scheduling [8-9]. In military interdiction problems, users of lines of communication networks are often limited by their supply of one or more vehicle types rather than their physical arc capacities [10-11].

A general situation, which includes some of the cases discussed above, is one in which the physical means of transport or vehicles are in short supply. Each mode of transportation would then impose a joint capacity constraint on all arcs which used it. Another general situation is operations under a fixed budget, which must be allocated to units of flow on all arcs. Still another is when physical arcs of a transportation system may be used for flow in either or both directions. Such an arc would need to be represented by two directed arcs and its capacity would be an upper bound on the flow of these two arcs.

The formulation treated in this paper handles all the above problems in addition to those treated previously. Specifically, certain subsets of arcs are identified. Rather than an upper bound being assigned to the flow of each arc, an upper bound is assigned to some positive linear combination of the arc flows of each subset. These types of bounds may be considered as joint capacity constraints. Individual arcs may belong to zero, one, or several subsets. Given these constraints, and unit flow costs on individual arcs, it is required to find a minimum cost multicommodity flow. Fulkerson [13-14] and Busacker and Gowan [15] have treated single commodity minimum cost flow problems involving one joint capacity constraint. The present paper has the advantage that more than one joint capacity constraint may be treated in multicommodity networks. It is surprising that while the scope of problems handled by this formulation is an order-of-magnitude greater than that handled by the individual arc capacity formulation, some of the algorithms which have previously been presented to solve the latter do not require extensive modifications to solve the former.

The remaining sections of this paper first present an arc subset-chain formulation of the problem, special cases of joint capacity constraints, and a method of solution based on an extension of Fulkerson's method for multicommodity flows with arc capacities [1]. Later a new method of solution is demonstrated that should be more efficient. Finally, special cases of the objective (cost) function are shown.

## II. AN ARC SUBSET-CHAIN LINEAR PROGRAM

Consider a network [N, A] of nodes $i = 1, \ldots, N$ and directed arcs connecting certain pairs of nodes for which flows of Q different commodities are to be assigned. Designate

| | |
|---|---|
| $(i,j)$ | the directed arc beginning at node i and ending at node j, |
| $k_{ij}$ | the cost of a unit of flow on arc $(i,j)$ |
| $s_q$ | the source for commodity q |
| $t_q$ | the sink (or terminal) node for commodity q. |

The topology of the network is specified by its arcs and nodes. Each commodity is identified by its source (the node where it originates) and its sink (the node where it terminates).

A chain is a sequence of distinct nodes $i_1, \ldots, i_n$ such that $(i_k, i_{k+1})$ is an arc for $k < n$. If $i_1 = s_q$ and $i_n = t_q$, the chain is said to be one for commodity q.

A cycle is defined as a chain, except that $i_1 = i_n$. Alternatively, a chain or cycle may be defined in terms of its arcs, $(i_k, i_{k+1})$. A chain or cycle is positive or negative depending on whether the sum of its arc costs are positive or negative respectively.

Given a network containing no negative cycles, it is required to find chain flows for all commodities, the number of which shall be designated by the symbol, Q, that minimizes total network cost, z, subject to the m nonnegative linear constraints, (1.2), where

(1.1) $$z = \sum_{i,j} k_{ij}\, y_{ij}$$

(1.2) $$\sum_{i,j} A^r_{ij}\, y_{ij} \le b_r \quad r = 1, \ldots, m$$

and

$b_r$ is the upper bound or capacity for the linear flow constraint r

$A^r_{ij}$ is the nonnegative* contribution of a unit of flow on arc (i,j) to flow constraint r

$y_{ij}$ is the flow on arc (i,j)

Expressions (1.2) may be thought of as joint capacity constraints on subsets of arcs, $S_1, \ldots, S_m$. Arc (i,j) is in subset r if and only if $A^r_{ij} > 0$ where $A^r_{ij}$ is the number of capacity units for subset r consumed by a unit of flow on (i,j) and $b_r$ is the total joint capacity for subset r. Any arc may belong to zero, to one, or to many subsets. For the special case where each individual arc has a capacity, the coefficient matrix for (1.2) would be an identity matrix.

This generalized multicommodity flow problem may be formulated as an arc subset-chain linear program. Specifically, if $C_1, \ldots, C_n$ are all the chains of the network for the various commodities and $x_j$ the flow on chain $C_j$, then the program may be formulated as follows:

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*The nonnegativity restriction $A^r_{ij}$ may be relaxed as follows. Suppose that, for each r, if (i,j) were to be assigned a cost of $A^r_{ij}$, the total cost on all cycles would be nonnegative; then all results which follow in this paper would still hold.

Find min z, $x_t \geq 0$ such that

$$z = \sum_{t=1}^{n} c_t x_t \tag{2.1}$$

$$\sum_{t=1}^{n} a_{rt} x_t + x_{n+r} = b_r \qquad r = 1, \ldots, m \tag{2.2}$$

where

$$c_t = \sum_{(i,j)\varepsilon C_t} k_{ij} \tag{3.1}$$

$$a_{rt} = \sum_{(i,j)\varepsilon C_t} A_{ij}^{r} \tag{3.2}$$

Thus the rows or constraints of the linear program correspond to the joint capacity constraints on the arc subsets while the columns or variables (other than the slacks, $x_{n+1}, \ldots, x_{n+m}$) correspond to the network commodity chains. The constraint coefficient, $a_{rt}$, represents the total amount of joint capacity on subset $S_r$ used by a unit of flow on chain $C_t$, and $c_t$, of course, is the total cost of a unit of flow on chain $C_t$.

## III. APPLICATIONS OF JOINT CAPACITY CONSTRAINTS

Some of the special applications involving networks with joint capacity constraints are those involving decisions as to the direction of traffic flow on streets, networks with node capacities, general resource constraints, and multimode networks with vehicle limitations.

### Directional Traffic Flow

In many actual network situations, arcs are undirected with capacities. Thus, one may be faced with a traffic situation where one would want to determine which streets should be one-way, which should be two-way and, for the two-way streets, the number of lanes there should be in each direction. However, these arcs need to be represented by oppositely directed one-way arcs. If one is dealing with single commodity networks with nonnegative arc costs, one is assured that an optimal flow pattern exists, using at most one of these arcs, and one may therefore assign to each of these directed arcs an individual capacity equal to that of the original undirected one. However, when more than one commodity is involved, situations may arise where any optimal solution involves flow in both directions of an undirected arc. For this reason, capacities on the individual arcs are insufficient to describe the undirected arc capacity. However, it can be handled by a joint capacity constraint. Specifically, if (i,j) is an undirected arc with capacity $b_{ij}$, then the joint capacity constraint for the two directed arcs replacing it is $x_{ij} + x_{ji} \leq b_{ij}$.

### Node Capacities

Nodes as well as arcs often have limited capacities. For example, one might be limited in an inventory situation by the capacity of a storage facility; or in a traffic network the capacity of an intersection may be more limiting than the capacity of all arcs leading into it. These situations may also be phrased in terms of joint arc capacities. For a source node, this could be expressed as the sum of flows on arcs leaving the source cannot exceed the source capacity; for a sink node, the sum of all flows on arcs entering it cannot exceed the sink capacity; and for any other node, its capacity may be expressed as an upper bound on the sum of flows, either on arcs entering it or leaving it. Thus a capacity of $b_i$ on an intermediate node, i, may be expressed either as

$$\sum x_{ij} \le b_i \text{ or } \sum x_{ji} \le b_i \quad ,$$

while the capacity of a source node would be the former and a sink node the latter of these two expressions. Node capacities for single commodity networks have previously been treated in [12].

Resource Constraints

Another situation that often occurs in practice is operations under resource or budget constraints. Specifically, there exists limited quantities of one or more resources, such as men, vehicles, dollars or other appropriate items. Suppose each unit of flow on arc (i,j) consumes $A_{ij}$ units of a particular resource; and b is the total supply of that resource. Then this could be expressed by the joint capacity constraint on all arcs $\Sigma A_{ij}x_{ij} \le b$. Such resource constraints frequently occur in aircraft allocation problems, including both military airlift such as Logair [6] and the Military Airlift Command (MAC) scheduling [7] and commercial airlift [8-9]. Here, the capacities of the air corridors are virtually unlimited, but the number of flights that can be assigned to routes is limited by the number of aircraft and crews available. For an aircraft constraint, b may represent the total flying hours available, and $A_{ij}$ the number of flying hours required for an aircraft to fly over arc (i,j). For a crew constraint, b may represent the number of crew flying hours available and $A_{ij}$ the flying hours required for aircraft and crew to fly over arc (i,j).

The special case of one resource constraint for a single commodity network flow has been treated before. It is usually attacked by assigning each arc, (i,j), a cost of $A_{ij}$, generating a profile of flow versus required cost, and choosing the point at which cost is equal to b [13-15], which would be more efficient than the method presented here. However, the method of this Memorandum has the advantages that more than one resource constraint may be handled, multicommodity networks treated, and the resource constraints separated from the cost or objective function.

Multimode Networks

A special case of resource constraints that deserves special attention is the situation where several modes of transport are involved and one or more vehicle types is in limited supply. No more than one vehicle type is allowed to carry units of flow on any one arc. This is really no restriction because a physical arc using more than one type of vehicle may be broken up into several arcs, each of which uses a single type. The number of vehicle units (i.e., hours, days, etc.) of a given type consumed by an arc using that vehicle is a linear function of its flow. Thus, if a unit of flow on arc (i,j) consumed $A_{ij}^{r}$ units of vehicle type r, and the number of vehicle r units is $b_r$, then one would have $\sum A_{ij}^{r} y_{ij} \le b_r$, where the summation is taken over all arcs using vehicle type r.

These types of constraints arise in interdiction problems because the user of a lines-of-communication network may often have several modes of transport available, but is often limited by a shortage of the different types of vehicles rather than by actual network throughput capacity [11].

## IV. A SOLUTION TECHNIQUE

The full representation of program (2) would require a complete enumeration of all chains for the Q commodities. The number of chains, even for relatively small networks, is large enough to make such enumeration impractical; also, an extension of Ford and Fulkerson [1] makes it unnecessary. Specifically, at least one optimal solution to (2) is a basic solution, for which the flow on no more than m chains is nonzero. Furthermore, the simplex method of solution proceeds from one basic solution to another, each time dropping one column and adding another to the basis, until optimality is reached. Thus, at any point of the calculation, it is only necessary to maintain the columns of the current basis, and to be able to generate the incoming column for the next basis. As will be shown later, the generation of the new column can be done by attaching appropriate costs or lengths to the arcs, and finding shortest chains for the Q commodities.

Let B be the current basis and $\pi$ the corresponding simplex multipliers. Then multiplying (2) by $B^{-1}$, one obtains for the cost coefficient of $x_j$, the quantity

$$\bar{c}_t = \begin{cases} c_t + \sum_{r=1}^{m} \pi_r a_{rt} & \text{for } t \le n \\ -\pi_r & \text{for } t > n \end{cases} \tag{4}$$

where $\pi_r$ is the simplex multiplier for row r. From (3) and (4), one obtains

$$(5.1) \qquad \bar{c}_t = \sum_{(i,j)\epsilon C_t} k_{ij} + \sum_r \Big( \pi_r \sum_{(i,j)\epsilon C_t} A^r_{ij} \Big) = \sum_{(i,j)\epsilon C_t} \ell_{ij}$$

where

$$(5.2) \qquad \ell_{ij} = k_{ij} + \sum_r \pi_r A^r_{ij}$$

Thus, if one were to assign each arc, $(i,j)$, a length of $\ell_{ij}$, $\bar{c}_t$ would merely be the total length of chain $C_t$. The current basis would be optimal if all $\pi_r \geq 0$ and the length of all chains were nonnegative. Otherwise, one may perform a pivot operation on a column, $n+r$, where $\pi_r < 0$, or on a column, $j \leq n$, where the length of $C_t$ is strictly negative, to obtain the next basis. The search for negative chains may be accomplished by finding the shortest chain for each commodity. Thus, a generalized minimum cost multicommodity flow may be found by the following algorithm.

1. Use as an initial starting basic solution, $x_{n+r} = b_r$ all other $x_t = 0$.
2. Let $\pi$ be the vector of the simplex multipliers for the current basis.
3. Let $\pi_q = \min_r \pi_r$. If $\pi_q < 0$, introduce $x_{n+q}$ into the basis by pivoting and return to 2. Otherwise continue.
4. Assign each arc $(i,j)$ a length $\ell_{ij}$ equal to that in expression (5.2). For each commodity, $q$, find the shortest chain $C_{j(q)}$ between its source and sink, and denote its length by $\ell_{j(q)}$.
5. Let $\ell_{\bar{q}} = \min_q \ell_{j(q)}$. If $C_{j(q)} \geq 0$, terminate as the program is optimal. Otherwise, introduce $x_{j(\bar{q})}$ into the basis by pivoting, and then go to Step 2.

There are several algorithms available for finding the shortest chain in a network, and many of these have been examined and compared by Dreyfus [16]. The first, and one of the most efficient, is one by Dykstra [17]. Hu [18] has some interesting procedures for finding shortest chains between all pairs of nodes, which may be quite useful when the number of commodities is large. Many of these procedures require that the network contain no cycles of negative length. This condition is assured in Step 4; because the cost portion ($k_r$'s) of all cycles are nonnegative by assumption, all $a_{ij} \geq 0$, and Step 3 assures all $\pi_r \geq 0$ before Step 4 is entered.

There are several possible minor modifications to the algorithm that may prove desirable and are easy to make. One possibility is to avoid finding shortest chains for all commodities by terminating the search as soon as a negative chain is found. If this is done, one may wish to examine the commodities in cyclical order. Thus, if commodity q is the last one examined in one iteration, commodity (q+1) mod Q is the first one examined for the next iteration. Another possibility is to add the shortest chain columns for all commodities to the basic columns, optimize the resulting program, and then drop all columns that do not appear in the new optimal basis.

## V. AN ALTERNATIVE METHOD OF SOLUTION

The algorithm presented in the last section finds a generalized multicommodity flow by solving an arc subset-chain linear program using simplex multipliers to generate the new column to enter the basis. The column generated is that with the most negative cost coefficient. Of course, one could choose any variable with a negative cost coefficient to enter the basis. Kuhn and Quandt [19] have performed computational experiments on several different column generating schemes for general linear programs; the most efficient ones proved to be those which chose the column with the greatest negative cost coefficient after the columns had been normalized. This normalization could be accomplished by dividing the column elements by their absolute sums, by the sum of their positive elements, by the sum of the squares of the elements, or other similar quantities. While many of these schemes would require one to enumerate all columns, modifications of them allow one to avoid this. The criterion proposed here is a modification of that by which normalization is performed by dividing the cost coefficient by the sum of the absolute values of all column elements.

Specifically, let B be the current basis and $\pi$ the corresponding vector of simplex multipliers, which is also the last row of $B^{-1}$. Furthermore, designate the element in row r and column t of $B^{-1}$ by $B^{-1}_{rt}$. Then, for the individual matrix elements, $\bar{a}_{rt}$, one has

(6.1) $$\bar{a}_{rt} = \sum_k B^{-1}_{rk}\, a_{kt}$$

(6.2)
$$\sum_r \bar{a}_{rt} = \sum_r \left( a_{rt} \sum_k B^{-1}_{kr} \right)$$

$$= \sum_r \left[ \sum_{(i,j) \epsilon C_t} A^r_{ij} \left( \sum_k B^{-1}_{kr} \right) \right]$$

Thus, if one were to assign each arc, (i,j), a length of

$$\sum_r A^r_{ij} \left( \sum_k B^{-1}_{kr} \right) ,$$

the sum of the elements of the column representing chain j would merely be the sum of the chain's arc lengths. Instead of normalizing the cost coefficient by dividing it by the sum of absolute values of its elements (i.e. terms on the left-hand side of (6.2)), as is often done, normalizing the cost coefficient in this paper will be done by dividing by the sum of the absolute value of the arc lengths in expression (7). Thus letting

(8.1)
$$\bar{\ell}_{ij} = \sum_r A^r_{ij} \left| \sum_k B^{-1}_{kr} \right|$$

(8.2)
$$N_t = \sum_{(i,j) \epsilon C_t} \bar{\ell}_{ij}$$

the next column or variable to introduce into the basis, if the current basis is not optimal, is column s, where

(9)
$$\frac{\bar{c}_s}{N_s} = \min_{\bar{c}_t < 0} \frac{\bar{c}_t}{N_t} .$$

A test to find whether a particular chain, s, for which $\bar{c}_s < 0$ satisfies (9) is suggested by the following theorem.

Theorem 1: Suppose $\bar{c}_s < 0$ and let $K = \left|\frac{\bar{c}_s}{N_s}\right|$. Then chain s satisfies (9) if and only if $\bar{c}_s + KN_s = 0 = \min_{C_t} \bar{c}_t + KN_t$. Furthermore, if this condition is not met, any chain with $\bar{c}_t + KN_t < 0$ satisfies $\bar{c}_t < 0$ and

$$\frac{\bar{c}_t}{N_t} < \frac{\bar{c}_s}{N_s}$$

Proof: Since $K = -\bar{c}_s/N_s$, $\bar{c}_s + KN_s = 0$. Also, because all $\bar{\ell}_{ij} \geq 0$, all chains with $\bar{c}_t \geq 0$ satisfy $\bar{c}_t + KN_t \geq 0$. Suppose $\bar{c}_t < 0$ and $\bar{c}_t/N_t \geq \bar{c}_s/N_s$. Then $\bar{c}_t/N_t \geq -K$ or $\bar{c}_t + KN_t \geq 0$. Similarly, suppose $\bar{c}_t < 0$ and $\bar{c}_t/N_t < \bar{c}_s/N_s$. Then $\bar{c}_t/N_t < -K$ or $\bar{c}_t + KN_t < 0$. QED

Noting that if one assigns arc (i,j) a length of $\ell_{ij} + K\bar{\ell}_{ij}$, the network contains no negative cycles, and the length of chain $C_t$ is $\bar{c}_t + KN_t$, it follows that Steps 4 and 5 of the algorithms, which find the column to enter the basis and determine when optimality is reached, may be modified as follows.

4. For each commodity q, do the following:

(a) Assign each arc (i,j) a length $\ell_{ij}$ as defined in expression (5.2), and find the shortest chain. If its length is nonnegative, designate it as $C_{t(q)}$ and go to 5. Otherwise designate it as $C_m$.

(b) Let $K = |\bar{c}_m| / N_m$ where $|\bar{c}_m|$ and $N_m$ are as defined in (5.1) and (6.2). Then assign each arc (i,j) a length of $\ell_{ij} + K\bar{\ell}_{ij}$ and find the shortest chain.

(c) If the length of the shortest chain found is nonnegative, let $t(q) = m$, and go to 5. Otherwise, let the shortest chain found replace $C_m$, and go back to 4(b).

5. If $\min_q c_{t(q)} = 0$, terminate, as the current basic solution is optimal. Otherwise, let $\bar{q}$ be such that $\bar{c}_{t(\bar{q})} / N_{t(\bar{q})} = \min_q \bar{c}_{t(q)} / N_{t(q)}$, introduce $x_{t(\bar{q})}$ into the basis, and go back to Step 2.

As was the case with the other criteria, the procedure may be modified to select a chain from the first commodity found to have a negative length chain and to consider commodities in cyclical order.

## VI. NODE-ARC FORMULATION AND DECOMPOSITION

The generalized multicommodity flow problem may be formulated in terms of its node-arc incidence matrix, as well as its arc subset-chain incidence matrix. Specifically, let

$y^q_{ij}$ = flow of commodity q on arc (i,j)

$r_q$ = the total flow on commodity q chains

Then the multicommodity flow problem is defined by the following constraints.

Find min z, $y^q_{ij} \geq 0$, $r_q \geq 0$ such that

$$z = \sum_q \sum_{(i,j)} k_{ij}\, y^q_{ij} \tag{10.1}$$

$$\sum_q \sum_{(i,j)} A^r_{ij}\, y^q_{ij} \leq b_r \qquad r = 1, \ldots, m \tag{10.2}$$

$$\sum_j y^q_{ij} - \sum_j y^q_{ji} = \begin{cases} r_q & i = s_q \\ 0 & i \neq s_q, t_q \\ -r_q & i = t_q \end{cases} \tag{10.3}$$

for q = 1, ..., Q

The transportation structure of (10.3) allows one to attack this program efficiently by the Dantzig-Wolfe decomposition algorithm [20], with (10.2) serving as the master program and the Q sets of equations of (10.3) forming Q subprograms.

Note that all constant terms in the Q subprograms are zero, and consequently none of these programs have extreme point solutions other than all $y_{ij}^{q} = 0$. Thus, if one lets $(\bar{y}_{ij}^{q1}), \ldots, (\bar{y}_{ij}^{qN(q)})$ be a set of vectors spanning the solution space for subprogram q, (10) is reformulated as follows when decomposition is applied.

Find min z, $x_{qt} \geq 0$, $s_r \geq 0$, $v_r \geq 0$ such that

(11.) $$z = \sum_{q,t} c_{qt}\, x_{qt}$$

(11.2) $$\sum_{q,t} P_{qt}^{r}\, x_{qt} + v_r = b_r \qquad r = 1, \ldots, m$$

where

(12.1) $$P_{qt}^{r} = \sum_{(i,j)} A_{ij}^{r}\, \bar{y}_{ij}^{qt}$$

(12.2) $$c_{qt} = \sum_{(i,j)} k_{ij}\, \bar{y}_{ij}^{qt}$$

Note, however, that any feasible solution to subprogram q (10.3) may be expressed as a nonnegative linear combination of commodity q chain flows* and that solutions corresponding to distinct chain flows are linearly independent. Thus, the $(\bar{y}_{ij}^{qt})$ may be defined more specifically as the N(q) solutions, corresponding to one unit of flow on a commodity q chain (i.e. $r_q = 1$). When this is done, program (11) is identical to program (2). Specifically, the $c_{qi}$ and $P_{qi}^{r}$ of (11) are equal to the $c_j$ and $a_{ij}$ of (2) respectively, and the $x_{qi}$ and $s_r$ of (11)

*Strictly speaking, such a solution may also contain cycles. However, as cycles are nonnegative by assumption, they cannot appear in an optimal solution, and consequently need not be considered.

are identical to the $x_i$ of (2). Furthermore, when the Dantzig-Wolfe decomposition is applied, not all columns of (11) are enumerated during the course of the calculation. Instead, only those corresponding to the current basis are maintained. Letting $\pi_r$ be the simplex multiplier corresponding to row r for the current basis, the next column to enter the basis (i.e. that with the lowest cost coefficient $c_{qt} - \sum_r \pi_r P^r_{qt} = \sum_{(i,j)} [k_{ij} + \sum_r \pi_r A^r_{ij}])$ if all $\pi_r \geq 0$ is found by solving the following Q programs. If $z_{\bar{q}} = \min_q (\min z_q) < 0$, then the new column is generated by applying the transformation (12) to the solution for program $\bar{q}$.

Find min $z_q$, $y^q_{ij} \geq 0$ such that

$$z_q = \sum_{(i,j)} \left( k_{ij} + \sum_r \pi_r A^r_{ij} \right) y^q_{ij} \tag{13.1}$$

$$\sum_j y^q_{ij} - \sum_j y^q_{ij} = \begin{cases} 1 & i = s_q \\ 0 & i \neq s_q, t_q \\ -1 & i = t_q \end{cases} \tag{13.2}$$

However, if arcs are assigned lengths equal to the $\ell_{ij}$ of (5.2), an optimal solution to (13) may be obtained by finding a shortest path from $s_q$ to $t_q$ and setting $y^q_{ij} = 1$ if (i,j) is on this path and $y^q_{ij} = 0$ otherwise. Of course, if some $\pi_r < 0$, then $v_r$ is the variable to enter the basis. The procedure terminates when all $\pi_r \geq 0$ and all min $z_q \geq 0$.

The above procedure, however, is identical to that given in Sec. III. Thus applying the Dantzig-Wolfe decomposition algorithm to the node-arc formulation is identical to using the arc subset-chain procedure given earlier.

## VII. FLOW MAXIMIZATION AND FEASIBILITY

Two multicommodity flow problems that often arise in practice are that of maximizing a linear combination of the commodity flows, and that of finding a feasible routing which will meet required flows. This section shows how these problems may be formulated in terms of finding generalized minimum cost flows.

### Maximizing a Linear Combination of the Commodity Flows

To maximize a linear combination of commodity flows, say $a_1P_1 + a_2P_2 + \dots + a_QP_Q$ where $P_i$ is the total flow of commodity i, one need only attach an artificial node, $s_i$, for each commodity, and an artificial arc directed from $s_i$ to node i's true source, and assign it a cost of $-a_i$. It can easily be seen that if all other arcs have zero costs and essentially infinite upper flow bounds on each, the cost minimization problem is the same as the linear combined flow maximization problem. If other arc costs are not zero, then care must be taken to insure that the negative of any artificial arc cost is greater than the length of any chain. This may be accomplished by multiplying all the $a_i$'s by a sufficiently large positive constant. It should be noted that none of the artificial arcs belong to cycles, and therefore their addition will not cause the restriction of no negative cycles to be violated. It should also be noted that since these arcs really have no capacity constraints, their addition adds no arc subset constraints, and hence no constraints to the linear program. Essentially their only effect is to terminate the program in Step 5, with each $\ell_{i(q)} \geq -a_q$ instead of $\ell_{j(q)} \geq 0$.

Meeting Required Flows

The problem of meeting required commodity flows, $P_1, \ldots, P_Q$, may be accomplished by adding the same artificial arcs and nodes as when maximizing a linear combination of commodity flows. However, the artificial arc connected to $s_i$ must be assigned an upper flow bound or capacity of $P_i$ and a cost sufficiently negative to insure that all chains for commodity i have negative length. For this problem, each of the artificial arcs does add a linear constraint to the program.

## LIST OF SYMBOLS

| | | |
|---|---|---|
| $(i,j)$ | = | the arc directed from node i to node j. |
| $k_{ij}$ | = | cost of a unit of flow on arc (i,j). |
| $s_q$ | = | source node for commodity q. |
| $t_q$ | = | sink node for commodity q. |
| $Q$ | = | the number of commodities. |
| $b_r$ | = | upper bound or capacity for linear flow constraint r. |
| $A_{ij}^{r}$ | = | contribution of a unit of flow on arc (i,j) to flow constraint r. |
| $y_{ij}$ | = | total flow on arc (i,j). |
| $y_{ij}^{q}$ | = | flow of commodity q on arc (i,j). |
| $S_r$ | = | subset of arcs for which $A_{ij}^{r} > 0$. |
| $C_t$ | = | network commodity chain t. |
| $c_t$ | = | sum of arc costs on chain t. |
| $a_{rt}$ | = | contribution of a unit of flow on chain t to flow constraint r. |
| $B$ | = | current basis for the linear program. |
| $\pi_r$ | = | simplex multiplier for row r. |
| $\ell_{uj}$ | = | length of arc (i,j) as determined by arc costs, constraint coefficients, and simplex multipliers. |
| $\bar{c}_t$ | = | total chain t cost relative to the basis B. |
| $\bar{a}_{rt}$ | = | the coefficient matrix element in row r column (chain) t with respect to the basis B |
| $B_{rt}^{-1}$ | = | the row r, column t element of $B^{-1}$. |
| $\bar{\ell}_{ij}$ | = | length of arc (i,j) as determined by the coefficient matrix (not cost row) with respect to basis B. |
| $r_q$ | = | total flow of commodity q. |

$p^r_{qt}$ = product of joint constraint row r and a feasible solution to the node-arc incidence program for commodity q.

$c_{qt}$ = product of the cost row of the multicommodity program and a feasible solution to the node-arc incidence program for commodity q.

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Wollmer