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RM-6175-PR
FEBRUARY 1970

EXACT SOLUTION OF THE RADIATION HEAT TRANSPORT EQUATION IN A GAS-FILLED SPHERICAL CAVITY

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PREFACE

This study is an outgrowth of Rand's investigation of the performance potential of new military propulsion and power systems. The Memorandum deals with radiation heat transfer in a gaseous medium, which has numerous applications to such high-temperature systems as military rockets, reentry vehicles, gaseous-fueled cavity reactors, and modern power plants.

A previous Memorandum, *The Feasibility of the Gaseous-Core Nuclear Reactor for Electric-Power Generation* (RM-5721-PR), discussed the feasibility of applying the gaseous-core reactor to electric-power production. The feasibility analysis relied in part on an approximate diffusion theory analysis of radiation heat transport in a spherical gaseous medium. In the present study, an exact solution to the radiative transport equation for a spherical gaseous medium is obtained using singular integral equation theory, and it is then compared to the diffusion theory solution in order to determine the accuracy and range of validity of diffusion theory.

SUMMARY

This Memorandum treats the problem of steady-state, radiative heat transport through an absorbing, emitting, and heat-generating gray gas contained inside a black-wall spherical cavity. An exact analytical solution to the radiative transport equation is obtained for the case of uniform heat generation throughout the gaseous medium.

The integral equation governing heat transfer by radiation is solved by introducing a complex function for the source distribution, which in turn leads to a singular integral equation of the principal-value type. This equation is solved by standard techniques. It is found that the exact solution involves the iteration of two Fredholm equations which have rapidly convergent solutions for all optical radii. A closed-form asymptotic solution is developed for the case of an optically thick medium. Numerical studies show that this closed-form solution is valid over a wide range of optical radii. As an example, for a cavity optical radius $\tau_o = 0.25$, a maximum error of 4 percent between the exact and asymptotic solutions is found for the distribution of the gas emissive power. For $\tau_o \geq 0.50$ the closed-form solution is essentially exact and can be used to calculate easily the spatial distribution of the gas emissive power to an accuracy of approximately 1 percent.

The exact transport theory solution is then compared with the Rosseland diffusion theory, and it is found that diffusion theory accurately describes the overall shape of the emissive power distribution in the interior of the gaseous medium. A maximum error of 3 percent between the exact and diffusion theory solutions is found for very small optical radii, $\tau_o \leq 0.1$. For intermediate values of optical radii, $0.1 \leq \tau_o \leq 1.0$, the maximum error between the exact and diffusion theory solutions in the interior of the gaseous medium increases to about 10 percent. As τ_o increases further, diffusion theory again becomes more accurate. For the case of the gaseous-core cavity reactor, where the optical radius is of the order of 10^3, the Rosseland approximation can therefore be used to describe the overall radiation heat transfer characteristics of the gaseous fuel. Within a few mean free paths of the outer boundary, however, diffusion theory breaks down and transport theory must be used to accurately describe the emissive power distribution in this region.

Applications of the theory to problems in plane geometry are also briefly discussed.

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CONTENTS

I. INTRODUCTION

The subject of radiation heat transfer in a gaseous medium has received increased attention recently from engineers concerned with various high-temperature system applications; e.g., gaseous-fueled cavity reactors, reentry vehicles, modern power plants, and rockets. In this Memorandum, we examine the problem of radiation heat transport in a spherical, gaseous-fueled, cavity reactor.

Initial studies utilized a diffusion theory approximation to the radiative transport equation.[1-3] To ascertain the accuracy of these approximations, other authors have made comparisons with various numerical solutions of the exact radiative transport equation. Sparrow, et al.,[4] appear to be the first to give a detailed description of heat transfer by radiation through an absorbing, emitting, heat-generating gas located between two concentric black spheres maintained at the same temperature. However, their solution utilized the method of successive approximations applied to the integral transport equation, which converges very slowly for large optical thicknesses. The transfer of radiation in a homogeneous spherical medium has been considered by Heaslet and Warming.[5,6] They obtain surface values of the emissive power in terms of Chandrasekhar's X and Y functions,[7] which have been tabulated extensively,[8] while numerical methods are used to obtain interior values.

Heat transfer by radiation between two concentric black spheres kept at different temperatures and separated by an absorbing, emitting gas has been examined by Ryhming,[9] while Viskanta and Crosbie[10] have extended this work to include gray walls. In these analyses the methods of undetermined coefficients and successive approximations were used to numerically solve the exact transport equation. Again, convergence to the exact solution is very slow when the gas is optically thick. Procedures for increasing the convergence rate for the method of successive approximations are given by Crosbie, et al.,[11] and Lee and Olfe.[12]

In this Memorandum, we consider the problem of an absorbing, emitting, and heat-generating gray gas contained inside a black-wall spherical cavity of radius R. The method of solution involves singular integral equations of the principal-value type as first developed for use in neutron transport theory[13,14] and Couette flow.[15] It consists of formulating the problem in terms of an integral equation with a difference kernel over a finite interval. A complex function is introduced for the source function, which in turn leads to a singular integral equation. This equation is then solved by the standard techniques of Muskhelishvili.[16] An exact solution is found which yields the entire spatial distribution of the gas emissive power. It involves the iteration of two Fredholm equations, the solutions of which are found to rapidly converge for all optical radii. In particular, for the case of an optically thick medium, such as the gaseous fuel in a cavity reactor, a closed-form asymptotic solution is found. It is found that this asymptotic solution is valid even at small optical radii; a maximum error of 4 percent between the exact and asymptotic solutions is found for optical radii τ_o as small as 0.25.

II. FORMULATION OF THE GOVERNING EQUATIONS

We choose to work with the integral equation governing steady-state radiative transfer in spherical geometry, which for the problem considered here is given by[4,5]

$$4k[e_g(r) - e_w] = \frac{2k^2}{r}\int_0^R r'[e_g(r') - e_w]$$
$$\times\ [E_1(k|r - r'|) - E_1(k|r + r'|)]\ dr' + S \qquad (1)$$

where $E_n(x)$ is the exponential integral defined by

$$E_n(x) = \int_0^1 t^{n-2}e^{-x/t}\ dt = \int_1^\infty e^{-xt}t^{-n}\ dt \qquad (2)$$

The gaseous medium is generating heat at a uniform rate per unit volume S, the absorption coefficient k is assumed to be independent of frequency and position, $e_g(r) = \sigma T(r)^4$ is the gas black-body emissive power, while $e_w = \sigma T_w^4$ is the black-body emissive power of the cavity wall, and σ is the Stefan-Boltzmann constant.

Defining the optical radial distance by $\tau = kr$ with the optical radius of the gaseous medium in the cavity given by $\tau_o = kR$ and substituting

$$\rho(\tau) = 4k\tau[e_g(\tau) - e_w] \qquad (3)$$

into Eq. (1) we obtain

$$\rho(\tau) = \frac{1}{2}\int_{-\tau_o}^{\tau_o} \rho(\tau')E_1(|\tau - \tau'|)\ d\tau' + \tau S \qquad (4)$$

where we have extended the definitions of $e_g(\tau)$ and S to be even functions of τ. Introducing the following transformations,

$$(\xi - \tau_o) = \tau, \qquad (\xi' - \tau_o) = \tau', \qquad \rho_o(\xi) = \rho(\xi - \tau_o) \tag{5}$$

the radiation transport equation takes the form

$$\rho_o(\xi) = \frac{1}{2}\int_0^{2\tau_o} \rho_o(\xi')E_1(|\xi - \xi'|)\, d\xi' + (\xi - \tau_o)S \tag{6}$$

This equation can also be interpreted as representing radiation transport through a slab of thickness $2\tau_o$ with a linearly varying source distribution.

We proceed with the solution of Eq. (6) by introducing a complex source function for the inhomogeneous source term. Consider the following integral equation,

$$\rho_A(\xi,z) = K(\rho_A)(\xi,z) + B(\xi,z) \tag{7}$$

where z is a complex variable, and the operator K and source function $B(\xi,z)$ are defined by

$$K(f)(\xi,z) = \frac{1}{2}\int_0^{2\tau_o} E_1(|\xi - \xi'|)f(\xi',z)\, d\xi' \tag{8}$$

$$B(\xi,z) = e^{-\xi/z} - e^{-(2\tau_o-\xi)/z} \tag{9}$$

Equation (7) is similar to Eq. (6) except that the inhomogeneous source term has been replaced by one which has z as a complex value. Recognizing that

$$\lim_{z\to\infty}\left[-\frac{z}{2}\, SB(\xi,z)\right] = S(\xi - \tau_o)$$

we see that our solution to Eq. (6) is given by

$$\rho_o(\xi) = \lim_{z\to\infty}\left[-\frac{z}{2}\, S\rho_A(\xi,z)\right] \tag{10}$$

Thus, once $\rho_A(\xi,z)$ is determined, $\rho_o(\xi)$ can easily be found.

From Eq. (7) we see that

$$\rho_A(\xi,z) = (I - K)^{-1}B(\xi,z) = B(\xi,z) + (I - K)^{-1}K(B)(\xi,z) \tag{11}$$

If we now consider the second term on the right-hand side of Eq. (11), we have for $z\notin[-1,1]$

$$\begin{aligned} K(B)(\xi,z) &= \frac{z}{2}\ln\left(\frac{z+1}{z-1}\right)B(\xi,z) + \frac{z}{2}\int_0^1 \frac{B(\xi,t')\,dt'}{(t'-z)} \\ &\quad + \frac{z}{2}\,e^{-2\tau_o/z}\int_0^1 \frac{B(\xi,t')\,dt'}{(t'+z)} \end{aligned} \tag{12}$$

This equation is seen to involve only integrals of the source function B over the parameter in B. Thus, applying the operation $(I - K)^{-1}$ to Eq. (12) and using Eq. (11), we obtain

$$\begin{aligned} (I-K)^{-1}K(B)(\xi,z) &= \frac{z}{2}\ln\left(\frac{z+1}{z-1}\right)\rho_A(\xi,z) + \frac{z}{2}\int_0^1 \frac{\rho_A(\xi,t)\,dt}{(t-z)} \\ &\quad + \frac{z}{2}\,e^{-2\tau_o/z}\int_0^1 \frac{\rho_A(\xi,t)\,dt}{(t+z)} \end{aligned}$$

Using this result in Eq. (11) and rearranging terms, we find

$$\Lambda(z)\rho_A(\xi,z) = B(\xi,z) + \frac{z}{2}\int_0^1 \frac{\rho_A(\xi,t)\,dt}{(t-z)} + \frac{z}{2}\,e^{-2\tau_o/z}\int_0^1 \frac{\rho_A(\xi,t)\,dt}{(t+z)} \tag{13}$$

where

$$\Lambda(z) = 1 - \frac{z}{2}\int_{-1}^{+1}\frac{dt}{(z-t)} = -\int_0^1 \frac{t^2\,dt}{(z^2-t^2)} = 1 - z\tanh^{-1}\frac{1}{z} \tag{14}$$

for $z\notin[-1,1]$.

Our analysis is based on the solution of Eq. (13). First, by examining the analyticity of Eq. (13) we obtain a linear constraint which must be satisfied by the solution $\rho_A(\xi,z)$. From Eq. (7), we see that excluding the essential singularity at $z = 0$, $\rho_A(\xi,z)$ is an analytic function of z for $|z| > 0$. Equation (14) shows that $\Lambda(z)$ has a double zero at infinity; therefore, the right-hand side of Eq. (13) must have at least a double zero at infinity. This requirement leads to the following constraint condition:

$$2(\tau_o - \xi) - \int_0^1 t\rho_A(\xi,t)\,dt - \tau_o \int_0^1 \rho_A(\xi,t)\,dt = 0 \qquad (15)$$

which can be determined by expanding the right-hand side of Eq. (13) in a power series in terms of $1/z$. Allowing z in Eq. (13) to take on values ν, where $-1 \le \nu \le +1$, we obtain the following singular integral equation for $\rho_A(\xi,\nu)$ on the real line $0 \le \nu \le 1$:

$$\lambda(\nu)\rho_A(\xi,\nu) = f(\xi,\nu) + \frac{\nu}{2}\int_0^1 \frac{\rho_A(\xi,t)\,dt}{(t - \nu)} \qquad (16)$$

where $f(\xi,\nu)$ is given by

$$f(\xi,\nu) = B(\xi,\nu) + \frac{\nu}{2}\, e^{-2\tau_o/\nu} \int_0^1 \frac{\rho_A(\xi,t)\,dt}{(t + \nu)} \qquad (17)$$

and

$$\lambda(\nu) = 1 + \frac{\nu}{2}\int_{-1}^{+1} \frac{dt}{(t - \nu)} \qquad (18)$$

Here, the singular integrals are to be taken in the Cauchy principal-value sense. The solution of this singular integral equation by standard techniques, along with the linear constraint condition, will yield $\rho_A(\xi,z)$. Knowing $\rho_A(\xi,z)$, it is possible to obtain $\rho_o(\xi)$ from Eq. (10) and hence the emissive power distribution of the gas.

III. SOLUTION OF THE SINGULAR INTEGRAL EQUATION

Following the methods of Muskhelishvili,[16] we introduce a sectionally holomorphic function of the complex variable z defined by

$$\varphi(\xi,z) = \frac{1}{4\pi i}\int_0^1 \frac{\rho_A(\xi,t)\,dt}{(t - z)} \tag{19}$$

Using Plemelj's[16] formulas, we find that we can write Eq. (16) as

$$\Lambda^-(\nu)\varphi^+(\xi,\nu) = \Lambda^+(\nu)\varphi^-(\xi,\nu) + \tfrac{1}{2}f(\xi,\nu) \tag{20}$$

where

$$\varphi^{\pm}(\xi,\nu) = \frac{1}{4\pi i}\int_0^1 \frac{\rho_A(\xi,t)\,dt}{(t - \nu)} \pm \frac{1}{4}\rho_A(\xi,\nu) \tag{21}$$

$$\Lambda^{\pm}(\nu) = \lambda(\nu) \pm \frac{i\pi\nu}{2} \tag{22}$$

Dividing Eq. (20) by $\Lambda^-(\nu)$ and rewriting, we obtain

$$\varphi^+(\xi,\nu) - \frac{\Lambda^+(\nu)}{\Lambda^-(\nu)}\varphi^-(\xi,\nu) = \frac{f(\xi,\nu)}{2\Lambda^-(\nu)} \tag{23}$$

The solution of the homogeneous equation

$$X^+(\nu) - \frac{\Lambda^+(\nu)}{\Lambda^-(\nu)}X^-(\nu) = 0 \tag{24}$$

is well known and is given by[17]

$$X(z) = \frac{1}{(1 - z)}\exp\left[\frac{1}{\pi}\int_0^1 \frac{\theta(t)\,dt}{(t - z)}\right] \tag{25}$$

where $\theta(t)$ is defined by

$$\theta(t) = \tan^{-1}\left[\frac{\pi t}{2\lambda(t)}\right] \tag{26}$$

and the branch of the $\tan^{-1}$ is chosen so that $0 \leq \theta(t) \leq \pi$. Inserting Eq. (24) into Eq. (23) and rearranging yield

$$\frac{\varphi^{+}(\xi,\nu)}{X^{+}(\nu)} - \frac{\varphi^{-}(\xi,\nu)}{X^{-}(\nu)} = \frac{f(\xi,\nu)}{2X^{+}(\nu)\Lambda^{-}(\nu)} \tag{27}$$

Then, the function

$$G(\xi,z) = \frac{\varphi(\xi,z)}{X(z)} - \frac{1}{2\pi i}\int_0^1 \frac{f(\xi,t)\, dt}{2X^{+}(t)\Lambda^{-}(t)(t - z)} \tag{28}$$

is analytic in the plane cut from 0 to 1 and, as shown by Eq. (27), is continuous across the cut. Therefore, $G(\xi,z)$ is analytic in the entire finite plane. Also, since $\varphi(\xi,z)$ and $X(z)$ are both proportional to $1/z$ as $z \to \infty$, we see that $G(\xi,z)$ is finite at infinity; hence by Liouville's theorem $G(\xi,z)$ is a constant (dependent, however, on the parameter ξ). After multiplication by $X(z)$, Eq. (28) can be rewritten as

$$\varphi(\xi,z) = \frac{X(z)}{4\pi i}\int_0^1 \frac{f(\xi,t)\, dt}{X^{+}(t)\Lambda^{-}(t)(t - z)} + \frac{C(\xi)}{4\pi i} X(z) \tag{29}$$

where C is an unknown constant to be determined later.

Using Eq. (19) in Eq. (17), we see that $f(\xi,z)$ is simply related to $\varphi(\xi,-z)$ and $B(\xi,z)$. The function $\varphi(\xi,-z)$ can also be determined from Eq. (29) and this yields the following equation for $f(\xi,z)$:

$$f(\xi,z) = B(\xi,z) + \frac{z}{2} e^{-2\tau_0/z} X(-z)\left[\int_0^1 \frac{f(\xi,t)\, dt}{X^{+}(t)\Lambda^{-}(t)(t + z)} + C(\xi)\right] \tag{30}$$

It will be shown later that $C(\xi)$ can also be written in terms of an integral involving $f(\xi,t)$ so that Eq. (30) is in fact a Fredholm integral equation in terms of $f(\xi,z)$. This equation, however, is not particularly suited for numerical calculations, since a computation must be made at every spatial point ξ. A superior numerical method for calculating $f(\xi,z)$ will be outlined in a later section of this Memorandum. We proceed now with the determination of $\rho_o(\xi)$.

Using $\rho_A(\xi,z)$ as given by Eq. (13) in Eq. (10), we obtain

$$\rho_o(\xi) = \lim_{z\to\infty} \left\{ -\frac{zs}{2\Lambda(z)} \left[B(\xi,z) + \frac{z}{2} \int_0^1 \frac{\rho_A(\xi,t)\, dt}{(t - z)} + \frac{z}{2}\, e^{-2\tau_o/z} \int_0^1 \frac{\rho_A(\xi,t)\, dt}{(t + z)} \right] \right\} \tag{31}$$

Carrying out this limit and applying the linear constraint given by Eq. (15), we find

$$\rho_o(\xi) = \frac{3}{2}\, S \left\{ \frac{1}{6}\left[(2\tau_o - \xi)^3 - \xi^3 \right] - \int_0^1 t^3 \rho_A(\xi,t)\, dt - \tau_o \int_0^1 t^2 \rho_A(\xi,t)\, dt - \tau_o^2 \int_0^1 t\rho_A(\xi,t)\, dt - \frac{2}{3}\, \tau_o^3 \int_0^1 \rho_A(\xi,t)\, dt \right\} \tag{32}$$

where for $z >> 1$, $\Lambda(z) = -1/3z^2 + O(1/z^4)$ has been used.

The moments of $\rho_A(\xi,t)$ appearing in Eq. (32) are related by Eq. (19) to the coefficients of the Laurent expansion of $\varphi(\xi,z)$ about $z = \infty$. We can also obtain these coefficients by expanding the right-hand side of Eq. (29) about infinity. First, we examine the expansion of $X(z)$. From Eq. (25) we have

$$X(z) = \frac{1}{(1 - z)} \exp\left[\frac{1}{\pi} \int_0^1 \frac{\theta(t)\, dt}{(t - z)} \right] = -\frac{1}{z} \exp\left\{ -\frac{1}{\pi} \int_0^1 \frac{[\pi - \theta(t)]\, dt}{(t - z)} \right\} \tag{33}$$

which can be expanded in the form

$$X(z) = -\frac{1}{z} \exp\left[\frac{1}{z}\left(\theta_0 + \frac{\theta_1}{z} + \frac{\theta_2}{z^2} + \frac{\theta_3}{z^3} + \ldots\right)\right] \tag{34}$$

where

$$\theta_i = \frac{1}{\pi}\int_0^1 [\pi - \theta(t)]t^i \, dt \tag{35}$$

Comparing coefficients of the Laurent expansion of $\varphi(\xi,z)$ in Eq. (19) about infinity with the coefficients of the expansion of the right-hand side of Eq. (29), we obtain the following equations for the moments of $\rho_A(\xi,t)$:

$$C(\xi) = \int_0^1 \rho_A(\xi,t) \, dt \tag{36}$$

$$\int_0^1 t\rho_A(\xi,t) \, dt = -f_0(\xi) + \theta_0 \int_0^1 \rho_A(\xi,t) \, dt \tag{37}$$

$$\int_0^1 t^2\rho_A(\xi,t) \, dt = -f_1(\xi) - \theta_0 f_0(\xi) + \left(\theta_1 + \frac{\theta_0^2}{2}\right)\int_0^1 \rho_A(\xi,t) \, dt \tag{38}$$

and

$$\begin{aligned}\int_0^1 t^3\rho_A(\xi,t) \, dt = {} & -f_2(\xi) - \theta_0 f_1(\xi) - \left(\theta_1 + \frac{\theta_0^2}{2}\right) f_0(\xi) \\ & + \left(\theta_2 + \theta_0\theta_1 + \frac{\theta_0^3}{6}\right)\int_0^1 \rho_A(\xi,t) \, dt\end{aligned} \tag{39}$$

where

$$f_n(\xi) = \int_0^1 \frac{f(\xi,t)t^n \, dt}{X^+(t)\Lambda^-(t)} \tag{40}$$

Using Eqs. (36) and (37) in Eq. (15), we can immediately solve for $C(\xi)$:

$$C(\xi) = \int_0^1 \rho_A(\xi,t) \, dt = \frac{2(\tau_o - \xi) + f_0(\xi)}{(\tau_o + \theta_0)} \tag{41}$$

All of the moments of $\rho_A(\xi,t)$ needed in Eq. (32) are known, and after some rearrangement we obtain the following for $\rho_o(\xi)$:

$$\rho_o(\xi) = \frac{3}{2} S \Bigg[(\tau_o - \xi)\tau_o^2 + \frac{1}{3}(\tau_o - \xi)^3 - \left\{ 2(\tau_o - \xi)\left[\frac{2}{3}\tau_o^2 + \frac{1}{3}\tau_o\theta_0 + \frac{1}{6}\theta_0^2 + \theta_1 + \frac{\theta_2}{(\tau_o + \theta_0)}\right] \right\} + f_2(\xi) + (\tau_o + \theta_0)f_1(\xi) + \left[\frac{1}{3}(\tau_o + \theta_0)^2 - \frac{\theta_2}{(\tau_o + \theta_0)}\right] f_0(\xi) \Bigg] \tag{42}$$

Applying the transformations given in Eq. (5) and using Eq. (3), the dimensionless emissive power distribution of the gas $\omega(\tau)$ is found to be

$$\omega(\tau) = \frac{[e_g(\tau) - e_w]}{\tau_o \frac{S}{k}} = \frac{\tau_o}{8}\left[1 - \left(\frac{\tau}{\tau_o}\right)^2\right] + \frac{\left(\frac{\theta_0^2}{2} + 3\theta_1\right)}{4\tau_o} + \frac{3\theta_2}{4\tau_o(\tau_o + \theta_0)} + \frac{\theta_0}{4} + \hat{f}(\tau) \tag{43}$$

where

$$\hat{f}(\tau) = \frac{3}{8\tau_o^2}\left(\frac{\tau}{\tau_o}\right)^{-1}\left\{f_2(\tau + \tau_o) + (\tau_o + \theta_0)f_1(\tau + \tau_o)\right.$$

$$\left. + \left[\frac{1}{3}(\tau_o + \theta_0)^2 - \frac{\theta_2}{(\tau_o + \theta_0)}\right]f_0(\tau + \tau_o)\right\} \qquad (44)$$

Thus, once the function $f(\xi,t)$ is determined, the general solution for the emissive power distribution is given by Eq. (43). However, since the determination of $f(\xi,t)$ involves the solution of a Fredholm equation, it is first instructive to examine an approximate form of Eq. (43) which will prove to be quite accurate for a wide range of optical thicknesses.

IV. ASYMPTOTIC FORM OF THE SOLUTION

To obtain a useful approximate solution in closed form to the radiative transport equation, we first write out in full the Fredholm equation satisfied by the function $f(\xi,\nu)$. Using Eq. (41) in Eq. (30), we find

$$f(\xi,\nu) = B(\xi,\nu) + \frac{\nu e^{-2\tau_o/\nu} X(-\nu)(\tau_o - \xi)}{(\tau_o + \theta_0)}$$

$$+ \frac{\nu}{2} e^{-2\tau_o/\nu} X(-\nu) \int_0^1 \frac{[(\tau_o + \theta_0) + (t + \nu)]f(\xi,t)\, dt}{X^+(t)\Lambda^-(t)(t + \nu)(\tau_o + \theta_0)} \tag{45}$$

An approximate solution for $f(\xi,\nu)$ can now be obtained by taking only the first term in the series expansion of the solution to the Fredholm equation, i.e., we neglect all integral terms. Thus, by using

$$f'(\xi,\nu) = B(\xi,\nu) + \frac{\nu e^{-2\tau_o/\nu} X(-\nu)(\tau_o - \xi)}{(\tau_o + \theta_0)} \tag{46}$$

for $f(\xi,\nu)$ in Eq. (40) instead of the exact value, we obtain the desired closed-form solution for $\omega(\tau)$, which will be called the "first asymptotic solution." Further terms in the series expansion of the Fredholm equation for $f(\xi,\nu)$ are of order $\exp(-2\tau_o/\nu)\cdot\exp[-(\tau_o-\tau)/\nu]$ and higher. For τ_o large, and $\tau \neq \tau_o$, only the first term in the expansion will be of importance and the rest are negligible. For $\tau = \tau_o$ numerical results show that the integral term of order $\exp(-2\tau_o/\nu)$, which is part of the second term in the Fredholm series expansion, can be neglected. The "first asymptotic solution" is found to give an accurate representation of the dimensionless emissive power distribution over a wide range of gas optical radii. This will be shown later in Section VII.

Another approximation to the exact solution can be obtained by neglecting all of the integral boundary correction terms involving the source function found in Eq. (43). By setting $\hat{f}(\tau) \equiv 0$ we obtain

$$\omega(\tau) = \frac{\tau_o}{8}\left[1 - \left(\frac{\tau}{\tau_o}\right)^2\right] + \frac{\left(\frac{\theta_0^2}{2} + 3\theta_1\right)}{4\tau_o} + \frac{3\theta_2}{4\tau_o(\tau_o + \theta_0)} + \frac{\theta_0}{4} \tag{47}$$

We will call this approximation the "second asymptotic solution."

The above approximations can be compared to the Rosseland diffusion theory with jump boundary conditions, as derived in Ref. 3 using the analysis of Deissler,(18) which in the notation of this Memorandum is given by

$$\omega(\tau) = \frac{\tau_o}{8}\left[1 - \left(\frac{\tau}{\tau_o}\right)^2\right] + \frac{1}{4\tau_o} + \frac{1}{6} \tag{48}$$

As can be seen by comparing Eq. (48) with Eq. (43), diffusion theory is capable of correctly describing the overall shape of the emissive power distribution for large values of τ_o. However, as the boundary between the gas and cavity wall is approached, i.e., within a few mean free paths, the transport theory correction terms become more important and must be included in order to accurately describe the characteristics of the distribution. The range of validity of the diffusion theory approximation and the more accurate first and second asymptotic solutions of transport theory will be discussed in more detail in Section VII.

V. SOLUTION OF THE FREDHOLM EQUATION

Since the value of $f(\xi,\nu)$ is to be determined for many values of ξ, it is not convenient to use Eq. (45). What is needed is a method of solution which requires the iteration of a Fredholm integral equation in terms of a single variable. To obtain this solution we follow the methods discussed in Refs. 14 and 15. This leads directly to a method for computing the moments $f_n(\xi)$ of $f(\xi,t)$ given by Eq. (40) so that $f(\xi,t)$ need not be found explicitly. By defining a linear operator L as

$$L(f)(\nu) = \frac{\nu}{2} e^{-2\tau_o/\nu} X(-\nu) \int_0^1 \frac{f(\xi,t)\, dt}{X^+(t)\Lambda^-(t)(t + \nu)} \tag{49}$$

it is possible to write Eq. (30) as the following set of integral equations:

$$F_0(\xi,\nu) = B(\xi,\nu) + L(F_0)(\xi,\nu) \tag{50}$$

$$F_1(\nu) = \frac{\nu}{2} e^{-2\tau_o/\nu} X(-\nu) + L(F_1)(\nu) \tag{51}$$

Because the operator L is linear we can write

$$f(\xi,\nu) = F_0(\xi,\nu) + C(\xi)F_1(\nu) \tag{52}$$

In Ref. 14, Leonard and Mullikin show that Eqs. (50) and (51) can be solved by iteration and that these iterations converge uniformly if

$$\max_{0\le\nu\le1} \left[\frac{\nu}{2} e^{-2\tau_o/\nu} X(-\nu) \int_0^1 \frac{dt}{X^+(t)\Lambda^-(t)(t + \nu)}\right] < 1 \tag{53}$$

By contour integration it can be shown that the following identity holds:

$$\frac{\nu}{2} X(-\nu) \int_0^1 \frac{dt}{X^+(t)\Lambda^-(t)(t + \nu)} = 1 - X(-\nu)\left[\nu + \frac{1}{X(0)}\right] \tag{54}$$

The quantity $1/X^+(t)\Lambda^-(t)$ appears in many expressions throughout this Memorandum. To determine the behavior of this quantity we consider the function $F(z)$ given by

$$F(z) = \frac{\Lambda(z)}{X(z)X(-z)} \tag{55}$$

The term $X(z)$ is analytic and nonvanishing in the plane cut from 0 to 1; therefore, $X(z)X(-z)$ is analytic and nonvanishing in the plane cut from -1 to $+1$. The term $\Lambda(z)$ is also analytic in this region, excluding the cut from -1 to $+1$; therefore, $F(z)$ is analytic in the cut plane. By Eq. (24) we see that the ratio condition $X^+(z)/X^-(z) = \Lambda^+(z)/\Lambda^-(z)$ ensures that $F(z)$ is continuous across the cut. Thus, $F(z)$ is analytic in the entire complex plane and is determined completely by its behavior near infinity. Using the known behavior of $\Lambda(z)$ near infinity, we find that

$$\lim_{z\to\infty} F(z) = \frac{1}{3}$$

and by Liouville's theorem, since $F(z)$ is analytic everywhere in the z plane and finite at infinity, it must equal a constant which we have shown to be 1/3. Therefore we have the identity

$$\frac{\Lambda(z)}{X(z)X(-z)} = \frac{1}{3} \tag{56}$$

which yields

$$\frac{1}{X^+(t)\Lambda^-(t)} = \frac{X(-t)}{3\Lambda^+(t)\Lambda^-(t)} = \frac{X(-t)}{3\left[\lambda^2(t) + \left(\frac{\pi t}{2}\right)^2\right]} \tag{57}$$

With this identity, we see that $X^+(t)\Lambda^-(t)$ is positive for $0 \le \nu \le 1$. Therefore, the left-hand side of Eq. (54) is ≥ 0, and since $X(-\nu)\left[\nu + \frac{1}{X(0)}\right] > 0$, then the right-hand side of Eq. (54) is < 1. Thus, it is found immediately that

$$0 \le \frac{\nu}{2} X(-\nu) \int_0^1 \frac{dt}{X^+(t)\Lambda^-(t)(t+\nu)} < 1 \quad \text{for } 0 \le \nu \le 1$$

and the inequality given by Eq. (53) holds for all $\tau_o \ge 0$ so that Eqs. (50) and (51) can be solved by iteration for all $\tau_o \ge 0$.

It is now necessary to determine the unknown constant $C(\xi)$. Using Eq. (52) in Eq. (41) and solving for $C(\xi)$, we obtain

$$C(\xi) = \frac{1}{D(\tau_o)}\left[2(\tau_o - \xi) + \int_0^1 \frac{F_0(\xi,t)\,dt}{X^+(t)\Lambda^-(t)}\right] \tag{58}$$

where

$$D(\tau_o) = (\tau_o + \theta_0) - \int_0^1 \frac{F_1(t)\,dt}{X^+(t)\Lambda^-(t)}$$

Once $F_1(t)$ is determined, by iteration of Eq. (51), $D(\tau_o)$ can be computed explicitly. However, the determination of $C(\xi)$ still requires an integral of $F_0(\xi,t)$ which must be calculated at each spatial point from Eq. (50). Again, this leads to a difficult and time-consuming numerical solution. To alleviate this problem, it is possible to develop a procedure for computing the numerator of Eq. (58), which involves only the determination of $F_1(\nu)$. We begin by rewriting Eq. (50) as

$$F_0(\xi,\nu) = B(\xi,\nu) + (I - L)^{-1}L(B)(\xi,\nu) \tag{59}$$

An operator $K(t,\nu)$ with parameter t is now defined such that

$$K(t,\nu) = (I - L)^{-1} \frac{\nu}{2} \frac{e^{-2\tau_o/\nu} X(-\nu)}{(t + \nu)} \qquad (60)$$

Thus, $K(t,\nu)$ satisfies the integral equation

$$K(t,\nu) - L(K)(t,\nu) = \frac{\nu}{2} \frac{e^{-2\tau_o/\nu} X(-\nu)}{(t + \nu)} \qquad (61)$$

Equation (59) can now be rewritten as

$$F_0(\xi,\nu) = B(\xi,\nu) + \int_0^1 \frac{K(t,\nu)B(\xi,t)\,dt}{X^+(t)\Lambda^-(t)} \qquad (62)$$

Using Eq. (50), it is seen that the numerator of Eq. (58) can be written as

$$\lim_{\nu\to\infty} 2[\nu F_0(\xi,\nu) - (\tau_o - \xi)] \qquad (63)$$

and therefore $K(t,\nu)$ need not be completely determined but only the quantity

$$\lim_{\nu\to\infty} \nu K(t,\nu) = k(t) \qquad (64)$$

Dividing Eq. (61) by $\frac{\nu}{t} e^{-2\tau_o/\nu} X(-\nu)$, using the following symmetry property of $K(t,\nu)$,

$$te^{-2\tau_o/t} X(-t)K(t,\nu) = \nu e^{-2\tau_o/\nu} X(-\nu)K(\nu,t) \qquad (65)$$

and taking the limit as $t \to \infty$ of the resulting equation, it is found that k can be expressed in terms of F_1. The result is

$$k(\nu) = \frac{1}{2}\left[1 + \int_0^1 \frac{F_1(\sigma)\, d\sigma}{X^+(\sigma)\Lambda^-(\sigma)(\sigma + \nu)}\right] \tag{66}$$

where we have found that

$$F_1(\sigma) = \sigma e^{-2\tau_o/\sigma} X(-\sigma)k(\sigma) \tag{67}$$

Equation (58) for $C(\xi)$ can now be rewritten as

$$C(\xi) = \frac{1}{D(\xi)}\left[2(\tau_o - \xi) + 2\int_0^1 \frac{k(t)B(\xi,t)\, dt}{X^+(t)\Lambda^-(t)}\right] \tag{68}$$

Therefore, since $D(\xi)$ and $k(t)$ depend only on the solution for $F_1(\nu)$, it is necessary to solve only a single Fredholm equation to obtain the spatial variation of C. However, since $F_0(\xi,\nu)$ is still unknown, $f(\xi,\nu)$ cannot be determined explicitly. A method for computing the moments of $f(\xi,t)$ given by Eq. (40) must now be developed in order to evaluate $\omega(\tau)$ in Eq. (43).

Consider the two equations for $F_0(\xi,\nu)$. By equating Eq. (50) and Eq. (62) we obtain

$$\frac{\nu}{2} e^{-2\tau_o/\nu} X(-\nu) \int_0^1 \frac{F_0(\xi,t)\, dt}{X^+(t)\Lambda^-(t)(t + \nu)} = \int_0^1 \frac{K(t,\nu)B(\xi,t)\, dt}{X^+(t)\Lambda^-(t)} \tag{69}$$

By expanding both sides of this equation in powers of $1/\nu$ and comparing coefficients, the moments of $f(\xi,t)$ are easily determined. It is convenient to first expand $K(t,\nu)$ in terms of the parameter t as follows:

$$K(t,\nu) = \frac{h_1(\nu)}{t} + \frac{h_2(\nu)}{t^2} + \ldots + \frac{h_n(\nu)}{t^n} + \ldots \tag{70}$$

Using this expansion in Eq. (61) and comparing coefficients of $1/t^n$, we find that the $h_n(\nu)$ satisfy the following integral equation:

$$h_n(\nu) = \frac{\nu}{2} e^{-2\tau_o/\nu} X(-\nu) \int_0^1 \frac{h_n(\sigma)\, d\sigma}{X^+(\sigma)\Lambda^-(\sigma)(\sigma + \nu)} + (-1)^{n+1} \frac{\nu^n}{2} e^{-2\tau_o/\nu} X(-\nu) \tag{71}$$

The symmetry property satisfied by $K(t,\nu)$ given by Eq. (65) yields

$$K(t,\nu) = \frac{\nu e^{-2\tau_o/\nu} X(-\nu)K(\nu,t)}{te^{-2\tau_o/t} X(-t)} = \frac{\nu e^{-2\tau_o/\nu} X(-\nu)}{te^{-2\tau_o/t} X(-t)} \left[\frac{h_1(t)}{\nu} + \frac{h_2(t)}{\nu^2} + \dots\right] \tag{72}$$

Inserting the series expansion of $K(t,\nu)$ given by Eq. (72) into Eq. (69), expanding the left-hand side of Eq. (69) in a power series in terms of $1/\nu^n$, and comparing coefficients of like powers yield the following expression for the moments of $f(\xi,t)$:

$$f_n(\xi) = \int_0^1 \frac{f(\xi,t)t^n\, dt}{X^+(t)\Lambda^-(t)} = (-1)^n \int_0^1 \frac{2h_{n+1}(t)B(\xi,t)\, dt}{te^{-2\tau_o/t} X(-t)X^+(t)\Lambda^-(t)}$$

$$+ C(\xi) \int_0^1 \frac{F_1(t)t^n\, dt}{X^+(t)\Lambda^-(t)} \qquad n = 0,1,2 \tag{73}$$

where we have used

$$F_0(\xi,t) = f(\xi,t) - C(\xi)F_1(t)$$

In Eq. (73) only the functions $h_1(t)$, $h_2(t)$, and $h_3(t)$ remain to be determined. Comparing Eq. (71) for n = 1 with Eq. (51), we see immediately that

$$h_1(\nu) \equiv F_1(\nu) \tag{74}$$

Using Eq. (71) and a similar equation with a negative integral operator given by

$$j_n(\nu) = -\frac{\nu}{2} e^{-2\tau_o/\nu} X(-\nu) \int_0^1 \frac{j_n(\sigma)\, d\sigma}{X^+(\sigma)\Lambda^-(\sigma)(\sigma + \nu)} + \frac{(-1)^{n+1}\nu^n e^{-2\tau_o/\nu} X(-\nu)}{2} \tag{75}$$

it is possible to derive a set of recursion formulas for the $h_n(\nu)$ in terms of the solutions to the two integral equations involving $h_1(\nu) \equiv F_1(\nu)$ and $j_1(\nu)$. We illustrate the procedure by solving for $h_2(\nu)$. Multiplying both sides of the equation for $j_1(\nu)$ by ν and writing $\nu = (\sigma + \nu) - \sigma$, we obtain

$$\nu j_1(\nu) = -\frac{\nu}{2} e^{-2\tau_o/\nu} X(-\nu) \int_0^1 \frac{j_1(\sigma)\, d\sigma}{X^+(\sigma)\Lambda^-(\sigma)}$$

$$+ \frac{\nu}{2} e^{-2\tau_o/\nu} X(-\nu) \int_0^1 \frac{\sigma j_1(\sigma)\, d\sigma}{X^+(\sigma)\Lambda^-(\sigma)(\sigma + \nu)} + \frac{\nu^2}{2} e^{-2\tau_o/\nu} X(-\nu) \tag{76}$$

or

$$(I - L)\nu j_1(\nu) = \frac{\nu^2}{2} e^{-2\tau_o/\nu} X(-\nu) + \frac{\nu}{2} e^{-2\tau_o/\nu} X(-\nu) K_1 \tag{76}$$

where

$$K_1 = \text{constant} = -\int_0^1 \frac{j_1(\sigma)\, d\sigma}{X^+(\sigma)\Lambda^-(\sigma)}$$

Treating this as an integral equation for $\nu j_1(\nu)$ with two inhomogeneous terms, we find the general solution to be

$$\nu j_1(\nu) = -h_2(\nu) + K_1 h_1(\nu) = -h_2(\nu) - h_1(\nu) \int_0^1 \frac{j_1(\sigma)\, d\sigma}{X^+(\sigma)\Lambda^-(\sigma)}$$

Solving for $h_2(\nu)$ yields

$$h_2(\nu) = -\left[\nu j_1(\nu) + h_1(\nu) \int_0^1 \frac{j_1(\sigma)\, d\sigma}{X^+(\sigma)\Lambda^-(\sigma)}\right] \tag{77}$$

In a similar manner we find the following relationship:

$$h_3(\nu) = -\left[\nu j_2(\nu) + h_1(\nu) \int_0^1 \frac{j_2(\sigma)\, d\sigma}{X^+(\sigma)\Lambda^-(\sigma)}\right] \tag{78}$$

where

$$j_2(\nu) = -\left[\nu h_1(\nu) - j_1(\nu) \int_0^1 \frac{h_1(\sigma)\, d\sigma}{X^+(\sigma)\Lambda^-(\sigma)}\right] \tag{79}$$

In summary, once the functions $h_1(\nu)$ and $j_1(\nu)$ are computed, by iteration of the Fredholm equations given by Eqs. (71) and (75) for $n = 1$, $h_2(\nu)$ and $h_3(\nu)$ are found from Eqs. (77) and (78), respectively. Equation (73) is then used to compute the moments of $f(\xi,t)$ where $C(\xi)$ is determined from Eq. (68). The moments of $f(\xi,t)$ are then used in Eqs. (43) and (44) to determine the exact spatial distribution of the dimensionless emissive power. In Section VII of this Memorandum, we give numerical results using the exact solution for a wide range of gas optical thicknesses. These results are then compared with the Rosseland diffusion approximation and the asymptotic solutions developed in Section IV to determine the range of validity of the various approximations.

VI. APPLICATION OF THE THEORY TO PLANE GEOMETRY

To illustrate the application of this method to problems in plane geometry, consider the simple case of an absorbing, emitting, and heat-generating gray gas contained between two parallel black plates which are held at the same temperature T_w. The geometry for this problem is shown in Fig. 1.

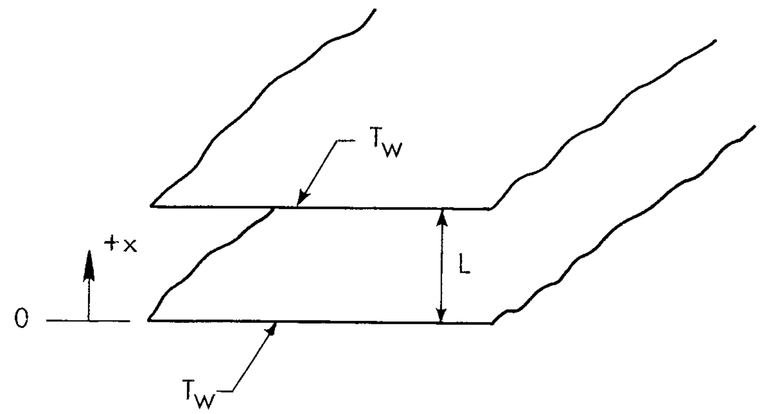

Fig. 1—Parallel plate geometry

This problem has received attention in the past by various authors. Several representative treatments are given by Usiskin and Sparrow[19] and Heaslet and Warming.[20,21] Again, the solutions are attained by numerically iterating the appropriate form of the radiative transport equation or by applying Sobolev's method of calculating the resolvent kernel.

The integral form of the radiative transport equation for this parallel plate problem is given by[19]

$$\int_0^L [e_g(x') - e_w]E_1(k|x - x'|)\, dx' + \frac{S}{2k^2} = \frac{2}{k}[e_g(x) - e_w] \tag{80}$$

The gaseous medium is assumed to be generating heat at a uniform rate per unit volume S. Introducing the optical path length $\tau = kx$ with the optical thickness of the plane layer given by $\tau_o = kL$ and substituting

$$\rho(\tau) = 4k[e_g(\tau) - e_w] \tag{81}$$

into Eq. (80), we obtain

$$\rho(\tau) = \frac{1}{2}\int_0^{\tau_o} \rho(\tau')E_1(|\tau - \tau'|)\, d\tau' + S \tag{82}$$

We now introduce the following integral equation with complex parameter z:

$$\rho_s(\tau,z) = K(\rho_s)(\tau,z) + B'(\tau,z) \tag{83}$$

where the complex source function and operator K are now given by

$$\begin{aligned} B'(\tau,z) &= e^{-\tau/z} + e^{-(\tau_o-\tau)/z} \\ K(f)(\tau,z) &= \frac{1}{2}\int_0^{\tau_o} E_1(|\tau - \tau'|)f(\tau',z)\, d\tau' \end{aligned} \tag{84}$$

Using this form, we immediately see that

$$\lim_{z\to\infty}\left[\frac{SB'(\tau,z)}{2}\right] = S$$

so that our solution to Eq. (82) is given by

$$\rho(\tau) = \lim_{z\to\infty}\left[\frac{S}{2}\,\rho_s(\tau,z)\right] \tag{85}$$

From Eq. (83) we can write

$$\rho_s(\tau,z) = B'(\tau,z) + (I - K)^{-1}K(B')(\tau,z) \tag{86}$$

Performing the indicated operations in the same manner as for the spherical case, we obtain, after rearranging, for $z \notin [-1,1]$,

$$\Lambda(z)\rho_s(\tau,z) = B'(\tau,z) + \frac{z}{2}\int_0^1 \frac{\rho_s(\tau,t)\,dt}{(t-z)} - \frac{z}{2} e^{-\tau_o/z}\int_0^1 \frac{\rho_s(\tau,t)\,dt}{(t+z)} \tag{87}$$

We can again obtain a linear constraint on the solution $\rho_s(\tau,z)$ by examining the analyticity of Eq. (87). We want a solution which is analytic for $|z| > 0$, and since $\Lambda(z)$ has a double zero at infinity, we find that the constraint condition is

$$\int_0^1 \rho_s(\tau,t)\,dt = 2 \tag{88}$$

Allowing z to take on values ν, where $-1 \leq \nu \leq +1$, the following singular integral equation for $\rho_s(\tau,\nu)$ is obtained:

$$\lambda(\nu)\rho_s(\tau,\nu) = f(\tau,\nu) + \frac{\nu}{2}\int_0^1 \frac{\rho_s(\tau,t)\,dt}{(t-\nu)} \tag{89}$$

where $f(\tau,\nu)$ is now given by

$$f(\tau,\nu) = B'(\tau,\nu) - \frac{\nu}{2} e^{-\tau_o/\nu}\int_0^1 \frac{\rho_s(\tau,t)\,dt}{(t+\nu)} \tag{90}$$

and the singular integrals are to be taken in the Cauchy principal-value sense.

The singular integral equation given by Eq. (89) has already been solved in Section III of this Memorandum (see Eq. (16) and Eq. (29)). Therefore, Eq. (29) can again be used to compute the required moments of $\rho_s(\tau,\nu)$. But first, let us determine $\rho(\tau)$. Using $\rho_s(\tau,z)$ from

Eq. (87) in Eq. (85), we find

$$\rho(\tau) = \lim_{z\to\infty}\left\{\frac{S}{2\Lambda(z)}\left[B'(\tau,z) + \frac{z}{2}\int_0^1 \frac{\rho_s(\tau,t)\ dt}{(t-z)}\right.\right.$$

$$\left.\left. - \frac{z}{2}\, e^{-\tau_o/z}\int_0^1 \frac{\rho_s(\tau,t)\ dt}{(t+z)}\right]\right\} \tag{91}$$

Performing the limiting operation and applying the linear constraint given by Eq. (88) yield

$$\rho(\tau) = -\frac{3}{4}\, S\left[\tau^2 + (\tau_o - \tau)^2 - 2\int_0^1 t^2\rho_s(\tau,t)\ dt\right.$$

$$\left. - \tau_o\int_0^1 t\rho_s(\tau,t)\ dt - \frac{\tau_o^2}{2}\int_0^1 \rho_s(\tau,t)\ dt\right] \tag{92}$$

The moments of $\rho_s(\tau,t)$ are obtained from Eqs. (36) to (38) by replacing $\rho_A(\xi,t)$ with $\rho_s(\tau,t)$. Using these moments in Eq. (92), we obtain, after some algebra,

$$\rho(\tau) = \frac{3}{2}\, S\left\{\tau(\tau_o - \tau) + \theta_0(\tau_o + \theta_0) + 2\theta_1\right.$$

$$\left. - \left[\left(\frac{\tau_o}{2} + \theta_0\right)f_0(\tau) + f_1(\tau)\right]\right\} \tag{93}$$

where $f_0(\tau)$ and $f_1(\tau)$ are again given by Eq. (40) with $f(\xi,t)$ replaced by $f(\tau,t)$ from Eq. (90). The dimensionless emissive power distribution is then easily found to be

$$\phi(\tau) = \frac{[e_g(\tau) - e_w]}{\frac{S}{2k}} = \frac{3}{4}\left\{\tau(\tau_o - \tau) + \theta_0(\tau_o + \theta_0) + 2\theta_1 - \left[\left(\frac{\tau_o}{2} + \theta_0\right)f_0(\tau) + f_1(\tau)\right]\right\} \tag{94}$$

It is clear that the exact determination of $f_0(\tau)$ and $f_1(\tau)$ again involves the iterative solution of two Fredholm equations. However, since the method of solution is similar to that presented for the spherical cavity, we will not include it here. The "second asymptotic solution," i.e., neglecting all integral terms involving the source function, is obtained by taking $f_0(\tau) = f_1(\tau) \approx 0$. Thus we find

$$\phi(\tau) = \frac{3}{4}\left[\tau(\tau_o - \tau) + \theta_0(\tau_o + \theta_0) + 2\theta_1\right] \tag{95}$$

We can compare this to the Rosseland diffusion theory approximation using jump boundary conditions, which for the problem outlined here yields

$$\phi(\tau) = \frac{3}{4}\left[\tau(\tau_o - \tau) + \frac{\tau_o}{2}\right] \tag{96}$$

As with the spherical cavity problem, diffusion theory is only capable of describing the overall spatial dependence of the emissive power distribution. Within a few mean free paths of either boundary, the transport correction terms become important and must be included to accurately describe the distribution.

VII. NUMERICAL RESULTS

The functions $h_1(\nu)$ and $j_1(\nu)$ were evaluated numerically by iterating the two Fredholm equations given by Eqs. (71) and (75). The calculations were performed on an IBM 360/65, and pointwise convergence to an error of less than 10^{-4} was obtained within one to four iterations. The exact spatial distribution of the gas emissive power was then calculated from Eqs. (43) and (44). All integrals were evaluated by using a ten-point Legendre Gauss quadrature on each of 10 subintervals.

In Fig. 2, the exact gas emissive power distribution as a function of position in a black-wall spherical cavity is shown for small values of optical radii, $\tau_o = 0.05$ and $\tau_o = 0.1$. Included also are the diffusion theory results given by Eq. (48). Diffusion theory is found to give very accurate results for small optical radii. This is because the spatial distribution of the emissive power is essentially flat for this case. Diffusion theory is based on the assumption that the emissive power distribution can be described by a Taylor series expansion in which only terms involving the first derivative of the distribution are retained. Thus, the flat distribution obtained for this problem at very small optical radii is approximated accurately by diffusion theory.

A comparison of the exact, first asymptotic, and diffusion theory solutions for several values of optical radii is shown in Fig. 3. The first asymptotic solution gives an accurate representation of the emissive power distribution even for small optical radii. For $\tau_o = 0.25$, a maximum error of 4 percent is found between the two theories, while even for $\tau_o = 0.1$ a maximum error of only 11 percent is found. For $\tau_o \geq 0.5$, the first asymptotic solution is essentially exact, having an error of less than 1 percent over the entire spatial distribution. Therefore, in this case, the emissive power distribution is given quite accurately by the closed-form first asymptotic solution, and there is no need to evaluate the more complicated Fredholm solutions. The accuracy of diffusion theory is found to decrease as τ_o increases from 0.1 to 1.0. At $\tau_o = 0.1$, a maximum error of 3 percent is found, while for $\tau_o = 1.0$ the error has increased to 10 percent.

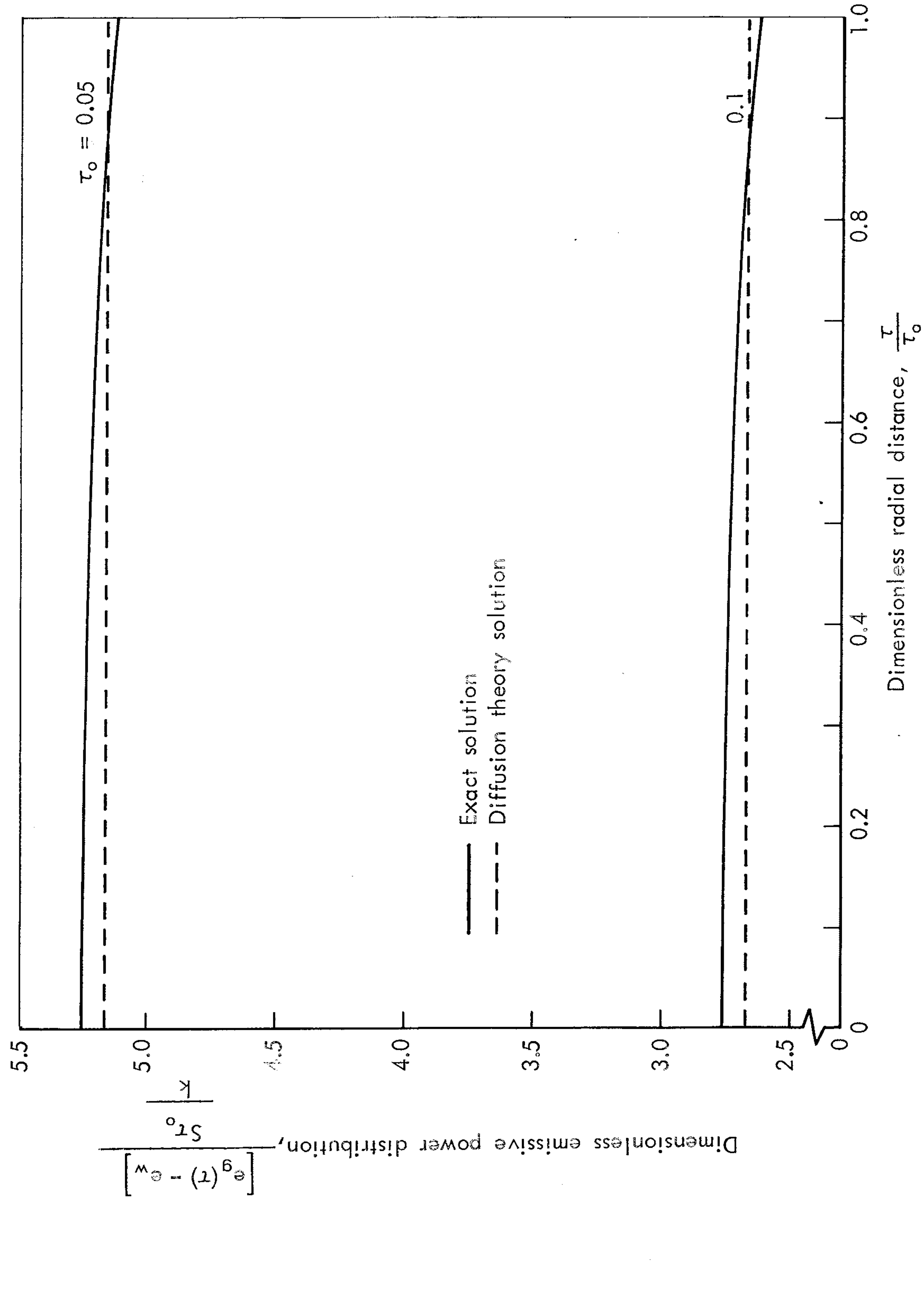

Fig. 2—Emissive power distribution for small optical radii

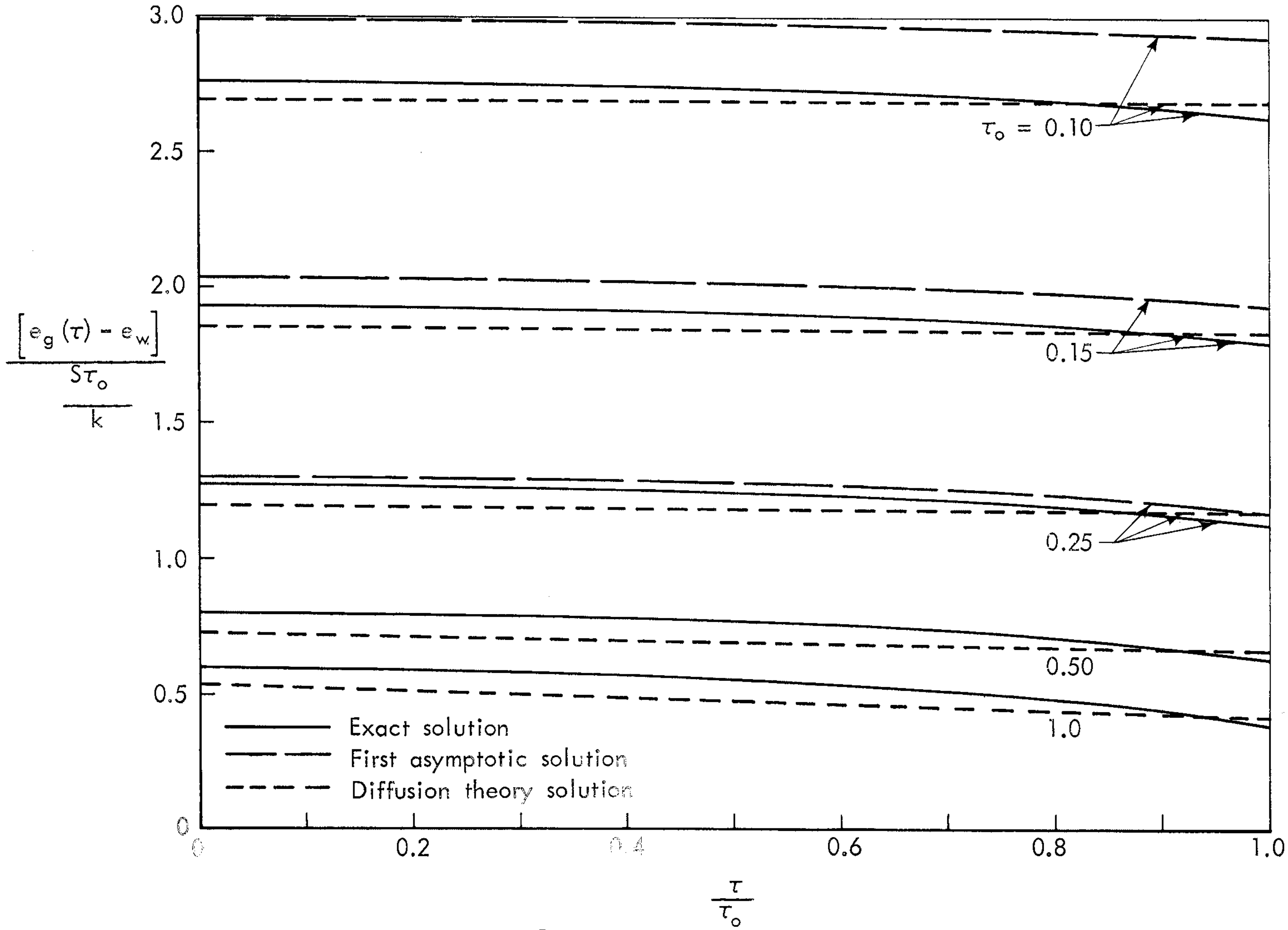

Fig. 3—Comparison of the exact, first asymptotic, and diffusion theory solutions for the gas emissive power in a spherical cavity

To assess the accuracy and range of validity of diffusion theory for larger values of τ_o, we have compared the exact solution for the gas emissive power to the diffusion theory solution in Figs. 4 through 6. For comparison, the second asymptotic solution given by Eq. (47) is also shown. Diffusion theory is seen to describe the overall shape of the emissive power distribution quite accurately. However, it does not give the correct shape near the outer boundary of the gaseous medium, i.e., near $\tau = \tau_o$, since the exact solution falls off much more rapidly. The second asymptotic solution yields a better approximation in the center of the cavity but also breaks down near the boundary. In both cases, this is due to the transport theory boundary correction terms given by integrals over the source distribution, which cannot be determined by diffusion theory. The rapidly varying distribution near the boundary cannot be approximated by a series expansion which includes only terms involving the first derivative; higher-order derivatives are important as well. Thus, diffusion theory is inaccurate in this region. As τ_o increases, the transport theory boundary correction terms die out a few mean free paths away from the boundary and diffusion theory becomes more accurate. This is clearly shown in Figs. 4 through 6. For the case of the gaseous-core reactor where $\tau_o \sim 10^3$, the Rosseland diffusion theory approximation can be used, except within several mean free paths of the boundary, with no practical loss in accuracy.

The dimensionless emissive power at the surface of the cavity, i.e., $\tau = \tau_o$, is given in Table 1. The exact solution is compared with the first and second asymptotic solutions as well as with the Rosseland diffusion theory solution. The results computed by Heaslet and Warming[5] using invariance principles as developed by Sobolev and Chandrasekhar are also included. The source function $\Omega(2\tau_o)$ used by Heaslet and Warming is related to our dimensionless emissive power distribution in the following manner:

$$\omega(\tau_o) = \frac{\Omega(2\tau_o)}{4\tau_o^2} \quad \text{for } e_w = 0$$

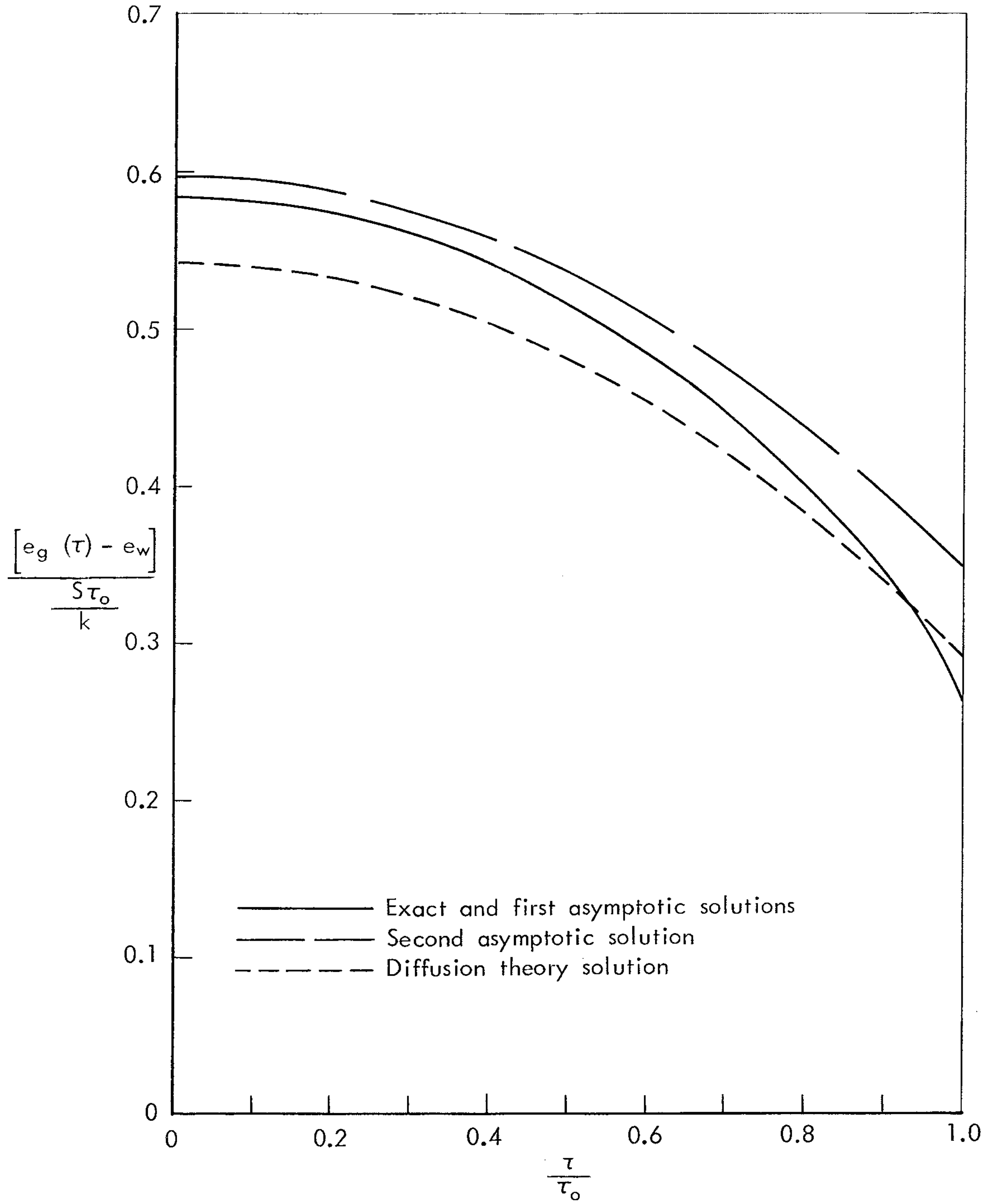

Fig. 4—Comparison of the exact, second asymptotic, and diffusion theory solutions for the gas emissive power in a spherical cavity ($\tau_o = 2.0$)

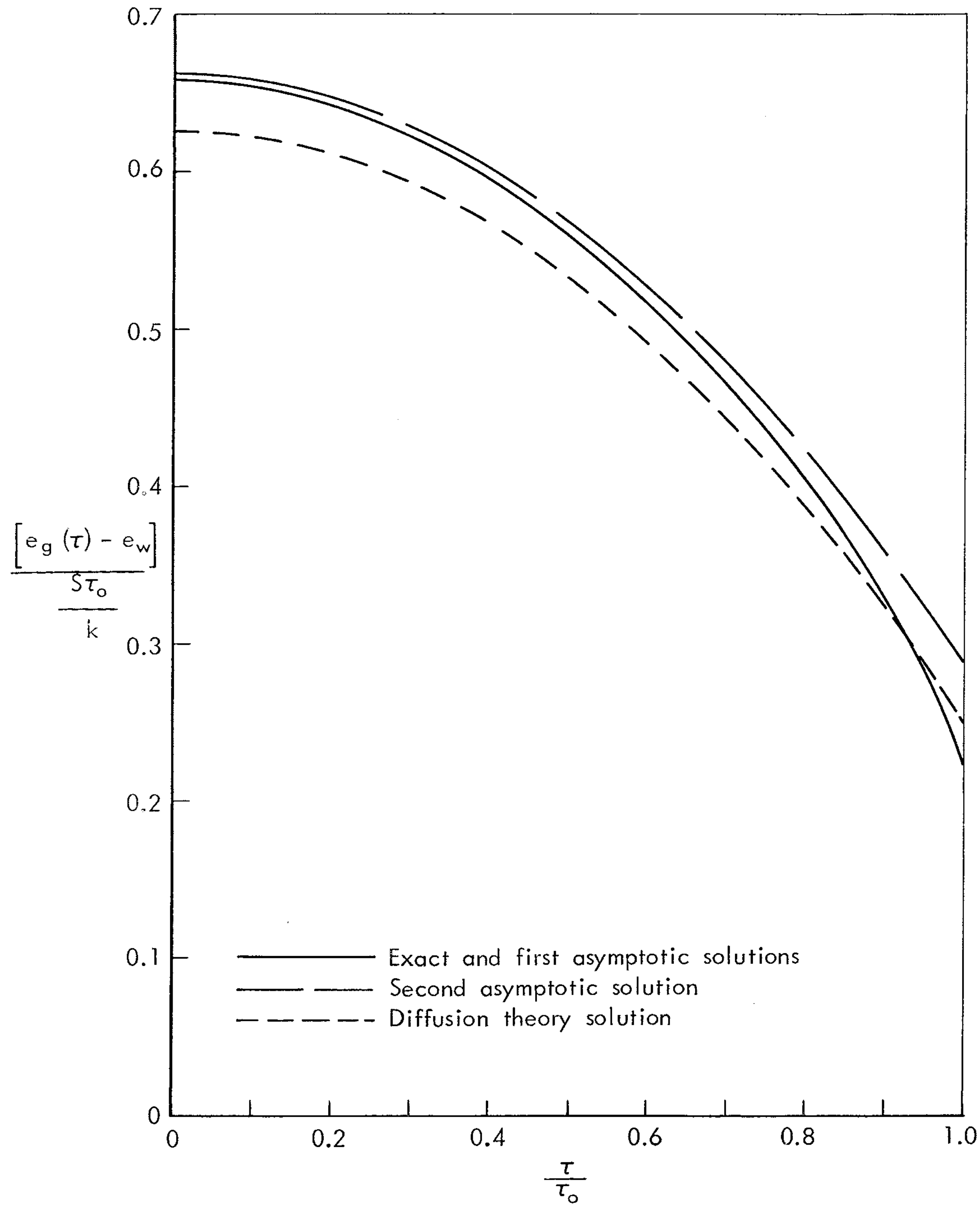

Fig. 5 —Comparison of the exact, second asymptotic, and diffusion theory solutions for the gas emissive power in a spherical cavity ($\tau_o = 3.0$)

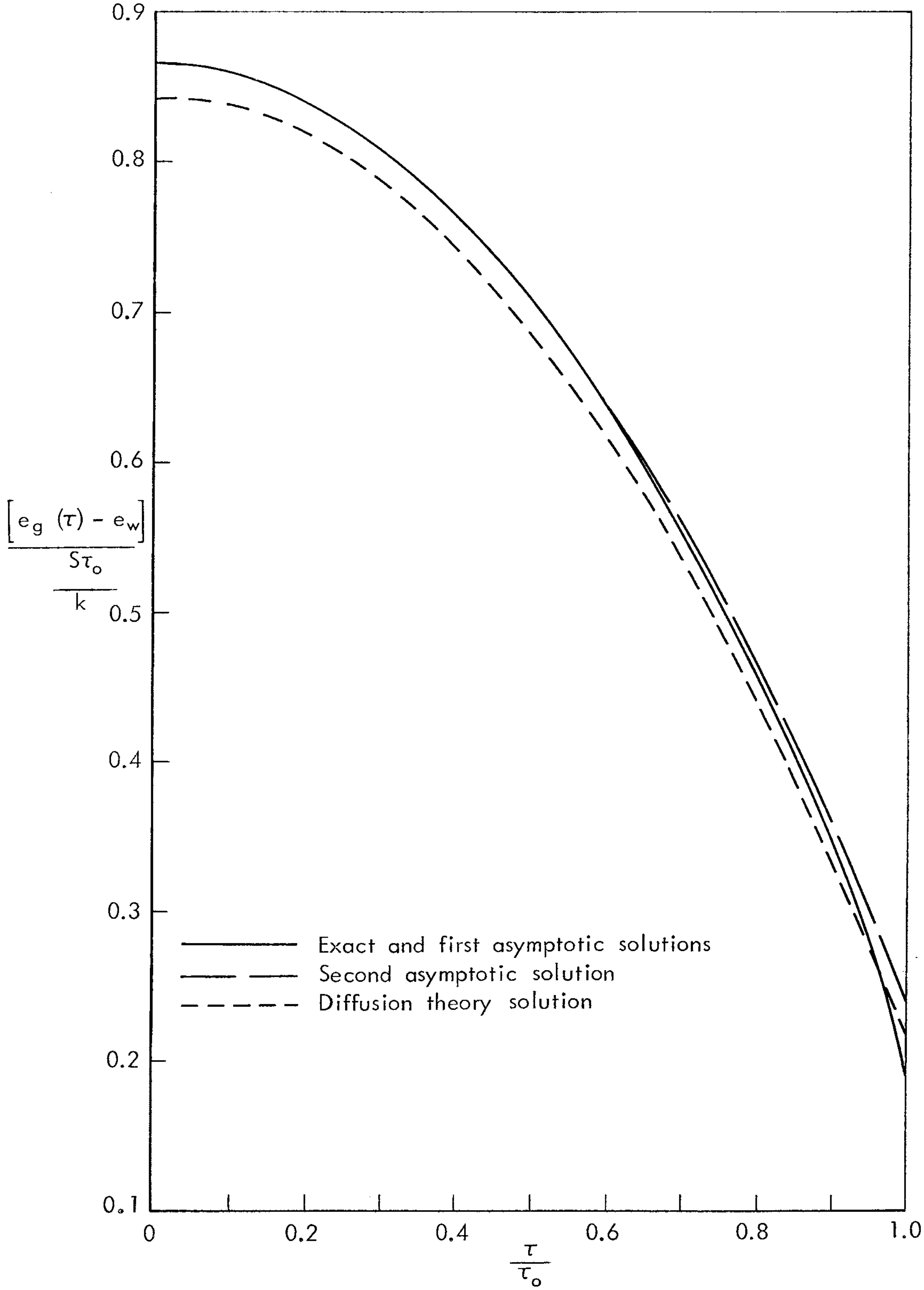

Fig. 6—Comparison of the exact, second asymptotic, and diffusion theory solutions for the gas emissive power in a spherical cavity ($\tau_o = 5.0$)

Table 1

VALUES OF THE SURFACE DIMENSIONLESS EMISSIVE POWER DISTRIBUTION $\omega(\tau_o)$
FOR A SPHERICAL GASEOUS MEDIUM

Solutions	Optical radius, τ_o						
	0.5	1.0	1.5	2.0	3.0	5.0	10.0
Exact	0.62770	0.37987	0.29827	0.25802	0.21847	0.18773	0.16550
First asymptotic	0.63435	0.38041	0.29835	0.25803	0.21848	0.18773	0.16550
Second asymptotic	0.97606	0.54435	0.41024	0.34647	0.28573	0.23994	0.20768
Diffusion theory	0.66667	0.41667	0.33333	0.29167	0.25000	0.21670	0.19167
Heaslet and Warming(5)	0.6280	0.3800	0.2978	0.2580	0.2185	0.1878	(a)

[a]Not given.

Again excellent agreement between the first asymptotic solution and the exact solution is clearly shown even for $\tau_o = 0.5$. As expected, diffusion theory and the second asymptotic solution are incapable of providing an accurate approximation to the exact solution at the boundary. Even for $\tau_o = 10.0$ an error of 15 percent exists between the exact and diffusion theory solutions.

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