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VALUES OF NON-ATOMIC GAMES, III: VALUES AND DERIVATIVES

R. J. Aumann and L. S. Shapley

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PREFACE

This is a continuation of RM-5468-PR and RM-5842-PR: "Values of Non-atomic Games, Part I: The Axiomatic Approach" and "Part II: The Random Order Approach." Non-atomic games are models for competitive situations in which there are many participants, none of whom has any appreciable influence as an individual. Such games have recently attracted attention as models for mass phenomena in economics.

Professor Aumann is Head of the Mathematics Department at Hebrew University in Jerusalem. This work was largely done in the summer of 1968 when Dr. Aumann was at RAND as a consultant to the Mathematics Department; it was completed by Aumann with support from various Office of Naval Research contracts.

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SUMMARY

The value of an n-person game is a function that associates to each player a number that, intuitively speaking, represents an a priori opinion of what it is worth to him to play in the game. A non-atomic game is a special kind of infinite-person game, in which no individual player has significance; such games have recently attracted attention as models for mass phenomena in economics. This is the third in a series of four papers in which the value concept is extended to certain classes of non-atomic games.

In this Memorandum, the characteristic function of the game is extended to a space that includes not only the ordinary coalitions (measurable sets) of players, but also certain "ideal" coalitions--intuitively, sets to which a player can "belong" at any intensity between 0 and 1. The value of the game is then redefined in terms of the derivatives of this extended function, following the principle that a player's value is based upon his incremental worth to coalitions. The close relation between this definition and those previously considered is established.

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20. INTRODUCTION TO PART III

This is a continuation of Parts I and II of "Values of Non-atomic Games," entitled "The Axiomatic Approach" and "The Random Order Approach" respectively.* Familiarity with parts I and II will be assumed throughout. Numeration of the sections will be continued serially here, to enable easy reference to the previous parts. Other conventions established previously will also be maintained here.

Let us recall formula (3.1) for the value of a "vector measure game," i.e. a game of the form $v = f \circ \mu$, where μ is an n-dimensional vector measure and f is a real function of n real variables. The formula reads

$$\varphi(f \circ \mu)(S) = \int_0^1 f_{\mu(S)}(t\mu(I))dt,$$

where $f_{\mu(S)}$ is the derivative of f in the direction $\mu(S)$. This formula is of central importance in the study of values. It is the purpose of this part to reformulate and generalize it, and thereby also gain a better insight into what the formula says.

Formula (3.1) may be intuitively understood as follows: Suppose the players could be ordered "at random." Then if T were an initial segment in such a random ordering, $\mu(T)$ would with probability 1 be on the diagonal $[0, \mu(I)]$, i.e. it would be of the form $t\mu(I)$ (compare the discussion at the beginning of Sec. 13, and the proofs of Propositions 19.3 and 19.7). Now let a coalition S be given. In a

*See [I, II] in the list of references.

random ordering, S would be "evenly spread" over the entire player set; in particular if T is an initial segment with $\mu(T) = t\mu(I)$, then we would have $\mu(S \cap T) = t\mu(S)$ (with probability 1). Suppose now that $T \cup \Delta T$ is another initial segment, where ΔT is a "small" segment disjoint from T; set $\mu(T \cup \Delta T) = (t + \Delta t)\mu(I)$. Then $\mu(S \cap [T \cup \Delta T]) = (t + \Delta t)\mu(S)$, and hence $\mu(S \cap \Delta T) = \mu(S)\Delta t$. Now let us appraise the contribution to v—over and above v(T)— of that portion of S that is in ΔT. We have

$$(f\circ\mu)(T \cup [S \cap \Delta T]) - (f\circ\mu)(T)$$

$$= \frac{f(t\mu(I) + \mu(S)\Delta t) - f(t\mu(I))}{\Delta t}\,\Delta t.$$

If we now think of Δt as an "infinitesimal" segment, then the right side of the above equation becomes, by definition,

$$f_{\mu(S)}(t\mu(I))dt.$$

The sum total of all these infinitesimal contributions of S is the total contribution of S in a random ordering, and it is exactly the right side of (3.1).

We wish to apply this reasoning to the more general situation in which v is not of the form $f\circ\mu$. The result will be a reformulation of (3.1) which is valid for a more general class of v's, and which is stated directly in terms of the set function v, rather than in terms of the representation of v in the form $f\circ\mu$.

The above intuitive reasoning is based on the notion

of an "evenly spread" measurable set. We know that no such set exists; nevertheless, it is this ideal that must somehow be embodied in the reformulation of (3.1) that we are seeking. Now we can generalize—or idealize—the notion of set by specifying for each point a weight between 0 and 1, which indicates the "degree" to which that point belongs to the set; ordinary sets are then characterized by weights which are 1 or 0, according as the point in question does or does not belong to the set. If we admit this more general—or ideal—kind of set, then we could say that a set which assigns a constant weight to all points of I is "evenly spread" over I; this could form a basis for a generalization and formalization of the above intuitive justification of (3.1).

Formally, let us define a measurable ideal subset of $(I,\mathcal{C})$ (or simply ideal set) to be a measurable function from $(I,\mathcal{C})$ to $([0,1],\mathcal{B})$. To an ordinary measurable set—i.e. member S of $\mathcal{C}$—there corresponds naturally an ideal set, namely its characteristic function χ_S. The family of all measurable ideal subsets* of $(I,\mathcal{C})$ will be denoted $\mathcal{I}$.

Although formally an ordinary set S is not an ideal set, we will find it convenient in intuitive discussion to identify it with the ideal set χ_S corresponding to it. Under this convention, $\mathcal{C}$ becomes a subset of $\mathcal{I}$. The situation is analogous to that in algebra, where a member

*Compare [Z]. Formally, our ideal sets are similar to the "fuzzy sets" of [Z], but intuitively the ideas are somewhat different. In particular, the topology we shall define on $\mathcal{I}$ (the "NA-topology") does not seem to fit in well with the intuitive explanation in [Z].

of a ring is sometimes identified—especially in intuitive discussion—with the principal ideal that it generates.

Since ideal sets are actually functions, they can, with certain restrictions, be multiplied by constants, added, and subtracted. Thus if S is an ideal set, and $0 \leq \alpha \leq 1$, then αS is also an ideal set; and if S and T are ideal sets, so is $S \pm T$, as long as the values of $S \pm T$ are in $[0,1]$. Furthermore, to the set-theoretic notions of union, intersection, and inclusion on ordinary sets, there correspond, respectively, the algebraic notions of max (or sup), min (or inf), and pointwise "less than or equal to" on ideal sets. When identifying ordinary and ideal sets, we may use either the set-theoretic or algebraic terminology, whichever is more convenient.

Now let v be a set function. A priori, v is defined on $\mathcal{C}$ only; however, since we are thinking of the ideal sets as being in some sense approximable by ordinary sets, let us suppose for the moment that v has been extended to all of $\mathcal{I}$. Let us return to the intuitive reasoning that we used above in our discussion of formula (3.1). If the players could be ordered at random, then with probability 1 each initial segment would be "evenly spread", i.e. it would be an ideal set of the form tI, where $0 \leq t \leq 1$. Now let S be an arbitrary set (ordinary or ideal). Let $(t + \Delta t)I$ be an "evenly spread" initial segment that is slightly larger than tI. The contribution to v—over and above v(T)—of that portion of S that is in $(t + \Delta t)I$ but

not in tI may be expressed by

$$v([S \cap (t+\Delta t)I] \cup tI) - v(tI)$$

$$= v(tI + (\Delta t)S) - v(tI)$$

$$= \frac{v(tI + (\Delta t)S) - v(tI)}{\Delta t}\,\Delta t.$$

If Δt is "infinitesimal", then

$$\frac{v(tI + (\Delta t)S) - v(tI)}{\Delta t} = \frac{d}{d\tau}\, v(tI + \tau S);$$

hence the contribution of that portion of S that we are looking at is

$$\frac{d}{d\tau}\, v(tI + \tau S)\, \Delta t.$$

The value $(\varphi v)(S)$ is the total contribution of S in a "random ordering", and hence is the sum total of all these infinitesimal contributions. Intuitively, therefore, we should have

$$(20.1) \qquad (\varphi v)(S) = \int_0^1 \left[\frac{d}{d\tau}\, v(tI + \tau S)\right] dt.$$

A rigorous statement and proof of formula (20.1) for v in pNA, is the main object of Part III. It is also possible to interpret formula (20.1) in terms of Fréchet differentials, and this will be done.

In Sec. 21, our main result will be stated. Of central importance in the statement of this result is the extension of v from $\mathcal{C}$ to $\mathcal{J}$; this extension will be studied

in detail in Sec. 22. Our main result will be proved in Sec. 23, and in Sec. 24 we will discuss the interpretation in terms of Fréchet derivatives. In Sec. 25 we will relate the notion of extension to the mixing and asymptotic approaches to value studied in Part II.

21. STATEMENT OF RESULTS

The set of measurable functions from the underlying space $(I, \mathcal{C})$ to the unit interval $[0,1]$ is denoted $\mathcal{J}$; the members of $\mathcal{J}$ are called ideal sets. We define a partial order on $\mathcal{J}$ by $f \geqq g$ if $f(s) \geqq g(s)$ for all $s \in I$. A real-valued function w on $\mathcal{J}$ with $w(0) = 0$ is called an ideal set function (i.e., a function on ideal sets); it is called monotonic if $f \geqq g \Rightarrow w(f) \geqq w(g)$. The characteristic function of a member S of $\mathcal{C}$ is denoted χ_S.

THEOREM D. There is a unique mapping that associates with each $v \in pNA$ an ideal set function v^*, so that

$$(21.1) \qquad (\alpha v + \beta w)^* = \alpha v^* + \beta w^*$$

$$(21.2) \qquad (vw)^* = v^* w^*$$

$$(21.3) \qquad \mu^*(f) = \int_I f d\mu$$

(21.4) $\quad v$ monotonic $\Rightarrow v^*$ monotonic

whenever $v, w \in pNA$, $\alpha, \beta \in E^1$, $\mu \in NA$ and $f \in \mathcal{J}$.

The ideal set function v^* is called the extension of v. The extension has other interesting and desirable properties, and in particular the property

$$v^*(\chi_S) = v(S),$$

which, together with the other properties, justifies its name. Similar extensions can be defined on spaces that are much larger than pNA, some of which are not even contained in BV. These matters will be investigated in Sec. 22, where also Theorem D will be proved.

We now come to the main object of this paper, namely a rigorous formulation of the idea embodied in formula (20.1). Denote

$$\partial v^*(t,S) = \frac{d}{dt}\, v^*(t\chi_I + t\chi_S),$$

where the derivative on the right is evaluated at $t = 0$ (of course no claim is being made about the existence of the derivative, we are merely introducing a notation). To relate this to previously defined notions, note that when $v = f\circ\mu$ and f is continuously differentiable on the range of μ, then

$$\partial v^*(t, S) = f_{\mu(S)}(t\mu(I))$$

(compare Theorem B in Sec. 3); hence when the range of μ has full dimension, we have

$$\partial v^*(t, S) = \Sigma_i\, \mu_i(S) f_i(t\mu(I)).$$

THEOREM E. For each v in pNA and each $S \in \mathcal{C}$, the derivative $\partial v^*(t_0 S)$ exists for almost all t in $[0,1]$, and is integrable over $[0,1]$ as a function of t; and if φ is the value on pNA, then

$$(\varphi v)(S) = \int_0^1 \partial v^*(t,S)\,dt.$$

22. EXTENSIONS: THE AXIOMATIC APPROACH

Our first object in this section is to prove Theorem D. Afterwards we will investigate the subject of extensions from a somewhat broader viewpoint.

An ideal set function is said to be of bounded variation if it is the difference of two monotonic ideal set functions. The space of all ideal set functions of bounded variation will be denoted IBV. For $v \in$ IBV, define

$$\|v\| = \inf(u(\chi_I) + w(\chi_I)),$$

where the inf ranges over all monotonic ideal set functions u and w such that

$$v = u - w.$$

The quantity $\|v\|$ will be called the variation of v; it is easily seen that it is a norm.

This definition of the variation of an ideal set function is completely analogous to the definition of variation for an ordinary set function given in Sec. 3. It also has analogous properties; in particular, IBV with the variation norm is a Banach space, in which the set of all monotonic ideal set functions forms a closed cone. Moreover, we may define the notions of chain, link, and subchain analogously to the definitions in Sec. 4, and the precise analogue of Proposition 4.1 holds. The proof is again entirely analogous.

We next define a topology on $\mathcal{I}$ which will be needed in the proof of the existence part of Theorem D. Each member μ of NA induces a function $\mu^{\#}$ on $\mathcal{I}$ defined by

$$\mu^{\#} f = \int_I f d\mu.$$

The NA-topology on $\mathcal{I}$ is defined to be the smallest topology for which all these linear functionals are continuous.

The following convention will be useful:

CONVENTION. If the range of integration of an integral is not specified, it shall be taken to be I. An integral with respect to a vector measure is the vector of integrals with respect to its components.

In the study of the NA-topology, we shall use the following lemma:

LEMMA 22.1. Let μ be a finite-dimensional vector of measures in NA. Let g_1 and g_2 be in $\mathcal{I}$, and $g_2 \geq g_1$. Then there are T_1 and T_2 in $\mathcal{C}$ with $T_2 \supset T_1$ such that for $i = 1, 2$,

$$\mu(T_i) = \int g_i d\mu.$$

Proof. First let g_1 and g_2 be simple* functions; in that case we may, w.l.o.g., write

$$g_i = \Sigma_{j=1}^{m} \alpha_{ij} \chi_{S_j},$$

where $0 \leq \alpha_{1j} \leq \alpha_{2j} \leq 1$ and the S_j are disjoint. By

*i.e., taking only finitely many values.

Lemma 5.4, we may find sets S_{1j} and S_{2j} with $S_{1j} \subset S_{2j} \subset S_j$ and

$$\mu(S_{ij}) = \alpha_{ij}\ \mu(S_j), \quad i = 1, 2.$$

If we then define

$$T_i = \cup_{j=1}^{m} S_{ij},$$

then $T_2 \supset T_1$ and $\mu(T_i) = \int g_i d\mu$ are easily verified.

Consider now the set X of all vectors (or matrices) $(\mu(T^1), \mu(T^2), \mu(T^3))$ as (T^1, T^2, T^3) ranges over all ordered partitions of I into three sets in $\mathcal{C}$; by Lemma 8.11, X is convex and compact. If we set $T^1 = T_1$, $T^2 = T_2 \backslash T_1$, $T^3 = I \backslash T_2$, where the T_i are as constructed above, then we see that

$$(22.2) \quad (\int g_1\, d\mu, \int (g_2 - g_1) d\mu, \int (1 - g_2) d\mu) \in X$$

whenever g_1 and g_2 are simple functions in $\mathcal{J}$ with $g_1 \leq g_2$. But if g_1 and g_2 are <u>any</u> function in $\mathcal{J}$ with $g_1 \leq g_2$, then they can be approximated in the supremum norm by simple functions of this kind; and then from the compactness of X we obtain (22.2) in this case as well. Thus there exist corresponding T^1, T^2, and T^3, and by setting $T_1 = T^1$ and $T_2 = T^1 \cup T^2$ we complete the proof of Lemma 22.1.

The following corollary shows that by extending the domain of a non-atomic vector measure to include ideal sets, as well as ordinary sets, we do not enlarge its range.

COROLLARY 22.3. Let μ be a finite dimensional vector of measures in NA, and let g be in $\mathcal{J}$. Then there is a $T \in \mathcal{C}$ with

$$\mu(T) = \int g d\mu.$$

Proof. In Lemma 22.1, set $g_1 = g_2$.

An important application of this corollary is

PROPOSITION 22.4. In the NA-topology on $\mathcal{J}$, the set of all characteristic functions χ_S is dense in $\mathcal{J}$.

Proof. If $g \in \mathcal{J}$, then every neighborhood of g contains a neighborhood of the form

$$\{f \in \mathcal{J} : \|\int (f - g) d\mu\| < \epsilon\}$$

where μ is a finite dimensional vector of measures in NA, $\epsilon > 0$, and $\| \quad \|$ is the maximum norm. Applying Corollary 22.3, we see that this neighborhood contains the characteristic functions χ_{S_2}. This completes the proof of Proposition 22.4.

We now proceed to the

Proof of Theorem D. First we prove uniqueness. Let $v \in pNA$. If, in fact, $v \in NA$, then v^* is determined by (21.3). Hence if v is a polynomial in measures, then v^* is determined by (21.1) and (21.2). Since the polynomials in measures are dense in pNA, it remains only to prove

that the mapping $v \to v^*$ is a continuous mapping from BV to IBV. To this end, let $v \in pNA$ and let $\varepsilon > 0$; by Proposition 7.25 we may find u and w in $(pNA)^+$ such that $v = u - w$ and

$$u(I) + w(I) \leqq \|v\| + \varepsilon.$$

By (21.4), both u^* and w^* are monotonic; hence

$$\|v^*\| = \|(u - w)^*\| = \|u^* - w^*\| \leqq \|u^*\| + \|w^*\|$$
$$= u^*(\chi_I) + w^*(\chi_I) = u(I) + w(I) \leqq \|v\| + \varepsilon.$$

Since ε may be chosen arbitrarily small, it follows that $\|v^*\| \leqq \|v\|$. The opposite inequality is readily established, and so we have $\|v^*\| = \|v\|$. In particular, the map $v \to v^*$ is continuous, and the proof of uniqueness is complete.

Next, we prove existence. Imbed $\mathcal{C}$ in $\mathcal{I}$ by identifying each S in $\mathcal{C}$ with its characteristic function χ_S. Impose the NA-topology on $\mathcal{I}$, and the induced subspace topology on $\mathcal{C}$. Our first aim is to show that every v in pNA is then continuous on $\mathcal{C}$.

If $v \in NA$, then the continuity of v on $\mathcal{C}$ follows from the definition of the NA topology. From this it follows that v is continuous also if it is a polynomial in measures. Suppose finally that v is an arbitrary member of pNA. Then we can find a sequence $v_1, v_2, \ldots$ of polynomials in nonatomic measures such that $v_i \to v$ in the variation norm, and hence also in the supremum norm, i.e. uniformly on $\mathcal{C}$.

Since the v_i are continuous on $\mathcal{C}$, it follows that v also is.

From the continuity of v on $\mathcal{C}$ and the denseness of $\mathcal{C}$ in $\mathcal{I}$ (Proposition 22.4) it follows that v has a unique continuous extension v^* to $\mathcal{I}$. It is this v^* that we claim satisfies the conditions of Theorem D. Indeed, to establish (21.1) through (21.3), we note that $(\alpha v + \beta w)^* - (\alpha v^* + \beta w^*)$, $(vw)^* - v^*w^*$, and $\mu^* - \mu^\#$ are continuous extensions of 0 and hence must vanish identically. It remains only to establish (21.4).

To this end, let v be monotonic, and let g_1 and g_2 in $\mathcal{I}$ be such that $g_2 \geqq g_1$. Every neighborhood of any g in $\mathcal{I}$ contains a neighborhood of the form

$$\{f \in \mathcal{I}: \|\int (f - g)d\mu\| < \epsilon\}$$

where μ is a finite dimensional vector of measures in NA, and $\epsilon > 0$. Since v^* is continuous on $\mathcal{C}$, it follows that for every $\delta > 0$ there are nonnegative measures μ_1 and μ_2 in NA and an $\epsilon > 0$ such that for $i = 1, 2$ and all $f \in \mathcal{I}$ we have

$$(22.5) \qquad \|\int (f - g_i)d\mu_i\| < \epsilon \Rightarrow |v^*(f) - v^*(g_i)| < \delta.$$

Consider now the vector measure $\mu = (\mu_1, \mu_2)$. By Lemma 22.1, there are sets T_1 and T_2 in $\mathcal{C}$ with $T_2 \supset T_1$ and

$$\mu(T_i) = \int g_i d\mu;$$

in particular, therefore,

$$\int \chi_{T_i} d\mu_i = \mu_i(T_i) = \int g_i d\mu_i.$$

Hence if in (22.5) we set $f = \chi_{T_i}$, then we obtain

$$|v(T_i) - v^*(g_i)| = |v^*(\chi_{T_i}) - v^*(g_i)| < \delta.$$

Using $T_2 \supseteq T_1$ and the monotonicity of v, we deduce

$$v^*(g_2) - v^*(g_1) > -2\delta;$$

and therefore, since δ may be arbitrarily chosen,

$$v^*(g_2) - v^*(g_1) \geq 0.$$

This completes the proof of (21.4), and with it, the proof of Theorem D.

Theorem D is all we used to know about extensions for the immediate chief purpose of this paper, namely the statement and proof of Theorem E; the subject of the extension of set functions to ideal sets is of some interest in its own right, though, and we would like to investigate this subject somewhat further. Our remarks fall into two classes: In the remainder of this section we will discuss some of the abstract properties of the operator $v \to v^*$, stressing in particular how this operator can be uniquely defined on spaces that are much larger than pNA; in Sec. 25 we will discuss some of the more "concrete" properties of v^* for individual v's, discussing how $v^*(f)$ can be calculated for given v and f and relating it to procedures

previously discussed, such as those leading to the mixing and asymptotic values.

First, let us recall two facts that were used in the proof of Theorem D, and that are worth remembering in their own right. The first is that in the NA-topology,

(22.6) $\qquad v^*$ is continuous on $\mathcal{I}$;

this was used in the existence part of the proof. The second is that the operator $v \to v^*$ is continuous and that, in fact, its norm is 1; that is

(22.7) $$\|v^*\| = \|v\|.$$

This was used in the uniqueness part of the proof. Now (22.7) was established when the variation norm is imposed both on BV and on IBV; but it is true also in another norm, namely the supremum norm $\| \ \|'$. Indeed, define

$$\|v\|' = \sup \{|v(S)| : S \in \mathcal{C}\}$$

for bounded set functions v, and

$$\|w\|' = \sup \{|w(f)| : f \in \mathcal{I}\}$$

for bounded ideal set functions w. Then using (22.6) and Proposition 22.4, we obtain

(22.8) $$\|v^*\|' = \|v\|',$$

from which we deduce that

(22.9) $v^* \to v$ is continuous in the supremum norm.

This suggests that it might be possible to define the extension operator $v \to v^*$ on a class of set functions much wider than pNA. Let BOUND be the Banach space of all bounded set functions with the supremum norm, and let pNA' be the subspace of BOUND spanned by the powers of nonatomic measures. pNA' is much larger than pNA; for example, it contains the set function of Example 5.8 and also set functions of the form $f \circ \mu$, where $\mu \in$ NA and f is a singular continuous function. It even contains set functions outside of BV; for example, the set function of Example 5.7 is in it.

PROPOSITION 22.10. There is a unique mapping that associates with each $v \in$ pNA' an ideal set function v^* so that (21.1), (21.2), (21.3) and (22.9) are satisfied. On pNA, this mapping coincides with that of Theorem D. Finally, this mapping obeys (22.6).

The proof of this proposition follows the same lines as that of Theorem D, and the reader will have no difficulty in reconstructing it. In some respects the proof is, in fact, easier than that of Theorem D.

It is possible to ring the changes on this proposition in a number of ways. First of all, (22.9) is interchangable with (22.8); this is readily verified. Next, (21.3) is interchangable with

(22.11) If $\mu \in NA$, then $\mu^*(\chi_S) = \mu(S)$ for all $S \in \mathcal{C}$ and $\mu^*(f + g) = \mu^*(f) + \mu^*(g)$ whenever f and g are ideal sets with $f + g \leq \chi_I$.

Condition (22.11) says that μ^* is, in a sense, a measure on ideal sets that extends the given measure; its interchangeability with (21.3) (in the presence of the other conditions, of course) is readily verified. Finally, (21.2) is interchangeable with

(22.12) If f is a continuous real-valued function, then $(f \circ v)^* = f \circ v^*$.

Indeed, to deduce (22.12) from (21.2) we approximate on the range of v to f in the supremum norm by polynomials. The reverse direction is also easy; it uses

$$vw = ((v + w)^2 - (v - w)^2)/4.$$

Still another version of (21.2), as is readily verified, is

(22.13) $(v^k)^* = (v^*)^k$ for all positive integers k.

These interchanges can be made independently of one another. Specifically, we have

Remark 22.14. Proposition 22.10 remains true if any one of (22.8) or (22.9) is substituted for (22.9), and

simultaneously any one of (21.3) or (22.11) is substituted for (21.3), and simultaneously any one of (21.2), (22.12) or (22.13) substituted for (21.2). All the mappings defined in this way coincide with each other.

23. PROOF OF THEOREM E

The subset S of I will be fixed throughout this section; to simplify the notation, therefore, we will write $\partial v^*(t)$ instead of $\partial v^*(t,S)$.

Define

$$|\partial v^*(t)|^+ = \limsup_{\tau \to 0} \left| \frac{v^*(t\chi_I + \tau\chi_S) - v^*(t\chi_I)}{\tau} \right| .$$

LEMMA 23.1. If v is in pNA, then $|\partial v^*(t)|^+$ is integrable over $[0,1]$, and we have

$$\int_0^1 |\partial v^*(t)|^+ \, dt \leq \|v\|.$$

Proof. First let v be monotonic. Then v^* is also monotonic, and for $\tau > 0$ we have

$$0 \leq v^*(t\chi_I + \tau\chi_S) - v^*(t\chi_I) \leq v^*((t+\tau)\chi_I) - v^*(t\chi_I).$$

For $\tau < 0$ we have

$$0 \geq v^*(t\chi_I + \tau\chi_S) - v^*(t\chi_I) \geq v^*((t+\tau)\chi_I) - v^*(t\chi_I).$$

In either case division by τ yields

$$(23.2) \quad 0 \leq \frac{v^*(t\chi_I + \tau\chi_S) - v^*(t\chi_I)}{\tau} \leq \frac{v^*((t+\tau)\chi_I) - v^*(t\chi_I)}{\tau} .$$

The function $g(t) = v^*(t\chi_I)$ is monotonic in t, and hence is a.e. differentiable. Hence if we let $\tau \to 0$ in (23.2), then the right side tends a.e. to the limit $g'(t)$. The middle term may not converge, but from (23.2) we obtain

(23.3) $$|\partial v^*(t)|^+ \leqq g'(t) \quad \text{a.e.}$$

From the monotonicity of g we obtain

(23.4) $$\int_0^1 g'(t) \leqq g(1) - g(0) = v(I) = \|v\|;$$

for example, this follows from decomposing g into an absolutely continuous and a singular part and then using, say, (8.4) (alternatively, see [T, § 11.54, p. 361]). The conclusion of Lemma 23.1 follows from (23.3) and (23.4).

In the general case, when v is not necessarily monotonic, let $\varepsilon > 0$ be given, and set

$$v = u - w,$$

where u and w in pNA are monotonic and

$$\|v\| + \varepsilon \geqq \|u\| + \|w\|;$$

such u and w exist by Proposition 7.25. We then have

$$v^* = u^* - w^*,$$

and hence

$$|\partial v^*(t)|^+ \leqq |\partial u^*(t)|^+ + |\partial w^*(t)|^+.$$

Integrating this inequality, and using the monotonic case of the lemma (which we have already proved), we obtain

$$\int_0^1 |\partial v^*(t)|^+ dt \leqq \|u\| + \|w\| \leqq \|v\| + \varepsilon;$$

and if we let $\varepsilon \to 0$, the proof of Lemma 23.1 is complete.

Define

$$(23.5)\quad \Delta_v(t) = \limsup_{\tau\to 0} \frac{v^*(t\chi_I + \tau\chi_S) - v^*(t\chi_I)}{\tau} - \liminf_{\tau\to 0} \frac{v^*(t\chi_I + \tau\chi_S) - v^*(t\chi_I)}{\tau}$$

and

$$\Delta_v = \int_0^1 \Delta_v(t)dt.$$

Clearly

$$0 \leq \Delta_v(t) \leq 2|\partial v^*(t)|^+,$$

and hence by Lemma 23.1,

$$(23.6)\quad \Delta_v \leq 2\int_0^1 |\partial v^*(t)|^+ dt \leq 2\|v\|.$$

Furthermore, since the lim sup is subadditive and the lim inf superadditive, we have

$$\Delta_{v+w}(t) \leq \Delta_v(t) + \Delta_w(t)$$

and hence

$$(23.7)\quad \Delta_{v+w} \leq \Delta_v + \Delta_w.$$

Now it is easily verified that $\Delta_v = 0$ when v is a power of a measure, and hence by (23.6), when it is any polynomial in measures. If now v is any member of pNA, let $\varepsilon > 0$ be given, and let w be a polynomial in measures such that $\|v - w\| < \varepsilon$. Then by (23.7) and (23.6),

$$\Delta_v \leq \Delta_w + \Delta_{v-w} = \Delta_{v-w} \leq 2\|v-w\| < 2\epsilon.$$

Letting $\epsilon \to 0$, we deduce that $\Delta_v = 0$ for all v in pNA. Hence $\Delta_v(t) = 0$ for almost all t, i.e. $\partial v^*(t)$ exists a.e. for all v. Whenever it exists we have $|\partial v^*(t)| = |\partial v^*(t)|^+$, and hence by Lemma 23.1,

$$\int_0^1 |\partial v^*(t)|dt \leq \|v\|; \tag{23.8}$$

in particular, this implies the integrability of $\partial v^*(t)$.

Now let

$$\theta v = \int_0^1 \partial v^*(t)dt.$$

θv is clearly linear in v, and from (23.8) we obtain

$$|\theta v| \leq \|v\|;$$

thus θ is a continuous linear functional on pNA. When v is a power of a measure, it is easily verified that $\theta v - (\varphi v)(S) = 0$. Hence this holds also when v is a polynomial in measures; and hence, since both θv and $\varphi v(S)$ are continuous in v, when v is any member of pNA. This completes the proof of Theorem E.

We close this section with another*

Alternative proof for Example 5.8. If $v \in$ pNA, then by (22.12),

$$v^*(f) = |\int f d\mu|.$$

*In addition to the original proof in Sec. 5, a proof was given in Sec. 17.

If we set $S = [0,1]$, then

$$\int (t\chi_I) d\mu = 0$$

and

$$\int (t\chi_I + \tau\chi_S) d\mu = t.$$

Hence

$$v^*(t\chi_I + \tau\chi_S) - v^*(t\chi_I) = |\tau|.$$

Therefore $\partial v^*(t) = \lim_{t \to 0} |\tau|/\tau$ never exists, and so by Theorem E, v cannot be in pNA. This completes the proof.

Note that though v is not in pNA, it <u>is</u> in pNA'.

24. FRÉCHET DIFFERENTIALS

It is possible to strengthen Theorem E as follows: Not only is it true that for each S, the derivative $\partial v^*(t,S)$ exists for almost all t, but we can reverse the quantifiers and assert that for almost all t, it is true that for each S, $\partial v^*(t,S)$ exists. Furthermore, we will be able to show that for almost all t, $\partial v^*(t,S)$ is a measure in S.

These ideas find their proper expression in terms of Fréchet differentials, and by replacing the ordinary sets S appearing in the derivative $\partial v^*(t,S)$ by ideal sets. First let us define the Fréchet differential; we follow [Dun - S], p. 92. Let X be a Banach space and let u be a function from an open subset of X to be reals. A Fréchet differential of u at a point x in X is a linear functional $Du(x,\cdot)$ from X to the reals such that

$$u(x + h) = u(x) + Du(x,h) + o(\|h\|)$$

as $\|h\| \to 0$. Of course it does not necessarily exist; but if it does exist, it is necessarily unique, since the difference of two Fréchet differentials is a linear functional in h that is $o(\|h\|)$, and so vanishes. The Fréchet differential is defined similarly when the range space of u is any Banach space rather than the reals; but we do not need this here. Roughly, we may say that u has a Fréchet differential at x if it can be approximated by a linear

functional in the neighborhood of x.

In our situation, let $\mathcal{M}$ be the Banach space of all bounded measurable functions from the underlying space $(I,\mathcal{C})$ to the reals, where the norm is defined by

$$\|f\| = \sup_{t\in\mathcal{I}} |f(t)|.$$

The space $\mathcal{I}$ of all ideal sets is a closed subset of $\mathcal{M}$. We wish to consider the Fréchet differential of v; however, since the domain of the function to be differentiated must be open, we will restrict our attention to the restriction of v^* to the interior $\mathcal{I}^\circ$ of $\mathcal{I}$. The set $\mathcal{I}^\circ$ may be characterized as the set of all $f \in \mathcal{I}$ that are bounded away from 0 and from 1 (i.e. such that here exist α and β with $0 < \alpha \leq f(t) \leq \beta < 1$ for all $t \in I$).

PROPOSITION 24.1. Let $v \in pNA$. Then for almost all $t \in (0,1)$, the following statements hold:

i) The extension v^* has a Fréchet differential $Dv^*(t\chi_I,\cdot)$ at $t\chi_I$.

ii) $\partial v^*(t,S)$ exists for all $S \in \mathcal{C}$.

iii) If we denote $\partial_t v^*(S) = \partial v^*(t,S)$, then $\partial_t v^*$ is a measure in NA, and for all $f \in \mathcal{I}$,

$$Dv^*(t\chi_I,f) = \int f d\partial_t v^*.$$

The integral $\int f d\partial_t v^*$ is best viewed as the $\partial_t v^*$-measure of the ideal set f. Thus our theorem states that for almost all t, the ideal set function v^* behaves like

a constant plus a measure in the neighborhood of the point t. According to Theorem E, the value is the integral of these measures over t.

Before proceeding with the proof of Proposition 24.1, we state a lemma which we will use on a number of occasions throughout this proof. Let us call a real function q on a linear space X subadditive if $q(x + y) \leqq q(x) + q(y)$ for all x and y in X.

LEMMA 24.2. Let q be a real nonnegative subadditive function on pNA such that for some constant α,

$$q(v) \leqq \alpha \|v\|$$

for all v, and such that q(v) vanishes whenever v is a polynomial in measures. Then q vanishes identically.

Proof. This was proved in Sec. 23 in the particular case $q(v) = \Delta_v$ and $\alpha = 2$; the proof is identical here.

Proof of Proposition 24.1. Throughout this proof the letters "a.e." (an abbreviation for "almost everywhere") will mean "for almost all t in (0,1)." Define

$$|Dv^*(t)|^+ = \limsup_{\|f\| \to 0} \left| \frac{v^*(t\chi_I + f) - v^*(t\chi_I)}{\|f\|} \right|.$$

When v is monotonic then this lim sup remains the same if we restrict f to be of the form $\tau\chi_I$, since by replacing f by $\|f\|\chi_I$ we cannot decrease the absolute value of the

ratio on the right. Hence reasoning as in the proof of Lemma 23.1, we obtain

$$\int_0^1 |\overline{Dv}^*(t)|^+ dt \leq \|v\|. \tag{24.3}$$

Again reasoning as in the proof of Lemma 23.1, we deduce (24.3) also when v is not necessarily monotonic.

Now let

$$\Gamma_v(t) = \sup_{\|f\| \leq 1} \left[\limsup_{\tau \to 0} \frac{v^*(t\chi_I + \tau f) - v^*(t\chi_I)}{\tau} - \liminf_{\tau \to 0} \frac{v^*(t\chi_I + tf) - v^*(t\chi_I)}{\tau} \right]$$

and

$$\Gamma_v = \int_0^1 \Gamma_v(t)dt.$$

Since $\|\tau f\| = |\tau| \|f\| \leq |\tau|$, we have

$$\Gamma_v(t) \leq 2|Dv^*(t)|^+$$

and hence

$$\Gamma_v \leq 2\int_0^1 |Dv^*(t)|^+ dt \leq 2\|v\|.$$

Since Γ_v is nonnegative and subadditive, it follows from Lemma 24.2 that it vanishes identically. Hence a.e. we have that for all f with $\|f\| \leq 1$—and therefore for all $f \in \mathcal{M}$—the limit

$$\lim_{\tau \to 0} \frac{v^*(t\chi_I + \tau f) - v^*(t\chi_I)}{\tau}$$

exists. This limit will be denoted $Dv^*(t,f)$; we are, however, not yet asserting that it is a Fréchet differential. It is easily verified that

$$|Dv^*(t\chi_I, f)| \leq \|f\| |Dv^*(t)|^+ \quad (24.4)$$

and that

$$Dv^*(t\chi_I, \chi_S) = \partial v^*(t,S) \quad (24.5)$$

for all $S \in \mathcal{C}$.

We now claim that a.e., $Dv^*(t\chi_I, f)$ is linear in f. To prove this, define.

$$\Phi_v(t) = \sup |Dv^*(t, f + g) - Dv^*(t, f) - Dv^*(t, g)|,$$

where the sup is taken over all f and g with $\|f\| \leq 1$, $\|g\| \leq 1$. Define

$$\Phi_v = \int_0^1 \Phi_v(t)dt;$$

from (24.4) and (24.3) it follows that Φ_v exists and in fact

$$\Phi_v \leq 4\|v\|.$$

Next, $\Phi_v(t)$ is clearly nonnegative and subadditive in v, and vanishes whenever v is a polynomial in measures (since $\partial v^*(t\chi_I, f)$ is linear then, as is easily verified); therefore Φ_v has the same properties. So by Lemma 24.2, Φ_v vanishes for all v. Hence $\Phi_v(t)$ vanishes a.e., i.e. $Dv^*(t\chi_I, f)$ is a.e. additive in f. Since

$$Dv^*(t\chi_I, \alpha f) = \alpha Dv^*(t\chi_I, f)$$

is easily verified, the desired linearity is established. By (24.4), it then follows that $Dv^*(t\chi_I, f)$ is a.e. a continuous linear functional in f.

We now use the fact that the adjoint of $\mathcal{M}$ is the set FA of bounded finitely additive measures on $(I,\mathcal{C})$ [Dun-S, Theorem 4.5.1, p. 258]. In our situation, this yields the fact that a.e. there is a member $\nu_t = \nu_t^v$ of FA such that for all f,

$$Dv^*(t\chi_I, f) = \int f d\nu_t.$$

Setting $f = t\chi_S$ and using (24.5), we get $\nu_t(S) = Dv^*(t\chi_I, \chi_S) = \partial v^*(t, S) = \partial_t v^*(S)$; thus $\nu_t = \partial_t v^*$, and we have

$$Dv^*(t\chi_I, f) = \int f d\partial_t v^*. \tag{24.6}$$

It remains to prove that a.e. $\partial_t v^*$ is completely additive and non-atomic, and that a.e. $Dv^*(t\chi_I, \cdot)$ is a Fréchet differential.

First we prove that $\partial_t v^*$ is a.e. completely additive. When v is monotonic, then $\partial_t v^*$ is nonnegative, and so when $S_1, S_2, \ldots$ is a sequence of disjoint sets in $\mathcal{C}$, we have for all k that

$$\Sigma_{i=1}^k \partial_t v^*(S_i) = \partial_t v^*(U_{i=1}^k S_i) \leq \partial_t v^*(U_{i=1}^\infty S_i); \tag{24.7}$$

hence $\Sigma_{i=1}^\infty \partial_t v^*(S_i)$ exists and is $\leq \partial_t v^*(\Sigma_{i=1}^\infty S_i)$.

When v is not necessarily monotonic, we may, for each $\varepsilon > 0$, find monotonic u and w in pNA with

$$v = u - w$$

and

$$\|v\| + \varepsilon \geq \|u\| + \|w\| \tag{24.8}$$

(see Proposition 7.25). Then $\partial_t v^* = \partial_t u^* - \partial_t w^*$, and applying (24.7) for u and w separately, we deduce that $\Sigma_{i=1}^{\infty} \partial_t v^*(S_i)$ converges. Using (24.5) and (24.4) we then deduce

$$\begin{aligned} |\Sigma_{i=1}^{\infty} \partial_t v^*(S_i)| &\leq \partial_t u^*(U_{i=1}^{\infty} S_i) + \partial_t w^*(U_{i=1}^{\infty} S_i) \\ &\leq |Du^*(t)|^+ + |Dw^*(t)|^+; \end{aligned} \tag{24.9}$$

similarly we have

$$|\partial_t v^*(U_{i=1}^{\infty})| \leq |Du^*(t)|^+ + |Dw^*(t)|^+. \tag{24.10}$$

Now let

$$\psi_v(t) = \sup |\partial_t v^*(U_{i=1}^{\infty} S_i) - \Sigma_{i=1}^{\infty} \partial_t w^*(S_i)|,$$

where the sup is taken over all sequences of disjoint sets $S_1, S_2, \ldots$ in $\mathcal{C}$, and let

$$\psi_v = \int_0^1 \psi_v(t)dt.$$

By (24.10) and (24.9) we have

$$\Psi_v(t) \leq 2(|Du^*(t)|^+ + |Dw^*(t)|^+),$$

and combining this with (24.3) and (24.8), we get

$$\Psi_v \leq 2(\|u\| + \|w\|) \leq 2(\|v\| + \epsilon);$$

since ϵ was arbitrarily chosen, we get

$$\Psi_v \leq 2\|v\|.$$

Also, $\Psi_v(t)$ is subadditive, nonnegative, and vanishes when v is a polynomial in measures; hence Ψ_v also has these properties. Thus from Lemma 24.2 we deduce that $\Psi_v(t)$ vanishes a.e., and it follows that $\partial_t v^*$ is a.e. completely additive.

To prove that $\partial_t v^*$ is a.e. nonatomic, set

$$\Xi_v(t) = \sup_{s \in I} |\partial_t v^*(\{s\})|;$$

then using Lemma 24.2 in a manner which is by now familiar, we deduce the desired nonatomicity.

It remains only to prove that a.e. $Dv^*(t\chi_I, \cdot)$ is indeed a Fréchet differential. This means that a.e. for every $\epsilon > 0$ there is an $\eta > 0$ such that

$$\|f\| < \eta \Rightarrow |v^*(t\chi_I + f) - v^*(t\chi_I) - Dv^*(t\chi_I, f)| \leq \epsilon \|f\|;$$

or, in other words, that if we set

$$\Theta_v(t) = \limsup_{\|f\| \to 0} \left| \frac{v^*(t\chi_I + f) - v^*(t\chi_I) - Dv^*(t\chi_I, f)}{\|f\|} \right|,$$

then a.e. $\Theta_v(t) = 0$. Now

$$\Theta_v(t) \leq \limsup_{\|f\|\to 0} \left| \frac{v^*(t\chi_I + f) - v^*(t\chi_I)}{\|f\|} \right|$$

$$+ \limsup_{\|f\|\to 0} \left| \frac{Dv^*(t\chi_I, f)}{\|f\|} \right| ;$$

and using the definition of $|Dv^*(t)|^+$, and (24.4), we obtain that

$$\Theta_v(t) \leq 2|Dv^*(t)|^+.$$

Now set

$$\Theta_v = \int_0^1 \Theta_v(t)dt;$$

then from 24.3 we obtain

$$\Theta_v \leq 2\|v\|.$$

Since $\Theta_v(t)$ is nonnegative, subadditive (in v), and vanishes when v is a polynomial in measures, it follows that Θ_v has the same properties; hence by Lemma 24.2, Θ_v vanishes, and hence $\Theta_v(t)$ vanishes a.e. This completes the proof of Proposition 24.1.

25. EXTENSIONS: THE MIXING AND ASYMPTOTIC APPROACHES

In this section we return to a study of the extension per se, and show how it is related to some of the ideas discussed in Part II.

To give an idea of how the extension is connected with the notion of mixing, let $\nu \in pNA' \cap AC$. Suppose μ to be a monotonic probability measure with $\nu << \mu$; let $\{\Theta_1, \Theta_2, \ldots\}$ be a μ-mixing sequence, and let S be a subset of I. Let τ be any subset of I with $\mu(\tau) = \frac{1}{2}$, say. If ν is any measure with $\nu << \mu$, then by Corollary 16.7,

$$\nu(S \cap \Theta_n T) \to \tfrac{1}{2}\, \nu(S). \tag{25.1}$$

If (25.1) were true for all ν, rather than just for $\nu << \mu$, then we would have $S \cap \Theta_n T \to \frac{1}{2}S$ in the NA-topology*, and it would follow from (22.6) that

$$\nu(S \cap \Theta_n T) \to \nu^*(\tfrac{1}{2}\chi_S). \tag{25.2}$$

As it is, although (25.1) is certainly not true for all ν, (25.2) nevertheless does hold. That is because measures that are singular w.r.f. μ have no "relevance" to ν^*.

To make this idea more precise, let us define the NA_μ-topology to be the smallest topology for which all the functions $\nu^\#$ on $\mathcal{J}$ are continuous, where ν ranges over all measures such that $\nu << \mu$. (Recall that $\nu^\# f = \int f d\nu$.)

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*Recall our convention, explained in the introduction, of identifying ideal with ordinary sets in intuitive discussion.

Of course the NA_μ-topology is a weaker topology than the NA-topology, i.e. it has fewer open sets. We know that v^* is continuous in the NA-topology; if we could prove the stronger statement that

(25.3) v^* is continuous in the NA_μ-topology,

then (25.2) would follow from (25.1).

To demonstrate (25.3), let $g \in \mathcal{G}$ and $\varepsilon > 0$ be given. Note that the NA-topology can be defined as the smallest topology such that all the $\nu^{\#}$ are continuous, where ν ranges over all measures in NA that are either absolutely continuous or singular w.r.t. μ; this follows from the fact that any measure in NA can be decomposed into two such measures. Therefore there is a finite vector ξ of measures that are $\ll \mu$, a finite vector ζ of measures that are $\perp \mu$, and a $\delta > 0$, such that

$$(25.4)\quad \left\| \int (f - g)d\xi, \int (f - g)d\zeta \right\| < \delta \Rightarrow |v^*(f) - v^*(g)| < \frac{\varepsilon}{2} .$$

Let $h \in \mathcal{G}$ be such that

$$(25.5)\qquad \left\| \int (h - g)d\xi \right\| < \delta .$$

Again, we can find finite vectors ξ' and ζ' whose components are $\ll \mu$ and $\perp \mu$ respectively, and a $\delta' > 0$, such that

$$(25.6)\quad \left\| \int (f - h)d\xi', \int (f - h)d\zeta' \right\| < \delta' \Rightarrow |v^*(f) - v^*(h)| < \frac{\varepsilon}{2} .$$

Now apply Corollary 22.3 to find a $T \in \mathcal{C}$ such that

(25.7) $$\nu(T) = \int h d\nu,$$

where ν is the vector measure (ξ, ξ', ζ'). Let U be a set that is μ-equivalent* to T and such that

(25.8) $$\zeta(U) = \int g d\zeta;$$

the existence of such a set U follows from the fact that the components of ζ are $\ll\mu$, together with another application of Corollary 22.3. Since $\xi \ll \mu$, it then follows from (25.7) that

$$\xi(U) = \xi(T) = \int h d\xi,$$

and so by (25.4), (25.5) and (25.8),

$$|\nu(U) - \nu^*(g)| = |\nu^*(\chi_U) - \nu^*(g)| < \frac{\varepsilon}{2}.$$

But since $\nu \ll \mu$, we have

$$\nu(U) = \nu(T).$$

Finally, from (25.6) and (25.7) we get

$$|\nu(T) - \nu^*(h)| = |\nu^*(\chi_T) - \nu^*(h)| < \frac{\varepsilon}{2}.$$

Combining the last three formulas, we get

$$|\nu^*(h) - \nu^*(g)| < \varepsilon.$$

But h was chosen to be any member of $\mathcal{G}$ satisfying (25.5), and the set of all such h is a neighborhood of g in the NA_μ-topology. This completes the proof of (25.3), and with

*i.e., differs from T by a set of μ-measure 0.

it the proof of (25.2).

A more general version of (25.2) is the following:

PROPOSITION 25.9. Let $v \in pNA' \cap AC$, and let $f = \Sigma_{i=1}^{k} \alpha_i \chi_{S_i} \in \mathcal{I}$, where $\{S_1, \ldots, S_k\}$ is a partition of I. Then for all μ with $v \ll \mu$, all μ-mixing sequences $\{\Theta_1, \Theta_2, \ldots\}$, and all sets $T_1, \ldots, T_k$ with $\mu(T_1) = \alpha_1, \ldots, \mu(T_k) = \alpha_k$, we have

$$v(\cup_{i=1}^{k} (S_i \cap \Theta_n T_i)) \to v^*(f)$$

as $n \to \infty$.

The proof uses the same ideas as above. Note that in the statement of Theorem E, we only use* ideal sets of the kind appearing in Proposition 25.9—in fact only a very special subclass of these ideal sets, with $k = 2$.

Next we investigate how the extension may be characterized in terms of the ideas leading up to the asymptotic value, in particular the notions of partition and approximation by finite games. Let $v \in pNA'$. Let $(\pi_1, \pi_2, \ldots)$ be an admissible sequence of partitions** of I all of which have an even number of members. From each partition π_j choose at random exactly half of the members, in such a way that all subsets of π_j having half the members of π_j have equal probability; for example, order them at random and choose the first half. Denote the union of the chosen

*via the definition of $\partial v^*(t,S)$).

**See Sec. 18.

members by U_j. Then we claim that $v(U_j) \to v^*(\frac{1}{2}\chi_I)$ with high probability; or more precisely, for each $\epsilon > 0$,

$$(25.10) \quad \text{Prob}\{|v(U_j) - v^*(\tfrac{1}{2}\chi_I)| < \epsilon\} \to 1 \text{ as } j \to \infty.$$

To prove this, first let $v \in NA^+$; then (25.10) follows from the same ideas as used in the proof of Proposition 19.7 (see in particular formula (19.11)*). The case when v is a power of an NA^+ measure follows from the case in which it is a measure, and the case when it is a sum of powers follows from this. Finally, the general case follows by approximation from the case of polynomials in measures, using (22.8). This proves (25.10).

If we wish to characterize $v^*(\frac{1}{2}\chi_S)$, where S is an arbitrary member of $\mathcal{C}$, we must use an admissible sequence $\{\pi_1, \pi_2, \ldots\}$ whose first member is $\{S, I\backslash S\}$ and such that in each π_j after the first, the number of subsets of S is even. Then if U_j is the union of one half of the subsets of S in π_j, randomly chosen, then for all $\epsilon > 0$,

$$\text{Prob}\{|v(U_j) - v^*(\tfrac{1}{2}\chi_S)| < \epsilon\} \to 1 \text{ as } j \to \infty.$$

The proof uses formula (25.10) applied to the underlying space S rather than the underlying space I.

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*There is a misprint in [II] and two lines between lines 8 and 9 on p. 75 are missing. These two lines should read as follows:

as $m \to \infty$, then for every positive number δ,

$$(19.11) \qquad \text{Prob}\{|X^{(i)} - \frac{s_p}{n}| \geq \delta\} \to 0.$$

More generally, we have

PROPOSITION 25.11. Let $v \in pNA'$, and let $f = \sum_{i=1}^{k} \alpha_i \chi_{S_i} \in \mathcal{S}$, where $\{S_1, \ldots, S_k\}$ is a partition of I. Let $\{\pi_1, \pi_2, \ldots\}$ be an admissible sequence whose first member is the partition $\{S_1, \ldots, S_k\}$. Let T_{ij} be the set of subsets of S_i belonging to π_j, and let $|T_{ij}|$ be its cardinality. Let m_{ij} be positive integers such that

$$\frac{m_{ij}}{|T_{ij}|} \to \alpha_i \quad \text{as } j \to \infty.$$

From each T_{ij}, choose m_{ij} members at random in such a way that all subsets of T_{ij} with exactly m_{ij} members have the same probability of being chosen. Let U_{ij} be the union of the m_{ij} sets so chosen, and let $U_j = U_i U_{ij}$. Then for each $\epsilon > 0$,

$$\text{Prob } \{|v(U_j) - v^*(f)| < \epsilon\} \to 1 \quad \text{as } j \to \infty.$$

The proof uses ideas similar to those used above.

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