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A MATHEMATICAL FOUNDATION FOR SELECTING RADAR LOCATIONS FOR OPERATIONAL TESTING OF INERTIAL SYSTEMS

John M. Bachar

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PREFACE

In July 1969, Rand initiated a Missile Test and Evaluation Study at the request of Major General S. J. Byerley, Director of Operations, Headquarters USAF. The objectives of this study are (1) to compare accuracy and cost of data obtained by different configurations of test equipments and sensor locations, and (2) to devise an orderly process for the best sequencing of missile tests for the highest assurance of success. The goal is the improvement of missile test accuracy while operating under reduced budgets.

This Memorandum is one of a series exploring techniques for operational use in ICBM testing; it specifically develops mathematical foundations for improving the test range configuration. A computer program derived from the results of this study might be used to optimally locate radar trackers on the missile test range. This study is not totally self-contained and should be read in conjunction with the following studies: R. Turn, *Real-Time Data Transmission and Processing for Missile Testing*, RM-6286-PR; T. B. Garber, R. L. Mobley, and D. S. Pass, *A Computer Program for Estimating Inertial Guidance System Error Parameters*, RM-6287-PR; R. H. Ball and A. B. Kahle, *Gravity Effects on the Missile Test Range*, RM-6289-PR, and T. B. Garber, *The Determination of Inertial Guidance System Error Parameters from Missile Flight Tests* (U), RM-6290-PR (Secret).

The material presented here is concerned with improving vehicle test accuracy by optimizing the range configuration through optimum placement of the radars.

This Memorandum and its four companion pieces should be of interest to those military and civilian personnel responsible for planning, evaluating, and administering missile tests and test ranges.

SUMMARY

This Memorandum contains the mathematical foundations necessary for the development of a computer program that could be used to locate the radar trackers on a missile test range. The problem considers the radar tracking, during boost phase, of a rocket vehicle whose navigation is provided by an inertial measurement unit (IMU). One criterion for selecting radar tracker locations is as follows: If the maximum likelihood estimate of the IMU parameters that produced a given set of noisy range data is determined, and if the maximum likelihood estimate of miss is determined from this parameter estimate, then it is evident that the vehicle miss covariance matrix is a function of the radar map locations. This Memorandum shows in detail an optimization scheme for locating the radars such that the maximum eigenvalue of the covariance matrix is minimized. This criterion results in improving overall vehicle test accuracy. In addition, this scheme has applications to the real-time navigation of rocket vehicles. Another criterion, briefly considered, for optimizing the radar locations is the minimization of the variance of a given IMU parameter as a function of radar locations, as outlined in the concluding paragraph of the Introduction.

CONTENTS

I. INTRODUCTION

In this study the following problem is solved: A vehicle, whose navigation is provided by an inertial measurement unit (IMU), is to be tracked in range and/or range change during powered test flight by a network of radars, and the sensed acceleration of the vehicle IMU is to be telemetered. We shall find the optimum physical location of the individual radars of the network such that a certain statistical estimate of miss, relative to the actual impact point, is minimized.

Assume that the IMU has a mathematical representation involving the sensed acceleration of the vehicle and a finite number of parameters which are assumed to be constant throughout a given powered flight. The (noisy) data of range and/or range change from the network for a given test flight is then used to obtain the maximum likelihood estimate of the IMU parameters that produced the noisy data. The maximum likelihood estimate of miss relative to the actual impact point is then derived from this estimate of the IMU parameters. If we let $E(xx')$ denote the covariance matrix of the maximum likelihood estimate of miss, where x is two-dimensional miss (column vector), then we can define a "standard" dispersion ellipse by the two-dimensional equation,

$$1 = x' \, E^{-1}(xx') \, x \quad \text{(where ' denotes transpose)}$$

This ellipse, centered about the actual impact point, has a semimajor axis, a, which is a function of the radar locations. The optimum physical location of the radars is defined to be the location such that a is minimized.* This is the criterion we shall study in depth.

A vector v is defined whose elements are map coordinates of the radar locations. Formulas defining the optimum locations are given in

*In fact, if we define a general "ρ" dispersion ellipse to be the one obtained by multiplying the "standard" dispersion ellipse axes by ρ, where $\rho > 0$, then the probability that the maximum likelihood miss estimate will be within the "ρ" dispersion ellipse (same center and orientation) will be $1 - \exp(-1/2\ \rho^2)$. Thus, if $C(r_\rho)$ is the circle of radius r_ρ (centered about the actual impact point) which contains $[1 - \exp(-1/2\ \rho^2)] \times 100$ percent of the miss estimate, then r_ρ is obviously bounded above by the semimajor axis of the ρ dispersion ellipse.

Eqs. (53) to (69) A technique for calculating the corresponding optimum best location vector v_{min} is given immediately following Eq. (69).

It is interesting to note the following. If, in real time, the maximum likelihood estimate of the IMU parameters is used in the mathematical navigation equations associated with the IMU (in which these parameters occur as variables), then the probability that a guided vehicle will impact (assuming no other error sources) within the ρ dispersion ellipse (centered at the desired impact point, i.e., the target) is $1-e^{-1/2\rho^2}$. This follows from the first part of Section VI.

A complete mathematical description of the "standard" dispersion ellipse, together with the minimization of its semimajor axis as a function of the radar location coordinates, is given below. Usually, in practice, the noise in range and/or range change measurements is assumed to obey a normal probability density law, with a covariance matrix that is uncorrelated (for any pair of time measurements) between different radars, but which may be correlated (for different times) for a given radar. In addition, the noise in range is assumed to be uncorrelated with respect to the noise in range change. There are three categories of measurements: (1) range only, (2) range change only, (3) range and range change. Independently of these categories we will allow some radar locations to be fixed (not to be optimized), while allowing the remaining locations to be optimized.

In the sections that follow, the mathematical models which express range and range change as functions of the state vector (i.e., the IMU parameters) do not include the effects of the troposphere. The models which are used herein employ the H_i and J_i matrices in Eqs. (33) and (39), respectively. If the tropospheric effects are to be included, then the H_i and J_i matrices must be modified by introducing the functional relationships between the *instrumented range* (*range change*, respectively) *measurements* and the range (range change, respectively), the unit vector from the radar beacon to the vehicle, the radar beacon bias, and the time-varying tropospheric delay. In this case, there will be a random error for the instrumented range (range change, respectively), and also a random error for the time-varying tropospheric delay.(1) The methods of this paper then apply to the tropospheric case in essentially the same way as presented here. In terms of the mathematical model described here, one can say that the major effect of including the troposphere is that the off-diagonal elements of the covariance matrices R_i and S_i of the range and range change, respectively, become non-zero. The reason is that any error in measurement due to the tropospheric effect is correlated with measurements nearby in time.

In Section VII, we present a summary of matrix inversion formulas for calculating the covariance matrices of the estimates (of either the miss or the IMU parameters) in a sequential manner. These formulas may be used either when the IMU parameter estimates are required in real-time navigation of the vehicle, or when the IMU parameter estimates are desired in real time when a test of an IMU system on a vehicle is being conducted. The formulas are sufficiently general for calculating the covariance matrix of the estimate (sequentially) even when the covariance matrix of the random noise (assumed Gaussian) in the basic measurement vector is *not* diagonal.

We briefly mention another criterion for optimizing radar locations. Instead of minimizing the semimajor axis, a, of the standard dispersion ellipse as a function of radar locations, we can minimize the variance of each IMU parameter as a function of radar locations. In terms of the covariance matrix, $E(\Delta\hat{p}\ \Delta\hat{p}')$, of the maximum likelihood estimate of the IMU parameters, this means minimizing the diagonal elements of $E(\Delta\hat{p}\ \Delta\hat{p}')$ as a function of radar locations. This amounts to omitting A and A' in Eq. (59), and then finding the radar locations such that the j^{th} diagonal elements of $\partial E/\partial\theta_i$ and $\partial E/\partial\phi_i$, $i = 1, \ldots, k_o$, are equal to zero. This yields the radar locations which minimize the covariance, σ_j, of the j^{th} IMU parameter. The subsidiary equations, Eqs. (60) through (67), are exactly the same as before in their use in Eq. (59).

II. NAVIGATION EQUATIONS OF THE IMU

We shall derive the equations of motion of a vehicle as a function of the output and the parameters of the IMU.

In an earth-centered nonrotating coordinate system, the equations of motion of the vehicle (assumed simply to be a "point mass") are

$$\dot{X} = V$$
$$\dot{V} = g(X) + a \tag{1}$$

where

$$X \equiv \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}, \quad V \equiv \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}, \quad g(X) \equiv \begin{bmatrix} g_1(X) \\ g_2(X) \\ g_3(X) \end{bmatrix}, \quad a \equiv \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

and g is the gravitational acceleration and a the nongravitational acceleration. The class of IMUs that we shall consider has the property that the output, a'', is of the form

$$a'' = f(p,a) \tag{2}$$

where from Eq. (1)

$$a = \dot{V} - g(X) = \ddot{X} - g(X)$$

and

$$f \equiv \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

is a vector function of a and

$$p \equiv \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$$

the n×1 column vector of IMU parameters (constant in time for a given powered flight). The components of p include misalignment angles (relative to some desired orientation) of the gyros and accelerometers, drift, nonlinearity, and other phenomena of the instruments, and scale factors that describe the output of each accelerometer. A "perfect" IMU is defined as one having all variables equal to zero, except the individual accelerometer scale factors, which are equal to one. Let us denote by p_o the value of p in a "perfect" IMU; in other terms, a "perfect" IMU satisfies

$$a = f(p_o, a) \tag{3}$$

Furthermore, we shall assume that the matrix of partials of f_i with respect to a_j, i, j = 1, 2, 3, evaluated at p_o and for all values of a in the operating range of the IMU, satisfies

$$\left(\frac{\partial f_i}{\partial a_j}\right) (p_o, a) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \tag{4}$$

For a given reference trajectory, $X(t) = X_{ref}(t)$, $V(t) = V_{ref}(t)$, which is obtained by solving Eq. (1) for a reference nongravitational function, $a(t) = a_{ref}(t)$, let us expand Eq. (2) linearly about $a(t) = a_{ref}(t)$, $p = p_1$, for each value of time, t, in powered flight $0 \le t \le t_{b.o.}$:

$$\Delta a''(t) = A(t)\Delta a(t) + B(t)\Delta p \tag{5}$$

$$= A(t)[\Delta\ddot{X}(t) - G(t)\Delta X(t)] + B(t)\Delta p$$

where

$$A(t) = \left(\frac{\partial f_i}{\partial a_j}\right) (p_1, a_{ref}(t)) \quad (3\times3 \text{ matrix}) \tag{6}$$

$$G(t) = \left(\frac{\partial g_i}{\partial X_j}\right) (X_{ref}(t)) \quad (3\times3 \text{ matrix}) \tag{7}$$

$$B(t) = \left(\frac{\partial f_i}{\partial p_k}\right) (p_1, a_{ref}(t)) \quad (3\times n \text{ matrix}) \tag{8}$$

$$\Delta a''(t) = a''(t) - a''_{ref}(t) \quad \left(a''_{ref}(t) = f(p_1, a_{ref}(t))\right) \tag{9}$$

$$\Delta a(t) = a(t) - a_{ref}(t) \tag{10}$$

$$\Delta p = p - p_1 \tag{11}$$

Note that if $p_1 = p_o$, then $a''_{ref}(t) = f(p_o, a_{ref}(t)) = a_{ref}(t)$, by Eq. (3). Furthermore, $A(t) \neq I$, in general, unless $p_1 = p_o$. From Eq. (5), by multiplying both sides on the left by $A(t)^{-1}$ (which we assume to exist) we obtain

$$\Delta\ddot{X}(t) = G(t)\Delta X(t) + A(t)^{-1} [\Delta a''(t) - B(t)\Delta p] \tag{12}$$

The solution to Eq. (12) is (we always assume $\Delta X(o) = \Delta V(o) = 0$)

$$\begin{bmatrix} \Delta X(t) \\ \Delta V(t) \end{bmatrix} = \Phi(t) \int_o^t \Phi(\tau)^{-1} \begin{bmatrix} 0_{3\times1} \\ A(\tau)^{-1}\Delta a''(\tau) \end{bmatrix} d\tau - \begin{bmatrix} \int_o^t \int_o^\tau A(\xi)^{-1} B(\xi) d\xi \, d\tau \\ \int_o^t A(\xi)^{-1} B(\xi) d\xi \end{bmatrix} \Delta p \tag{13}$$

where $\Phi(t)$ is the 6X6 matrix solution of

$$\frac{d\Phi(t)}{dt} = \begin{bmatrix} 0_{3\times3} & I_{3\times3} \\ G(t) & 0_{3\times3} \end{bmatrix} \Phi(t) \tag{14}$$

having initial condition $\Phi(o) = I_{6\times6}$ (the 6x6 identity matrix), $0_{3\times3}$ is the 3X3 zero matrix, $I_{3\times3}$ is the 3x3 identity matrix, and $0_{3\times1}$ is the 3x1 zero column matrix. If we partition the 6X6 matrices, $\Phi(t)^{-1}$ and $\Phi(t)$, into 3X3 block matrices by defining

$$\Phi(t) = \begin{bmatrix} \Phi_{11}(t) & \Phi_{12}(t) \\ \Phi_{21}(t) & \Phi_{22}(t) \end{bmatrix}, \quad \Phi(t)^{-1} = \begin{bmatrix} \Phi^*_{11}(t) & \Phi^*_{12}(t) \\ \Phi^*_{21}(t) & \Phi^*_{22}(t) \end{bmatrix} \tag{15}$$

then $\Delta X(t)$ in Eq. (13) is given by

$$\Delta X(t) = \int_o^t [\Phi_{11}(t)\Phi^*_{12}(\tau) + \Phi_{12}(t)\Phi^*_{22}(\tau)]A(\tau)^{-1}\Delta a''(\tau)d\tau$$

$$- \left[\int_o^t \int_o^\tau A(\xi)^{-1}B(\xi)d\xi d\tau\right]\Delta p \tag{16}$$

Equation (16) is the linearized approximation of vehicle position (relative to $X_{ref}(t)$) as a function of small changes, Δp (relative to $p = p_1$), in the IMU parameters, and of $\Delta a''(t)$, the output of the IMU relative to $a''_{ref}(t)$ (see Eq. (9)).

If Eq. (16) is integrated by parts, then $\Delta X(t)$ will be given as a function of perturbations $\Delta v''(t)$, from reference values $v''_{ref}(t) = \int_o^t a''_{ref}(\tau)d\tau$, and of Δp. Note that $v''(t) = \int_o^t a''(\tau)d\tau$, the integrated value of a'' from o to t, is actually the natural output of an IMU that consists of pendulous integrating gyro accelerometers. Let

$$\Delta v''(t) = \int_o^t \Delta a''(\tau)d\tau \tag{17}$$

$$C(t,\tau) = [\Phi_{11}(t)\Phi_{12}^{*}(\tau) + \Phi_{12}(t)\Phi_{22}^{*}(\tau)]A(\tau)^{-1} \quad (18)$$

We deduce that

$$\int_{o}^{t} C(t,\tau)\Delta a''(\tau)d\tau = C(t,\tau)\Delta v''(\tau)\Bigg|_{\tau=o}^{\tau=t} - \int_{o}^{t} \frac{\partial C(t,\tau)}{\partial \tau}\Delta v''(\tau)d\tau$$

$$= C(t,t)\Delta v''(t) - \int_{o}^{t} \frac{\partial C(t,\tau)}{\partial \tau}\Delta v''(\tau)d\tau \quad (19)$$

since $\Delta v''(o) = \int_{o}^{o} \Delta v''(\xi)d\xi = 0$.

Substituting this into Eq. (16), we finally obtain

$$\Delta X(t) = C(t,t)\Delta v''(t) - \int_{o}^{t} \frac{\partial C(t,\tau)}{\partial \tau}\Delta v''(\tau)d\tau$$

$$- \left[\int_{o}^{t}\int_{o}^{\tau} A(\xi)^{-1}B(\xi)d\xi d\tau\right]\Delta p \quad (20)$$

III. RANGE MEASUREMENTS AS A FUNCTION OF INTEGRATED OUTPUT AND THE IMU PARAMETERS

The location at time t of the i^{th} radar, $P_i(t)$, $i = 1, \ldots, k$ in an earth-centered nonrotating coordinate system is

$$P_i(t) = |P_i(o)| \; U_i(t, \theta_i, \phi_i) \tag{21}$$

where

$$U_i(t, \theta_i \; \phi_i) = \mathrm{Col}(\cos \phi_i \cos (\theta_i + \omega_E t), \cos \phi_i \sin (\theta_i + \omega_E t), \sin \phi_i) \tag{22}$$

θ_i is Greenwich referenced longitude, and ϕ_i is the latitude.

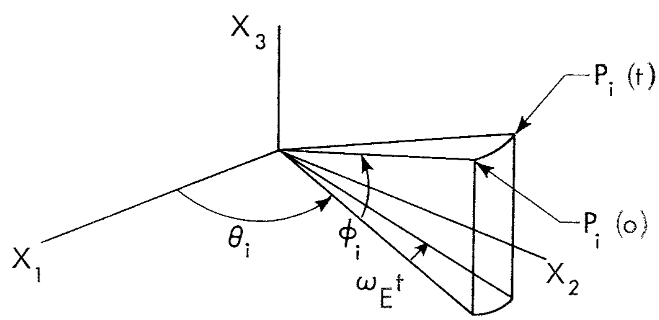

Fig. 1—Radar location

The range, $r_i(t)$, from $P_i(t)$ to the vehicle, X(t) (at time t), is

$$r_i(t) = |X(t) - P_i(t)| \tag{23}$$

Perturbations in range about a reference trajectory $X(t) = X_{ref}(t)$, are given by

$$\Delta r_i(t) = u'_{iref}(t) \Delta X(t) \tag{24}$$

where

$$u_{iref}(t) = \left|X_{ref}(t) - P_i(t)\right|^{-1} \left[X_{ref}(t) - P_i(t)\right] \qquad (25)$$

The vector in Eq. (25) is the unit vector from the i^{th} radar to $X_{ref}(t)$ at time t. Using Eq. (20), we obtain

$$\Delta r_i(t) = u'_{iref}(t) \left[C(t,t)\Delta v''(t) - \int_0^t \frac{\partial C(t,\tau)}{\partial \tau} \Delta v''(\tau)d\tau - \left(\int_0^t \int_0^\tau A(\xi)^{-1} B(\xi) d\xi d\tau \right) \Delta p \right] \qquad (26)$$

IV. MAXIMUM LIKELIHOOD ESTIMATE OF IMU PARAMETERS DETERMINED BY A GIVEN SET OF NOISY RANGE AND/OR RANGE CHANGE DATA

We now suppose (hypothetically) we are to make a test of a vehicle a large number of times, and that the vehicle trajectory and IMU parameters are identical for each of these tests. The only thing that will change from test to test will be the noise in the range and range change data. This is tantamount to saying that for a given test, the noise in range and range change are random variables. In mathematical terms, we assume $X(t) = X_{ref}(t)$ for each test. This is equivalent to $a(t) = a_{ref}(t)$, which in turn is equivalent to $a''(t) = a''_{ref}(t)$, $p = p_{ref}$, and, finally, this is equivalent to $\Delta v''(t) \equiv 0$, $p = p_{ref}$. We now first examine the case of noise in the range measurement.

Let $n_i(t)$ be the error (i.e., the noise) at time t that exists in the actual (imperfect) range measurement, $r_{i_a}(t)$. Thus, in general

$$n_i(t) = \Delta r_{i_a}(t) - \Delta r_i(t) \tag{27}$$

where

$$\Delta r_{i_a}(t) = r_{i_a}(t) - r_{i_{ref}}(t), \quad \Delta r_i(t) = r_i(t) - r_{i_{ref}}(t) \tag{28}$$

By Eq. (27) and Eq. (26) we have (remembering that $\Delta v''(t) \equiv 0$)

$$n_i(t) = \Delta r_{i_a}(t) - u'_{i_{ref}}(t) \left[- \int_0^t \int_0^\tau A(\xi)^{-1} B(\xi) d\xi d\tau \right] \Delta p \tag{29}$$

Note that $A(\xi)$ and $B(\xi)$ (see Eqs. (6) and (8)) are to be evaluated at $p = p_{ref}$, $a(t) = a_{ref}(t)$. In general $A(\xi) \neq I$, $0 \leq \xi \leq t_{b.o.}$, except when $p_{ref} = p_o$.

For convenience, define the 3Xn matrix

$$Q(t) = - \int_0^t \int_0^\tau A(\xi)^{-1} B(\xi)\, d\xi d\tau \tag{30}$$

Assume that for the i^{th} radar, range measurements are taken for a given test at $0 \le t_{1i} < t_{2i} < \ldots < t_{N_i i} \le t_{b.o.}$, $i = 1, \ldots, k$. Then the range noise vector (a random variable), n_i, and actual range measurement vector, Δr_{i_a}, have N_i components:

$$n_i = \text{col}\left(n_i(t_{1i}),\ n_i(t_{2i}),\ \ldots,\ n_i\left(t_{N_i i}\right)\right) \tag{31}$$

$$\Delta r_{i_a} = \text{col}\left(\Delta r_{i_a}(t_{1i}),\ \ldots,\ \Delta r_{i_a}\left(t_{N_i i}\right)\right), \quad i = 1, \ldots, k$$

We then have, by Eqs. (27) and (30),

$$n_i = \Delta r_{i_a} - H_i \Delta p, \quad i = 1, \ldots, k \tag{32}$$

where H_i is the $N_i \times n$ matrix

$$H_i = \begin{bmatrix} u'_{i_{ref}}(t_{1i})Q(t_{1i}) \\ \vdots \\ u'_{i_{ref}}\left(t_{N_i i}\right)Q\left(t_{N_i i}\right) \end{bmatrix} \tag{33}$$

written in block form. Under our test assumptions, $\Delta p = 0$, so that $\Delta r_i = H_i \Delta p = 0$ and $n_i = \Delta r_{i_a}$. We also emphasize that H_i depends upon θ_i, ϕ_i, and $|P_i(o)|$ (see Eqs. (21), (22), and (25)). We assume that if a large number of tests are made then the random vectors n_i have covariance matrices $R_i = \text{cov}(n_i, n'_i) = E(n_i n'_i)$, and that $E(n_i) = 0$.

We next examine the noise m_i (M_i-1 components) in the range change measurements, $r_{c_{i_a}}(t_1, t_2)$, in the time interval t_1 to t_2. It is defined by

$$m_i(t_1, t_2) = \Delta r_{c_{i_a}}(t_1, t_2) - \Delta r_{c_i}(t_1, t_2) \tag{34}$$

where

$$\Delta r_{c_{i_a}}(t_1,t_2) = r_{c_{i_a}}(t_1,t_2) - r_{c_{i_{ref}}}(t_1,t_2)$$

$$\Delta r_{c_i}(t_1,t_2) = r_{c_i}(t_1,t_2) - r_{c_{i_{ref}}}(t_1,t_2)$$

$$r_{c_i}(t_1,t_2) = r_i(t_2) - r_i(t_1)$$

$$r_{c_{i_{ref}}}(t_1,t_2) = r_{i_{ref}}(t_2) - r_{i_{ref}}(t_1) \qquad (35)$$

Thus we have, from Eqs. (26), (34), and (35) (using $\Delta v''(t) \equiv 0$ in Eqs. (26) and (30)),

$$\begin{aligned} m_i(t_1,t_2) &= \Delta r_{c_{i_a}}(t_1,t_2) - \Delta r_{c_i}(t_1,t_2) \\ &= \Delta r_{c_{i_a}}(t_1,t_2) - \left[r_{c_i}(t_1,t_2) - r_{c_{i_{ref}}}(t_1,t_2)\right] \\ &= \Delta r_{c_{i_a}}(t_1,t_2) - \left\{[r_i(t_2) - r_i(t_1)] - \left[r_{i_{ref}}(t_2) - r_{i_{ref}}(t_1)\right]\right\} \\ &= \Delta r_{c_{i_a}}(t_1,t_2) - [\Delta r_i(t_2) - \Delta r_i(t_1)] \\ &= \Delta r_{c_{i_a}}(t_1,t_2) - \left[u'_{i_{ref}}(t_2)Q(t_2) - u'_{i_{ref}}(t_1)Q(t_1)\right]\Delta p \qquad (36) \end{aligned}$$

Using the time points $0 \le \tau_{1i} \le \tau_{2i} \le \dots \le \tau_{M_i i} \le t_{b.o.}$ for the i^{th} radar in order to take range change measurements (note that this division may be different from that used for range measurements), the range change noise vector (a random variable), m_i, and actual range change measurement vector, $\Delta r_{c_{i_a}}$, have (M_i-1) components:

$$m_i = \text{col}\left(m_i(\tau_{1i}, \tau_{2i}), \ldots, m_i\left(\tau_{(M_i-1)i}, \tau_{M_i i}\right)\right)$$

$$\Delta r_{c_{i_a}} = \text{col}\left(\Delta r_{c_{i_a}}(\tau_{1i}, \tau_{2i}), \ldots, \Delta r_{c_{i_a}}\left(\tau_{(M_i-1)i}, \tau_{M_i i}\right)\right) \qquad (37)$$

Then by Eqs. (34) and (36), we have

$$m_i = \Delta r_{c_{i_a}} - J_i \Delta p, \; i = 1, 2, \ldots, k \qquad (38)$$

where J_i is the $(M_i-1)\times n$ matrix

$$J_i = \begin{bmatrix} u'_{i_{ref}}(\tau_{2i})Q(\tau_{2i}) - u'_{i_{ref}}(\tau_{1i})Q(\tau_{1i}) \\ \vdots \\ u'_{i_{ref}}\left(\tau_{M_i i}\right)Q\left(\tau_{M_i i}\right) - u'_{i_{ref}}\left(\tau_{(M_i-1)i}\right)Q\left(\tau_{(M_i-1)i}\right) \end{bmatrix} \qquad (39)$$

As in the case for n_i, we assume the random vectors m_i have the covariance matrices $S_i = \text{cov}(m_i, m_i') = E(m_i, m_i')$ and that $E(m_i) = 0$.

What we seek for a given test is the *most likely value*, $\Delta p = \Delta\hat{p}$ ($= \hat{p} - p_{ref}$), which produces the actual range measurements, $r_{i_a}(t)$, and/or range change measurements, $r_{c_{i_a}}(t_1, t_2)$, $i = 1, \ldots, k$.

The joint probability density for $n_1, \ldots, n_k$ and $m_1, \ldots, m_k$ is the normal density function

$$P(n_1, \ldots, n_k, m_1, \ldots, m_k) = \prod_{i=1}^{k} (2\pi)^{-N_i/2} (\det R_i)^{-1/2} \exp\left(-1/2 \; n_i' R_i^{-1} n_i\right)$$

$$\times (2\pi)^{-(M_i-1)/2} (\det S_i)^{-1/2} \exp\left(-1/2 \; m_i' S_i^{-1} m_i\right) \qquad (40)$$

Using Eq. (32), we deduce the joint probability density function for $\Delta r_{1_a}, \ldots, \Delta r_{k_a}, \Delta r_{c_{1_a}}, \ldots, \Delta r_{c_{k_a}}$ (using the *true* value Δp and assuming $\Delta v''(t) = 0$) is

$$L\left(\Delta r_{1_a}, \ldots, \Delta r_{k_a}, \Delta r_{c_{1_a}}, \ldots, \Delta r_{c_{k_a}}, \Delta p\right) = \prod_{i=1}^{k} (2\pi)^{-N_i/2} (\det R_i)^{-1/2}$$

$$\times \exp\left[-1/2\left(\Delta r_{i_a} - H_i\Delta p\right)' R_i^{-1}\left(\Delta r_{i_a} - H_i\Delta p\right)\right] (2\pi)^{-(M_i-1)/2} (\det S_i)^{-1/2}$$

$$\times \exp\left[-1/2\left(\Delta r_{c_{i_a}} - J_i\Delta p\right)' S_i^{-1}\left(\Delta r_{c_{i_a}} - J_i\Delta p\right)\right] \tag{41}$$

Thus, in order to maximize the probability density that the actual collection of vectors, $\Delta r_{1_a}, \ldots, \Delta r_{k_a}, \Delta r_{c_{1_a}}, \ldots, \Delta r_{c_{k_a}}$, were produced by *some* value of $\Delta p = \Delta\hat{p}$ (note that $\Delta v''(t) \equiv 0$ and that the *true* value is $\Delta p = 0$), then the *maximum likelihood estimate* is the $n\times 1$ vector, $\Delta\hat{p}$, that minimizes, as a function of Δp, the exponent which appears in L. This is equivalent to minimizing

$$J = \sum_{i=1}^{k}\left[\left(\Delta r_{i_a} - H_i\Delta p\right)' R_i^{-1}\left(\Delta r_{i_a} - H_i\Delta p\right) + \left(\Delta r_{c_{i_a}} - J_i\Delta p\right)' S_i^{-1}\left(\Delta r_{c_{i_a}} - J_i\Delta p\right)\right] \tag{42}$$

as a function of Δp. The result is

$$\Delta\hat{p} = \left[\sum_{i=1}^{k}\left(H_i' R_i^{-1} H_i + J_i' S_i^{-1} J_i\right)\right]^{-1}\left[\sum_{i=1}^{k}\left(H_i' R_i^{-1}\Delta r_{i_a} + J_i' S_i^{-1}\Delta r_{c_{i_a}}\right)\right] \tag{43}$$

If for each test we know *a priori* that $\Delta r_i(t) \equiv 0$, then $n_i(t) = \Delta r_{i_a}(t)$ and $m_i(t_1,t_2) = \Delta r_{c_{i_a}}(t_1,t_2)$, and so this reduces to

$$\Delta\hat{p} = \left[\sum_{i=1}^{k}\left(H_i' R_i^{-1} H_i + J_i' S_i^{-1} J_i\right)\right]^{-1}\left[\sum_{i=1}^{k}\left(H_i' R_i^{-1} n_i + J_i' S_i^{-1} m_i\right)\right] \tag{44}$$

V. DENSITY FUNCTIONS FOR MAXIMUM LIKELIHOOD ESTIMATE OF PARAMETERS AND FOR DERIVED MISS ESTIMATE

It is a basic fact in probability theory that the maximum likelihood estimate, $\Delta\hat{p}$, given in Eq. (44) satisfies $E(\Delta\hat{p}) = 0$ (i.e., has expected value zero) and that the joint probability density for the estimate $\Delta\hat{p}$ is

$$P(\Delta\hat{p}) = (2\pi)^{-n/2}[\det E(\Delta\hat{p}\Delta\hat{p}')]^{-1/2} \exp[-1/2\ \Delta\hat{p}'E^{-1}(\Delta\hat{p}\Delta\hat{p}')\Delta\hat{p}] \tag{45}$$

where

$$E(\Delta\hat{p}\Delta\hat{p}') = \left[\sum_{i=1}^{k} H_i'R_i^{-1}H_i + J_i'S_i^{-1}J_i\right]^{-1} \tag{46}$$

If we know that miss $m = A\Delta p$, where A is the 2Xn "miss" matrix relative to the impact point produced by $\Delta p = 0$ and $\Delta v''(t) = 0$ (these two conditions being equivalent to $\Delta X(t) = 0$), then the probability density function for the miss estimate, $\hat{m}$, is

$$P(\hat{m}) = (2\pi)^{-1}[\det E(\hat{m}\ \hat{m}')]^{-1/2} \exp[-1/2\ \hat{m}'E^{-1}(\hat{m}\ \hat{m}')\hat{m}] \tag{47}$$

where

$$E(\hat{m}\ \hat{m}') = A\left[\sum_{i=1}^{k} H_i'R_i^{-1}H_i + J_i'S_i^{-1}J_i\right]^{-1} A' \tag{48}$$

VI. OPTIMIZATION OF RADAR LOCATIONS

We take up the optimization criterion for k_o of the radar locations ($k_o \leq k$) while fixing the remaining $k-k_o$ radar locations.

Let us diagonalize the positive definite matrix, $E(\hat{m}\,\hat{m}')$, and denote the result by

$$D = \begin{pmatrix} \sigma_1^2 & o \\ o & \sigma_2^2 \end{pmatrix} = U^{-1}E(\hat{m}\,\hat{m}')U$$

where $\sigma_1 > 0$, $\sigma_2 > 0$, and U is the 2×2 orthogonal change of coordinates $\hat{m} = U\hat{y}$. We then have $UDU^{-1} = E(\hat{m}\,\hat{m}')$, and $\det E(\hat{m}\,\hat{m}') = \det U \cdot \det D \cdot \det U^{-1} = \sigma_1^2\,\sigma_2^2$. The probability density for the estimate $\hat{y}$ in the new coordinate system is

$$P(\hat{y}) = (2\pi)^{-1}(\sigma_1\sigma_2)^{-1} \exp\left[-1/2\left(\frac{\hat{y}_1^2}{\sigma_1^2} + \frac{\hat{y}_2^2}{\sigma_2^2}\right)\right] \tag{49}$$

If we calculate the probability that the derived miss estimate (based on $\Delta\hat{p}$) falls within the ellipse, E_ρ, whose axes are $\rho\sigma_1$, $\rho\sigma_2$, respectively, $\rho > o$, then, if we make the change of variables $y_1 = r\sigma_1 \cos\theta$, $y_2 = r\sigma_2 \sin\theta$,

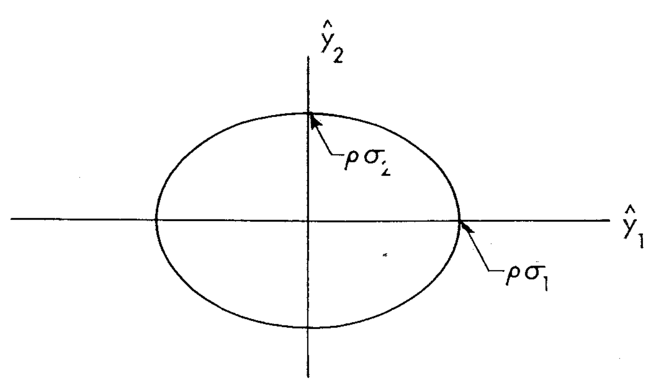

Fig. 2—The ellipse, E_ρ

we find that $d\hat{y}_1 d\hat{y}_2 = \sigma_1\sigma_2 r dr d\theta$, and that

$$P(\hat{y}\epsilon\xi\rho) = (2\pi)^{-1}(\sigma_1\sigma_2)^{-1} \iint_{\hat{y}=(\hat{y}_1,\hat{y}_2)\epsilon E_\rho} \exp\left[-1/2\left(\frac{\hat{y}_1^2}{\sigma_1^2} + \frac{\hat{y}_2^2}{\sigma_2^2}\right)\right] d\hat{y}_1 d\hat{y}_2$$

$$= 4(2\pi)^{-1}(\sigma_1\sigma_2)^{-1} \int_0^{\pi/2}\int_0^{\rho} \exp\left[-1/2\left(\frac{r^2\sigma_1^2\cos^2\theta}{\sigma_1^2} + \frac{r^2\sigma_2^2\sin^2\theta}{\sigma_2^2}\right)\right]$$

$$\times \sigma_1\sigma_2 r dr d\theta$$

$$= 4(2\pi)^{-1} \int_0^{\pi/2}\int_0^{\rho} \exp(-1/2\, r^2) r dr d\theta$$

$$= 4(2\pi)^{-1} \int_0^{\pi/2}\int_0^{(-1/2)\rho^2} (-e^u) du d\theta$$

$$= 4(2\pi)^{-1} \int_0^{\pi/2} \left(-e^u \Big|_0^{(-1/2)\rho^2}\right) d\theta = 1-e^{(-1/2)\rho^2} \qquad (50)$$

The semimajor axis, $a = \max(\sigma_1,\sigma_2)$, of the "standard" dispersion ellipse E_1 (obtained by letting $\rho = 1$), is a function of θ_i, ϕ_i, $i = 1, \ldots, k_o$ (we take $|P_i(o)|$ equal to the earth radius, i.e., the radars are constrained to lie on the surface of the earth). The criterion that we take for selecting the radar locations is that a shall be minimized as a function of θ_i and ϕ_i. Here, θ_i is the longitude of the i^{th} radar, relative to Greenwich, and ϕ_i is its latitude. Notice that a is a minimum if and only if the semimajor axis, ρa, of *every* ρ-dispersion ellipse, E_ρ, $\rho > 0$ is a minimum.

Now that we have shown that $P(\hat{m}\epsilon E_\rho) = 1-e^{(-1/2)\rho^2}$, let us note the relation between a and the radius, R_ρ, of the circular disk, $D_{R\rho}$, centered about the actual impact point, which contains $1-e^{(-1/2)\rho^2}$ of the miss distance. Clearly, we have

$$P(\hat{m}\varepsilon D_{\rho a}) \geq P(\hat{m}\varepsilon D_{R_\rho}) = 1-e^{(-1/2)\rho^2} = P(\hat{m}\varepsilon E_\rho)$$

where $D_{\rho a}$ is the disk of radius ρa centered about the actual impact point. From this it follows that $R_\rho \leq \rho a$. Thus, by minimizing a as a function of θ_i, ϕ_i, $i = 1, 2, \ldots, k_o$, we will then minimize the upper bound for R_ρ.

We proceed to the details for finding the value of the set θ_i, ϕ_i, $i = 1, \ldots, k_o$ which minimizes a. Calling

$$E(\hat{m}\,\hat{m}') \equiv \begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix}$$

(which is positive definite), we see that c_{ij}, $i, j = 1, 2$, are functions of the 2k variables θ_i, ϕ_i, $i = 1, \ldots, k_o$. We must express $a = \max(\sigma_1, \sigma_2)$ as a function of the c_{ij}'s in $E(\hat{m}\,\hat{m}')$. It is well known from the theory of positive definite matrices that the eigenvalues of $E(\hat{m}\,\hat{m}')$ are positive--call them λ_1, λ_2--and that

$$\begin{aligned} \lambda_1 &= \frac{1}{2}\left[\text{tr } E(\hat{m}\,\hat{m}') + \sqrt{\left\{\text{tr } E(\hat{m}\,\hat{m}')\right\}^2 - 4 \det E(\hat{m}\,\hat{m}')}\right] \\ \lambda_2 &= \frac{1}{2}\left[\text{tr } E(\hat{m}\,\hat{m}') - \sqrt{\left\{\text{tr } E(\hat{m}\,\hat{m}')\right\}^2 - 4 \det E(\hat{m}\,\hat{m}')}\right] \end{aligned} \qquad (51)$$

(tr denotes trace, and det denotes determinant). Furthermore, since

$$D = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} = U^{-1}E(\hat{m}\,\hat{m}')U$$

where $\hat{m} = U\hat{y}$, and

$$U = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

is the orthogonal coordinate change, it is also known that the semi-major axis we seek is given by

$$a = \max\ (\sigma_1, \sigma_2) = \max\left(\sqrt{\lambda_1}, \sqrt{\lambda_2}\right)$$

$$= \sqrt{\lambda_1} = \frac{1}{\sqrt{2}} \sqrt{\mathrm{tr}\ E(\hat{m}\ \hat{m}') + \sqrt{\{\mathrm{tr}\ E(\hat{m}\ \hat{m}')\}^2 - 4\ \det\ E(\hat{m}\ \hat{m}')}}$$

$$= \frac{1}{\sqrt{2}} \sqrt{c_{11} + c_{22} + \sqrt{(c_{11} - c_{22})^2 + 4c_{12}^2}} \tag{52}$$

It is also known that a matrix (square) is positive definite if and only if every principal minor is positive, so that $c_{11} > 0$, $c_{11}c_{22} - c_{12}^2 > 0$, which implies $c_{22} > 0$, so that $\mathrm{tr}\ E(\hat{m}\ \hat{m}') = c_{11} + c_{22} > 0$.

A necessary condition that a be a minimum is that

$$0 = \frac{\partial a}{\partial \theta_i} = (4a^{-1}) \left[\frac{\partial \mathrm{tr}E}{\partial \theta_i} + \left(\{\mathrm{tr}E\}^2 - 4\ \det\ E\right)^{-1/2} \left(\mathrm{tr}E\ \frac{\partial \mathrm{tr}E}{\partial \theta_i} - 2\ \frac{\partial \det E}{\partial \theta_i}\right)\right]$$

$$0 = \frac{\partial a}{\partial \phi_i} = (4a^{-1}) \left[\frac{\partial \mathrm{tr}E}{\partial \phi_i} + \left(\{\mathrm{tr}E\}^2 - 4\ \det\ E\right)^{-1/2} \left(\mathrm{tr}E\ \frac{\partial \mathrm{tr}E}{\partial \phi_i} - 2\ \frac{\partial \det E}{\partial \phi_i}\right)\right] \tag{53}$$

$i = 1, \ldots, k_o$, and where, for convenience, $E = E(\hat{m}\ \hat{m}')$. Using Eq. (52) it can easily be deduced that Eq. (53) holds if and only if

$$a^2\ \frac{\partial \mathrm{tr}E}{\partial \theta_i} - \frac{\partial \det E}{\partial \theta_i} = 0$$

$$a^2\ \frac{\partial \mathrm{tr}E}{\partial \phi_i} - \frac{\partial \det E}{\partial \phi_i} = 0 \tag{54}$$

$i = 1, \ldots, k_o$, which is equivalent to (using $\mathrm{tr}E = c_{11} + c_{22}$, $\det E = c_{11}c_{22} - c_{12}^2$)

$$\mathrm{tr}\left[\frac{\partial E}{\partial \theta_i}\begin{pmatrix} c_{22} - a^2 & -c_{12} \\ -c_{12} & c_{11} - a^2 \end{pmatrix}\right] = 0$$

$$\mathrm{tr}\left[\frac{\partial E}{\partial \phi_i}\begin{pmatrix} c_{22} - a^2 & -c_{12} \\ -c_{12} & c_{11} - a^2 \end{pmatrix}\right] = 0 \tag{55}$$

$i = 1, \ldots, k_o$, which, in turn, is equivalent to

$$0 = \mathrm{tr}\left[\frac{\partial E}{\partial \theta_i}\left(E^{-1} - \lambda_2 I_{2\times 2}\right)\right]$$

$$0 = \mathrm{tr}\left[\frac{\partial E}{\partial \phi_i}\left(E^{-1} - \lambda_2 I_{2\times 2}\right)\right] \tag{56}$$

$i = 1, \ldots, k_o$, where $I_{2\times 2}$ is the 2X2 identity matrix.

Of the various equivalent forms Eqs. (53) to (56), let us work with Eq. (53). Since the partial derivatives of trE and detE are needed, this requires that we must find the partials of c_{ij}, $i \leq j$, i, j = 1, 2, the entries in E. This leads to calculating the partials $\partial E/\partial \theta_i$, $\partial E/\partial \phi_i$, $i = 1, \ldots, k_o$. The necessary calculations are as follows: Define

$$Z = \sum_{i=1}^{k}\left(H_i' R_i^{-1} H_i + J_i' S_i^{-1} J_i\right) \tag{57}$$

Then from Eq. (48) we have

$$E = AZ^{-1}A' \tag{58}$$

Then

$$\frac{\partial E}{\partial \theta_i} = A \frac{\partial (Z^{-1})}{\partial \theta_i} A' = - AZ^{-1} \frac{\partial Z}{\partial \theta_i} Z^{-1} A'$$

$$\frac{\partial E}{\partial \phi_i} = - AZ^{-1} \frac{\partial Z}{\partial \phi_i} Z^{-1} A', \quad i = 1, \ldots, k_o \tag{59}$$

Next,

$$\frac{\partial Z}{\partial \theta_i} = H_i' R_i^{-1} \frac{\partial H_i}{\partial \theta_i} + \left(H_i' R_i^{-1} \frac{\partial H_i}{\partial \theta_i} \right)' + J_i' S_i^{-1} \frac{\partial J_i}{\partial \theta_i} + \left(J_i' S_i^{-1} \frac{\partial J_i}{\partial \theta_i} \right)'$$

$$\frac{\partial Z}{\partial \phi_i} = H_i' R_i^{-1} \frac{\partial H_i}{\partial \phi_i} + \left(H_i' R_i^{-1} \frac{\partial H_i}{\partial \phi_i} \right)' + J_i' S_i^{-1} \frac{\partial J_i}{\partial \phi_i} + \left(J_i' S_i^{-1} \frac{\partial J_i}{\partial \phi_i} \right)' \tag{60}$$

$$i = 1, \ldots, k_o$$

Using the definition of H_i in Eq. (33) and the definition of $U_{i_{ref}}'$ in Eq. (25), we obtain (in block form)

$$\frac{\partial H_i}{\partial \theta_i} = \begin{bmatrix} \frac{\partial u_{iref}'(t_{1i})}{\partial \theta_i} Q(t_{1i}) \\ \vdots \\ \frac{\partial u_{iref}'\left(t_{N_i i}\right)}{\partial \theta_i} Q\left(t_{N_i i}\right) \end{bmatrix}$$

$$\frac{\partial H_i}{\partial \phi_i} = \begin{bmatrix} \frac{\partial u_{iref}'(t_{1i})}{\partial \phi_i} Q(t_{1i}) \\ \vdots \\ \frac{\partial u_{iref}'\left(t_{N_i i}\right)}{\partial \phi_i} Q\left(t_{N_i i}\right) \end{bmatrix} \tag{61}$$

$$\frac{\partial J_i}{\partial \theta_i} = \begin{bmatrix} \dfrac{\partial u'_{iref}(\tau_{2i})}{\partial \theta_i} Q(\tau_{2i}) - \dfrac{\partial u'_{iref}(\tau_{1i})}{\partial \theta_i} Q(\tau_{1i}) \\ \vdots \\ \dfrac{\partial u'_{iref}\left(\tau_{M_i i}\right)}{\partial \theta_i} Q\left(\tau_{M_i i}\right) - \dfrac{\partial u'_{iref}\left(\tau_{(M_i-1)i}\right)}{\partial \theta_i} Q\left(\tau_{(M_i-1)i}\right) \end{bmatrix}$$

and similarly for $\partial J_i/\partial \phi_i$ (replace θ_i by ϕ_i in the expression for $\partial J_i/\partial \theta_i$), $i = 1, 2, \ldots, k_o$.

The partials of $u'_{iref}(t)$ are given by

$$\frac{\partial u'_{iref}(t_{mi})}{\partial \theta_i} = \frac{|P_i(o)|}{r_{iref}(t_{mi},\theta_i,\phi_i)} \times \left\{- V'_i(t_{mi},\theta_i,\phi_i) + [V'_i(t_{mi},\theta_i,\phi_i)u_{iref}(t_{mi})]u'_{iref}(t_{mi})\right\}$$

$$\frac{\partial u'_{iref}(t_{mi})}{\partial \theta_i} = \frac{|P_i(o)|}{r_{iref}(t_{mi},\theta_i,\phi_i)} \times \left\{- W'_i(t_{mi},\theta_i,\phi_i) + [W'_i(t_{mi},\theta_i,\phi_i)r_{iref}(t_{mi})]u'_{iref}(t_{mi})\right\} \tag{62}$$

$1 \le m \le N_i$, $i = 1, \ldots, k_o$, and similarly for $t = \tau_{mi}$, $1 \le m \le M_i$, where (see Eq. (23))

$$r_{iref}(t_{mi},\theta_i,\phi_i) = |X_{ref}(t_{mi}) - P_i(t_{mi})| \tag{63}$$

$$P_i(t_{mi}) = P_i(o)U_i(t_{mi},\theta_i,\phi_i) \tag{64}$$

(see Eq. (21)).

$$U_i(t_{mi},\theta_i,\phi_i) = \text{col}[\cos\theta_i \cos(\theta_i+\omega_E t_{mi}), \cos\phi_i \sin(\theta_i+\omega_E t_{mi}), \sin\phi_i] \tag{65}$$

(see Eq. (22)).

$$V_i(t_{mi},\theta_i,\phi_i) = \text{col}[-\cos\phi_i \sin(\theta_i+\omega_E t_{mi}),\ \cos\phi_i \cos(\theta_i+\omega_E t_{mi}),\ 0] \tag{66}$$

$$W_i(t_{mi},\theta_i,\phi_i) = \text{col}[-\sin\phi_i \cos(\theta_i+\omega_E t_{mi}),\ -\sin\phi_i \sin(\theta_i+\omega_E t_{mi}),\ \cos\phi_i] \tag{67}$$

Similar expressions hold when $t = \tau_{mi}$, $1 \le m \le M_i$.

These calculations enable us to obtain the partials of a with respect to θ_i, ϕ_i, $i = 1, \ldots, k_o$. Let us define the gradient of a at the 2k-vector $v = \text{col}(\theta_1, \ldots, \theta_k, \phi_1, \ldots, \phi_k)$ by

$$\nabla(a)(v) = \left(\frac{\partial a}{\partial\theta_1}, \ldots, \frac{\partial a}{\partial\theta_k}, \frac{\partial a}{\partial\phi_1}, \ldots, \frac{\partial a}{\partial\phi_k}\right) \tag{68}$$

Then, at points $v'' = \text{col}(\theta_1'', \ldots, \theta_1'', \phi_1'', \ldots, \phi_k'')$ nearby v, we have approximately

$$a(v'') - a(v) = \nabla(a)(v) \cdot \Delta v \tag{69}$$

where $\Delta v = v'' - v$.

An illustrative iteration procedure for determining the value $v = v_{min}$ such that $a(v_{min})$ is a minimum is as follows:

1. Define the unit vector, $v_1 = \nabla(a)(v)/\|\nabla(a)(v)\|$.
2. For $\epsilon > 0$, calculate $a(v + \epsilon v_1)$, $a(v)$, $a(v - \epsilon v_1)$ using Eq. (52).
3. Quadratically interpolate in order to find a new value, v_{new}, with which we calculate $\nabla(a)(v_{new})$
4. Repeat until the gradient of a, $\nabla(a)(v_{new})$, is sufficiently close to the 2k-zero vector.

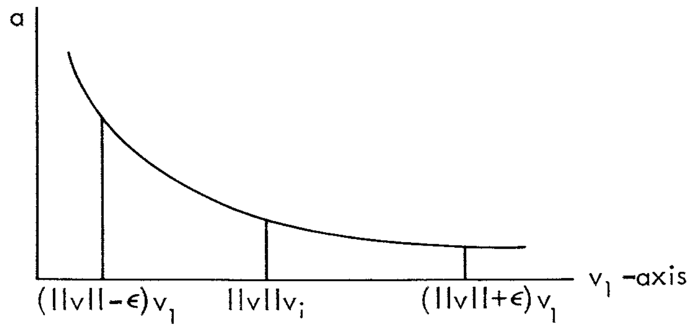

Fig. 3—Quadratic interpolation for minimizing a

We comment that once v_{min} is found such that $\nabla(a)(v_{min}) = 0$, this is only a *necessary* condition in order that a be a minimum. It is possible to have a saddle point for a at v_{min}. A *sufficient* condition for a local minimum is that the matrix of second partials of a with respect to the 2k-vector v, denoted

$$\left(\frac{\partial^2 a}{\partial v_i \partial v_j}\right)(v_{min}), \ i, j = 1, \ldots, 2k \tag{70}$$

which is a symmetric matrix (this follows provided all the second partials are continuous so that $\partial^2 a/\partial v_i \partial v_j = \partial^2 a/\partial v_j \partial v_i$, i, j = 1, ..., k), is positive definite. This will be true if and only if $g_i > 0$ for i = 1, ..., 2k, where

$$g_i = \det \begin{bmatrix} \frac{\partial^2 a}{\partial v_1^2} & \cdots & \frac{\partial^2 a}{\partial v_1 \partial v_i} \\ \vdots & & \vdots \\ \frac{\partial^2 a}{\partial v_1 \partial v_i} & \cdots & \frac{\partial^2 a}{\partial v_i^2} \end{bmatrix} \tag{71}$$

i.e., every principal minor is positive.

This sufficient condition requires, ultimately, the second partials $\partial^2 E/\partial\theta_i \partial\phi_j$, i, j = 1, ..., k_o, but we shall not carry this work further.

The essential features of a computer program for implementing either this criterion or the alternate criterion briefly considered in Section I will now be outlined. The main function of such a computer program is to minimize a as a function of the radar coordinates (given in Eq. (52)), or in the case of the alternate criterion, to minimize the jth diagonal element (the covariance of the jth IMU parameter) of the $E(\hat{m}\,\hat{m}')$ matrix as a function of the radar coordinates (given in Eq. (48)). For both criteria, a numerical iterative algorithm is needed, such as the illustrative classical steepest descent method given in the paragraph following Eq. (69). Since linear interpolation is generally a part of any iteration scheme, the partials of the H_i and J_i matrices are needed with respect to θ_i and ϕ_i, as given in Eq. (61). These partials, in turn, depend upon the Q matrix described in Section IV, Eq. (30). A computer program for the generation of the Q matrix is described in Ref. 1.

VII. MATRIX INVERSION FORMULAS FOR SEQUENTIAL CALCULATION OF COVARIANCE MATRIX OF STATE VECTOR ESTIMATE

The sequential calculation of the covariance matrix of the state vector estimate, such as $E(\Delta\hat{p}\Delta\hat{p}')$ in Eq. (46), essentially comes to the following considerations. It suffices to illustrate by assuming one radar station and one type of measurement--for example, range. We have the covariance matrix of the state estimate, $[H'R^{-1}H]^{-1}$, after gathering n data points. Here, H is $n\times\ell$, ℓ is the dimension of the state vector (see Eq. (32), for example), R is the $n\times n$ covariance matrix of the noise in the measurement vector (here, range). Next, we gather k more data points and then wish to calculate the covariance matrix of the state estimate,

$$\left[(H',p')\begin{pmatrix} R & q \\ q' & r \end{pmatrix}^{-1}\begin{pmatrix} H \\ p \end{pmatrix}\right]^{-1}$$

where p is $k\times\ell$ (note that $\begin{pmatrix} H \\ p \end{pmatrix}$ is the new transition matrix after n+k time points), q is $n\times k$, r is $k\times k$. Note that the measurement noise covariance matrix,

$$\begin{bmatrix} R & q \\ q' & r \end{bmatrix}$$

is not necessarily diagonal. By straightforward calculation, the following holds:

$$\left\{[H',p']\begin{bmatrix} R & q \\ q' & r \end{bmatrix}^{-1}\begin{bmatrix} H \\ p \end{bmatrix}\right\}^{-1} = \Big\{H'R^{-1}H + (q'R^{-1}H)'(r-q'R^{-1}q)^{-1}(q'R^{-1}H)$$

$$+ p'r^{-1}p + (qr^{-1}p)'R^{-1}(qr^{-1}p)$$

$$+ (q'R^{-1}qr^{-1}p)(r-q'R^{-1}q)^{-1}(q'R^{-1}qr^{-1}p)$$

$$- \left[p'(r-q'R^{-1}q)^{-1}q'R^{-1}H + (p'(r-q'R^{-1}q)^{-1}q'R^{-1}H)'\right]\Big\}^{-1} \tag{72}$$

The method used to invert the right-hand side in Eq. (72) is to use repeatedly Schur's identity:

$$\left(A_1 + A_2' A_3^{-1} A_2\right)^{-1} = A_1^{-1} - A_1^{-1} A_2' \left(A_3 + A_2 A_1^{-1} A_2'\right)^{-1} A_2 A_1^{-1} \qquad (73)$$

To start the process, we would let $A_1 = H' R^{-1} H$, $A_2 = q' R^{-1} H$, $A_3 = r - q' R^{-1} q$, and use Schur's identity and thus obtain $\left(A_1 + A_2' A_3^{-1} A_2\right)^{-1}$. We would then let the first two terms in the right-hand side of Eq. (72) be A_1, and then A_2 would be p, and A_3 would be r. Continuing in this way, we eventually arrive at the desired inversion.

The sequential calculation of the covariance matrix of the state estimate is especially simple in case $q \equiv 0$, i.e., when the covariance matrix is diagonal. In the nondiagonal case, however, the calculations by the method of Eqs. (72) and (73) become significantly more complicated.

REFERENCE

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Bachar

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