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EQUIVALENCE OF GAMES IN EXTENSIVE FORM

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Summary: Four simple transformations are characterized which are sufficient to carry any two equivalent games in extensive form one into the other. Application is made to the problem of simplification of a game in extensive form.

EQUIVALENCE OF GAMES IN EXTENSIVE FORM

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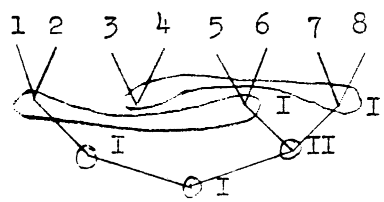
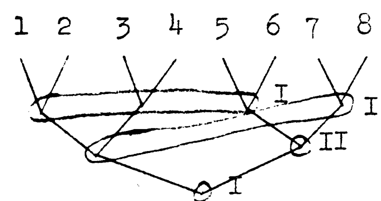
Informal Discussion: Given a game, each player has available to him a set of pure strategies. Considerations involving the playing of the game in its so called *extensive form* can often be simplified by passing to its *normal form* in which each player is to choose a (pure) strategy in ignorance of the choices of his opponents, and the appropriate payoff determined for these choices. It may happen that among the strategies available to a given player there are several which would yield the same payoffs against all combinations of strategies of his opponents. In such cases it may be desirable to consider a *reduced normal form* in which each player chooses an equivalence class of strategies, two equivalent strategies giving the same results against all strategies of the opponents [1]. The reduced normal form matrix will consequently have no repeating rows or columns. Examples can easily be found where the reduced normal form has far fewer choices open to the players and thus is much easier to deal with in carrying out computations involved in solving the game.

In considering the differences between games it seems natural to distinguish between games which differ only in payoff function and games which differ in other ways. We shall consider here the notion of game structure defined so that, loosely speaking, two games will have the same game structure if they differ only in payoff functions. One can define the corresponding notions of normal form and reduced normal form for game structures. Here a selection of a strategy for each player determines a play of the game rather than a numerical payoff. McKinsey has considered a special class of games and their patterns of information, a notion closely related to our notion of game structure [2]. His considerations can be stated in terms of game structures as follows. Two game structures of his class are to be considered equivalent if for every payoff function for one of them there is defined a function for the other such that the two resulting games have the same value ("value" is used here in the sense of von Neumann [3]). It is then proved [4] that two structures are equivalent if and only if they have isomorphic reduced normal forms. This theorem is easily extended to general game structures, and infact the notion of equivalence of game structures used in both the Krentel-McKinsey-Quine paper [4] and by Dalkey in his paper [5] is essentially that of having the same reduced normal form.

The motivation behind the papers of Krentel-McKinsey-Quine and of Dalkey is this. Given a game in extensive form, its normal form may be very large relative to its reduced normal form, the number of repetitions of rows or columns in its normal form matrix being very large. "In such a case, it becomes desirable to transform the given game in extensive form, so as to reduce the number of repetitions in its matrix."[4]. The best results in this direction have been achieved by Dalkey.[5]. However the transformations he considered involved changes in the information partition alone. He obtained an elegent characterization of those changes in information patterns which would transform a game structure into an equivalent game structure. If one carries out on a given game structure the process of deflation which he describes one obtains an equivalent structure whose normal form will be a simplification of that of the original. Unfortunately he was able to find examples of game structures which could not be transformed in this way into structures whose normal form matrix have no repeated rows and columns.

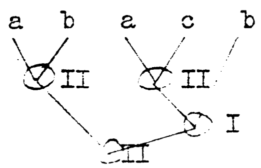
The last mentioned fact raised a question which was put to me by McKinsey in conversation. Are there game structures where the stepwise deflation process of Dalkey can be carried out in two ways so that the resulting two completely deflated game structures do not have the same size matrix? The answer to this question is "yes", as seen from the following example. We shall simply exhibit in graphical form two games. That they are completely deflated in the sense of Dalkey and that they are equivalent can easily be verified.





The matrix for the first is 2×16 while that for the second is 2×8.

The problem then still remains to find transformations on a game structure which will carry it into an equivalent structure whose normal form will in fact be its reduced normal form. In attacking this problem we have found it desirable to modify the notion of game structure in the following way. Two games will have the same structure if they differ only in payoff function <u>and</u> if the payoff function for one of them assigns the same payoff to two plays then the payoff function for the other does likewise. Thus a game structure will involve an equivalence relation on the set of plays; if a game has a certain structure then its payoff function assigns the same value to two plays if they are equivalent. The normal form for a game structure will be such that a selection of a strategy for each player determines an equivalence class of plays rather than a single play. A structure in this new sense can be presented in graphical form. For example we exhibit the following structure and both its normal form and reduced normal form matricies:



a	a	b	b	a	c	a	c
a	a	b	b	b	b	b	b

a	b	a	c
a	b	b	b

where "a", "b", "c" can be considered as names of equivalence classes of plays. We shall consider two game structures as equivalent if they have isomorphic reduced normal forms.

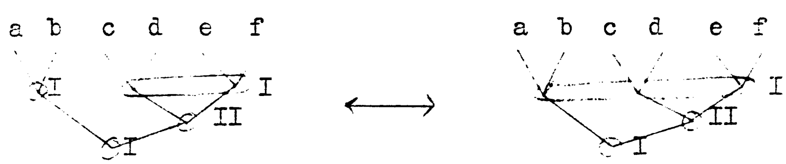
The two, closely related problems we wish to attack are these: Find a simple set of transformations on game structures which:

i) will carry one game structure into another if and only if they are equivalent;

ii) will carry a game structure into another which is equivalent to it and whose normal form matrix is also its reduced normal form matrix.

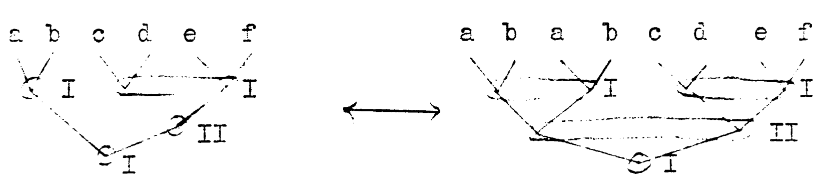
Now it is easy to find transformations which will carry one game structure into an equivalent one. Consider the four transformations exemplified below:

α:



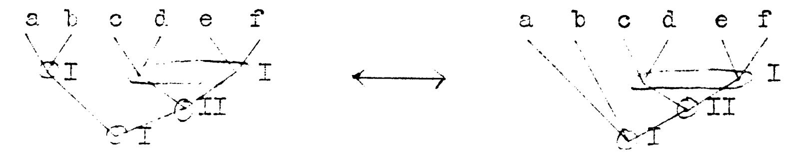
Inflation-deflation

β:



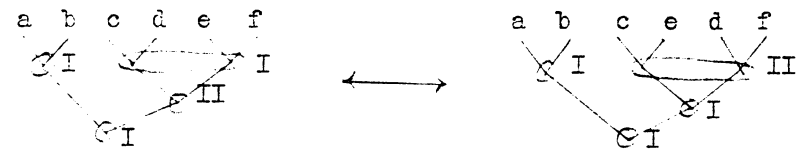
Addition of a superfluous move

π:



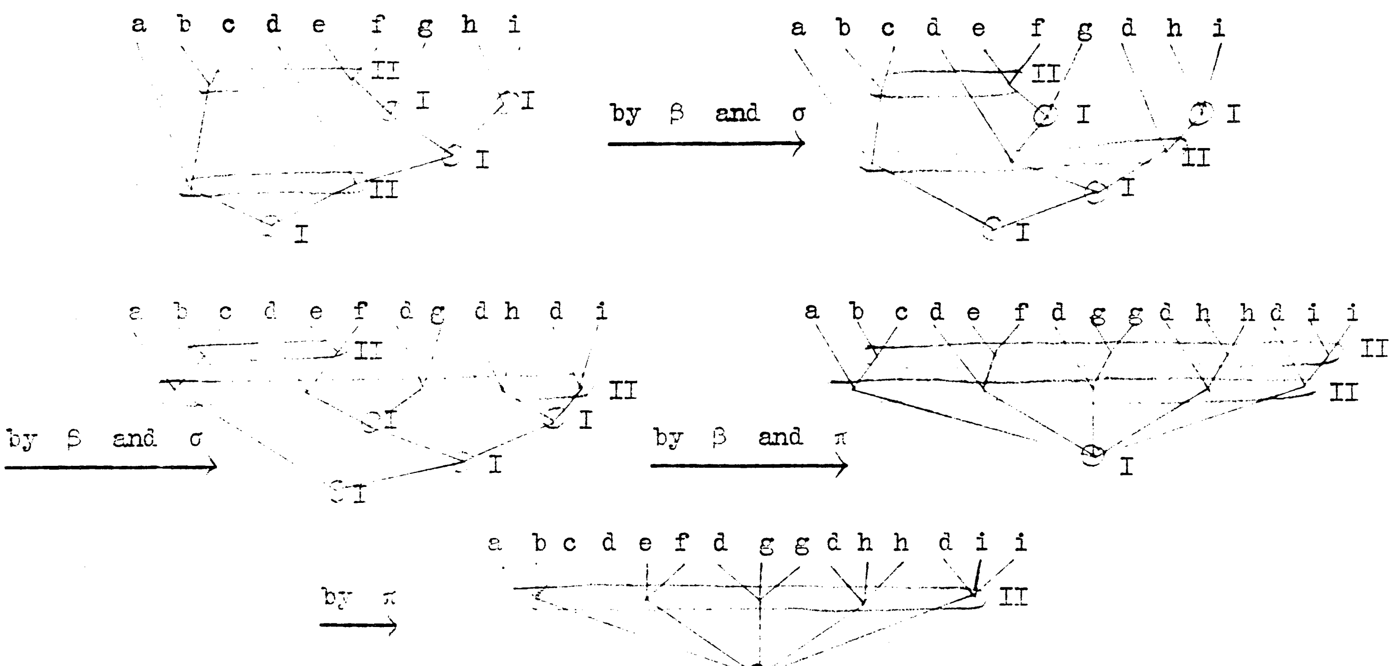
Coalescing of moves

σ:



Interchange of moves

One easily checks, either by inspection or by writting out their reduced normal form matrix, that the two structures in each of these pairs are equivalent. We shall characterize the four transformations of which the above examples are typical. It will then be shown that these four transformations are in fact sufficient to solve both of our problems. Indeed, given a game structure, we can transform it stepwise using only transformations of the above types into a structure whose normal form will be the reduced normal form of the original. We end this informal part of our paper by an example of such a reduction.



The original structure has a normal form matrix which is 16×4 while the normal form matrix of the simplified structure is 5×3.

Formal Presentation: Let G be a finite, non-empty set partially ordered by a relation $\leq$ in such a manner that G has a least element and, for $a, b, c \in G$, $a \leq c$ and $b \leq c$ imply $a \leq b$ or $b \leq a$. For $a, b \in G$, a covers b if $b < a$ and $b < c \leq a$ implies $c = a$; let Aa be the set of all elements which cover a. If Aa is empty then a will be called an "end-point", otherwise a will be called a "move". Let E be the set of end points.

Let P be an equivalence relation on G such that the set of all end-points is an equivalence class. A "player" of the game is an equivalence class a/P for which a is a move. Let "$\underline{P}^*$" denote the set of all players.

Let R be an equivalence relation on G such that, for all $a, b \in G$, $R \cap (Aa \times Ab)$ either is empty or is a biunique function on Aa onto Ab. R plays the role of both the alternative partition and the counter-clockwise ordering of alternatives which appear in the Kuhn formalization [1].

Let I be an equivalence relation on G such that: (i) $I \subseteq P$, (ii) if aIb then $R \cap (Aa \times Ab)$ is not empty, (iii) if aIb then $a \not< b$. If a is a move, then the equivalence class a/I which contains a will be called "an information set for the player a/P".

Definition 1: $\underline{G} = \langle G, \leq, P, R, I \rangle$ is a game structure if $G, \leq P, R, I$ are as described above.

Definition 2: $\underline{G} = \langle\langle G, \leq, P, R, I, h \rangle\rangle$ is a game if $\langle G, \leq, P, R, I \rangle$ is a game structure and h is such a function on $E \times \underline{P}^*$ to the reals that for $a, b \in E$, aIb implies $h(a,p) = h(b,p)$ for all $p \in \underline{P}^*$. h is called the payoff function for the game. $\underline{G}$ will be called an n-person game if $\underline{P}^*$ has just n-elements.

Definition 3: Let $\underline{G}$ be a game structure, $p \in \underline{P}^*$ a player. α is a strategy for p if α is such a function on p that:

i) for $a \in p$, $\alpha(a) \in Aa$;

ii) for $a, b \in p$, if aIb, then $\alpha(a) R \alpha(b)$.

Let S_p be the set of strategies for player p. Let $S = \prod_{p \in \underline{P}^*} S_p$ be the Cartesian product of the S_p's ; S will be called the strategy space for $\underline{G}$.

Each element $\vec{\alpha}$ of S uniquely determines in a natural way an end-point $e = e(\vec{\alpha})$. In fact, let a_o be the least element of G . Suppose $a_o, a_1, \ldots, a_k$ are defined. If a_k is an end-point, let $e(\vec{\alpha}) = a_k$. Otherwise, let $a_{k+1} = \vec{\alpha}_{a_k/P}(a_k)$. Clearly $\{a_o, \ldots, a_k\}$ will be a chain, for each k . Thus for some k , a_k will be an end-point. The set $\{a_o, \ldots, a_k\}$ is usually called a "play".

Definition 4: Let e be the function defined above. e can be considered as the matrix of the game structure.

Definition 5: Let $\underline{G}$ be a game with payoff h ; let $p \in \underline{P}^*$. The payoff matrix of $\underline{G}$ for p is the function H_p on S such that for $\vec{\alpha} \in S$, $H_p(\vec{\alpha}) = h(e(\vec{\alpha}), p)$.

Definition 6: Let $\underline{G}$ be a game structure, $p_1 \in \underline{P}^*$ and $\sigma_1, \sigma_2 \in S_{p_1}$. Then $\sigma_1 \sim \sigma_2$ (σ_1 is equivalent to σ_2) if, for any $\vec{\alpha}, \vec{\beta} \in S$ such that $\vec{\alpha}_{p_1} = \sigma_1$, $\vec{\beta}_{p_1} = \sigma_2$ and $\vec{\alpha}_p = \vec{\beta}_p$ for $p \in \underline{P}^* - \{p_1\}$, then $e(\vec{\alpha}) I e(\vec{\beta})$.

Definition 7: Let $\underline{G}$ be a game structure. For $p \in \underline{P}^*$, let S_p^* be the family of equivalence classes of S_p under $\sim$. Thus an element of S_p^* is a set of equivalent strategies for p . Let S^* be the Cartesian product of the S_p^*'s for $p \in \underline{P}^*$. Then the reduced normal matrix of $\underline{G}$ is the function K on S^* such that for $\vec{\alpha}^* \in S^*$, and $\vec{\alpha} \in \vec{\alpha}^*$, $K(\vec{\alpha}^*) = e(\vec{\alpha})/I$.

A little reflection will convince one that definition 7 formalizes the notion of reduced normal form matrix mentioned on page 4.

Definition 8: Two game structures $\underline{G}_1$ and $\underline{G}_2$ are equivalent, $\underline{G}_1 \sim \underline{G}_2$, if there are biunique functions u, v, w such that:

i) u is on $\underline{P}_1^*$ onto $\underline{P}_2^*$;

ii) v is such a function on $\underline{P}_1^*$ that for $p \in \underline{P}_1^*$, $v(p) = v_p$ maps in a one-one way the set S_p^* onto $S_{u(p)}^*$;

iii) v is on the set E_1/I_1 onto E_2/I_2 , where E_i/I_i is the family of all equivalence classes of elements of E_i under the equivalence relation I_i ;

iv) for $\vec{\alpha}^* \in S_1^*$, $w(K_1(\vec{\alpha}_1^*)) = K_2(\vec{\beta}^*)$ where $v(\vec{\alpha}_p^*) = \vec{\beta}_{u(p)}^*$ for $p \in \underline{P}^*$.

Definition 9: Two game structures $\underline{G}_1$ and $\underline{G}_2$ are isomorphic under the mapping Φ , $\underline{G}_1 \underset{\Phi}{\cong} \underline{G}_2$, if Φ maps $\underline{G}_1$ isomorphically onto $\underline{G}_2$ in the sinse of general algebra, i.e. Φ maps G_1 one to one onto G_2 such that for a, b G , $a \leq_1 b$ implies $\Phi(a) \leq_2 \Phi(b)$, etc.

Definition 10: If $\underline{G} = < G, \leq, P, R, I >$ is a game structure, $a \in G$, then $\underline{G}^a$ is the sequence $< G \cap \{b | a \leq b\} , \leq, P, R, I >$.

Lemma 11: If $\underline{G} = < G, \leq, P, R, I >$ is a game structure, $a \in G$, then $\underline{G}^a$ is a game structure.

Definition 12: Let $\underline{G}$ be a game structure, let $\underline{P}^*$ be ordered so that $\underline{P}^* = \{p_1, p_2, \ldots, p_k\}$. Then $\underline{G}$ has normal form relative to this ordering if:

i) for all moves a and b , aPb implies aIb (i.e. each player has just one information set);

ii) for $a \in E$, $p \in \underline{P}^*$, there is a $b \in p$ such that $b < a$, (i.e. each player has a move in every play);

iii) if $a \in p_i$, $b \in p_j$ and $i < j$, then $a < b$;

iv) for $a \in G$, $b, c \in Aa$, there are $a' \in a/I$, $b', c' \in Aa'$ such that $b'Rb$, $c'Rc$ and $\underline{G}^{b'} \not\cong \underline{G}^{c'}$.

Theorem 13: For every game structure $\underline{G}$, and every ordering of its players, there is a $\underline{G}'$ which has normal form relative to a fixed ordering of its players and $\underline{G} \sim \underline{G}'$ under mapping functions u, v, w where u

preserves the respective orderings of their players.

Theorem 14: Let $\underline{G}, \underline{G}'$ be game structures in normal form relative to fixed orderings of their players. Then $\underline{G} \sim \underline{G}'$, under mapping functions u, v, w where u preserves the respective orderings of their players, if and only if $\underline{G} \cong \underline{G}'$.

The proofs of theorems 13 and 14 are straightforward. If a game structure has normal form, then its matrix, the function e, has no "repeated rows or columns".

We have now defined equivalence classes of game structures essentially in terms of their reduced matricies, (though this last notion has not been explicitly defined), and we have characterized certain canonical members of each class. We now turn to our major task: to characterize the equivalence classes of game structures in terms of four elementary transformations and independently of the notion of strategy.

Definition 15: Let $\underline{G}$ be a game structure, $p \in \underline{P}^*$, a, $b \in G$. Then $a \underset{p}{|} b$ if there are $a', b' \in p$, $a'' \in Aa'$, $b'' \in Ab'$ such that $a'Ib'$, $a'' \leq a$, $b'' \leq b$ and not $a''Rb''$.

Definition 16: Let $\underline{G}, \underline{G}'$ be a game structures, $\underline{G} = \langle G, \leq, P, R, I \rangle$. Then:

i) (Inflation-deflation) $\underline{G} \overset{1}{\sim} \underline{G}'$ if, for some a, $b \in G$,

1) for $a'Ia$, $b'Ib$, $a' \underset{a/P}{|} b'$,

2) $\underline{G}' \cong \rangle G, \leq, P, R, I \cup \{\langle c,d \rangle | cIa$ and dIb or dIa and $cIb\} \rangle$;

ii) (Addition of superfluous moves) $\underline{G} \overset{2}{\sim} \underline{G}'$ if, for some a, $b_1, b_2, \in G$ and some Φ:

1) $Aa = \{b_1, b_2\}$;

2) $\underline{G}^{b_1} \underset{\Phi}{\cong} \underline{G}^{b_2}$;

3) for $c \geq b_1$, $cI\Phi(c)$;

4) $\underline{G}' \cong \langle G - (\{a\} \cup \{c \mid b_2 \leq c\}), \leq, P, R, I \rangle$.

iii) (Coalescing of moves) $\underline{G} \overset{3}{\sim} \underline{G}'$ if, for some $\{a_1, \ldots, a_k\} = a_1/I$, $\{b_1, \ldots, b_k\} = b_1/I$:

1) $b_i \in Aa_i$ for $1 \leq i \leq k$;

2) $b_i R b_j$ for $1 \leq i, j \leq k$;

3) $\underline{G}' \cong \langle G - \{b_1, \ldots, b_k\}, \leq, P, R, I \rangle$.

iv) (Interchange of moves) $\underline{G} \overset{4}{\sim} \underline{G}'$ if, for some $a, b_1, b_2, c_1, c_2, d_1, d_2 \in G$, and some $\leq', P', I'$:

1) $Aa = \{b_1, b_2\}$, $Ab_1 = \{c_1, c_2\}$, $Ab_2 = \{d_1, d_2\}$;

2) $b_1 I b_2$, $c_1 R d_1$, $c_2 R d_2$;

3) $A'b_1 = \{c_1, d_1\}$, $A'b_2 = \{c_2, d_2\}$;

4) $(a/I - \{a\}) \cup \{b_1, b_2\} = b_1/I'$, $(b_1/I - \{b_1, b_2\}) \cup \{a\} = a/I'$;

5) $\underline{G}' = \langle G, \leq', P', R', I' \rangle$;

6) $\langle G' - G'^{a}, \leq', P', R', I' \rangle \underset{\Phi}{\cong} \langle G - G^{a}, \leq, P, R, I \rangle$, where Φ is the identity map;;

7) $\underline{G}^{c_i} \cong \underline{G}'^{c_i}$, $\underline{G}^{d_i} \cong \underline{G}'^{d_i}$ for $i = 1, 2$.

In the remainder of this paper we shall have need of two functions which we now define. Let $\underline{G}$ be a game structure, $\underline{U}$ the family of all of its information sets. (i) For $U \in \underline{U}$, let $A(U)$ be the number of alternatives at any move in U. We note that $A(U) \geq 2$. Let $\pi(\underline{G}) = \sum_{U \in \underline{U}} (A(U) - 2)$. ii) Let V be an ordering of $\underline{U}$. Thus for some k, $\underline{U} = \{V_1, \ldots, V_k\}$. We shall say that $b \in G$ is out of order with respect to V if there are $i > j \geq k$, $a \in G$, such that $a \in V_j$, $b \in V_i$ and $a > b$. Let B be the set of elements which are out of order. For $a \in G$, let $\rho(a)$ be the number of elements which precede a, and $r = \max_{a \in G} \{\rho(a)\}$. Let

$$\sigma(\underline{G}, V) = \sum_{a \in B,\ a \in V_i} 4^{r + i - \rho(a)} .$$

Lemma 17: Let $\underline{G}$ be a game structure. Then there are game structures $\underline{G}_1, \ldots, \underline{G}_t$ such that $\underline{G} \cong \underline{G}_1$, $\underline{G}_i \overset{3}{\sim} \underline{G}_{i+1}$ for $1 \leq i < t$, and $\pi(\underline{G}_t) = 0$.

Proof: Let $\mathscr{A}(n)$ mean: If $\underline{G}$ is a game structure such that $\pi(\underline{G}) \leq n$, then there are game structures $\underline{G}_1, \ldots, \underline{G}_t$ such that $\underline{G} \cong \underline{G}_1$, $\underline{G}_i \overset{3}{\sim} G_{i+1}$ for $1 \leq i < t$, and $\pi(\underline{G}_t) = 0$. The lemma follows by induction on n.

Lemma 18: Let $\underline{G}$ be a game structure. Then there are game structures $\underline{G}_1, \ldots, \underline{G}_t$ such that $\underline{G} \cong \underline{G}_1$, $\underline{G}_i \overset{j}{\sim} \underline{G}_{i+1}$ for $1 \leq i < t$ and $j = 2, 3, 4$, $\pi(\underline{G}_t) = 0$ and, for some ordering V, $\sigma(\underline{G}_t, V) = 0$.

Proof: First we get a $\underline{G}'$ as given by lemma 17. Let V be an ordering of its information sets. We complete our proof by supposing $\sigma(\underline{G}', V) \leq n$ and proceeding by induction on n.

Theorem 19: If $\underline{G}$ is a game structure then there are game structures $\underline{G}_1, \ldots, \underline{G}_t$ such that $\underline{G} \cong \underline{G}_1$, $\underline{G}_i \overset{j}{\sim} \underline{G}_{i+1}$ or $\underline{G}_{i+1} \overset{j}{\sim} \underline{G}_i$ for $1 \leq i < t$ and $j = 1, 2, 3, 4$, and $\underline{G}_t$ is in normal form.

Proof: We first get a $\underline{G}'$ as given by lemma 18. Using def. 16 ii, we get a $\underline{G}''$, preserving the properties of $\underline{G}'$ and such that each information set intersects every maximal chain. Now we inflate whereever possible. We easily check that several applications of def. 16 iii give the desired result.

Theorem 20: If $\underline{G}_1, \ldots, \underline{G}_t$ are game structures such that $\underline{G}_i \overset{j}{\sim} \underline{G}_{i+1}$ or $\underline{G}_{i+1} \overset{j}{\sim} \underline{G}_i$ for $1 \leq i < t$ and $j = 1, 2, 3, 4$, then $\underline{G}_1 \sim \underline{G}_t$.

Proof: It is clearly sufficient to prove that $\underline{G} \overset{j}{\sim} \underline{G}'$ implies $\underline{G} \sim \underline{G}'$ for $j = 1, 2, 3, 4$. The checking of the four cases is straight forward.

Theorem 21: $\underline{G} \sim \underline{G}'$ if and only if $\underline{G}'$ can be obtained from $\underline{G}$ by stepwise application of the four transformations of def. 16.

Proof: This follows from theorems 13, 14, 19 and 20.

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