SIMPLIFIED MODEL OF A SYMMETRIC TACTICAL AIR WAR

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SUMMARY

A set of modified Lanchester's Equations are used to describe the interactions between opposing air and ground forces. In Part I a model describing the rate of growth or decay of two air forces is developed. In Part II this model is extended to include the ground forces. Several applications of the equations are contained in both Part I and Part II.

In Part III some of the capabilities and limitations of this model are discussed, along with recommendations for future improvements.

INTRODUCTION

In searching for a method to study problems of a Tactical Air Command, a set of modified Lanchester's Equations were studied. Since equations of this form might be useful in further studies, the methods and concepts are presented in this preliminary report. It must be emphasized that the model has not yet been developed to the degree of refinement needed to furnish reliable information about an air ground war.

In Part I, two differential equations and an integral payoff function describing a model of a tactical air force is presented. This model describes the dependence of growth or decay of an air force upon rate of replacement, tactics of each side, and general efficiency.

In Part II the model is extended to include both air and ground phases. Two differential equations for the air war are retained and two additional differential equations are introduced to describe the ground war. Several examples of the types of information that can be obtained from this model are illustrated by a pilot study.

Part III contains a discussion of some possible extensions and improvements that might be made in a more exhaustive study.

Our purpose in developing this model is not to describe the course of battle, but rather to form an easily manageable mechanism with which we can study a few aspects of air and ground warfare. It is found that, as an interesting corollary,
the course of warfare within one theater is determined from the forces involved. In the spirit of checking our model with reality, we employ this descriptive aspect of the model to study WW II campaigns in the ETO. Although this descriptive aspect is of interest, one should keep in mind that the purpose of this model is to provide a suitable payoff function with which we can evaluate tactical air power. This modest goal has not as yet been reached.
I. AIR BATTLE

A simple model of a tactical air war can be developed with two differential equations and one integral equation. The differential equations, patterned after the well known Lanchester's Equations, describe the time rate of increase or decrease of the size of the two forces. The integral equation, which serves as a payoff function, determines the number of ground support sorties that one side furnishes in excess of the number furnished by the other side.

For simplicity we designate one side as the red and the other as the blue force. The red and blue commanders can use their aircraft for any of the three following types of missions:

1. The aircraft may engage in offensive air battle strikes, such as attacking enemy air fields.
2. The aircraft may engage in defensive air battle strikes, such as intercepting enemy attackers.
3. The aircraft may engage in ground cooperation strikes.

Missions of type one and two reduce the enemy aircraft strength, while missions of type three contribute to the payoff function.

For simplicity it is assumed that each air force is composed of one type of aircraft that can be used for any one of the three types of missions. The utilization rate is assumed to be one sortie per day per aircraft.

The differential equations (1) for blue and (2) for red air forces and the payoff function (3) that are used in mode No. 1 are of the form

\[ \frac{dm}{dT} = p - l_1 \left( s_1 n - k_2 r_2 m \right) - l_2 s_2 n - cm \]  \hspace{1cm} (1)

\[ \frac{dn}{dT} = q - k_1 \left( r_1 m - l_2 s_2 n \right) - k_2 r_2 m - dn \]  \hspace{1cm} (2)

\[ G = \int_{0}^{T} \left[ k_3 \left( r_3 m - l_4 s_2 n \right) - l_3 \left( s_3 n - k_4 r_2 m \right) \right] \, dT \]  \hspace{1cm} (3)
The meaning of the symbols are:

\[ \begin{align*}
    m &= \text{number of blue aircraft} \\
    n &= \text{number of red aircraft} \\
    r_1 &= \text{fraction of blue a/c used to attack red airfields} \\
    r_2 &= \text{fraction of blue a/c used for air defense} \\
    r_3 &= \text{fraction of blue a/c used for interdiction} \\
    r_4 &= \text{fraction of blue a/c use for ground support} \\
    r_{34} &= r_3 + r_4 \\
    s_1 &= \text{fraction of red a/c used to attack blue airfields} \\
    s_2 &= \text{fraction of red a/c used for air defense} \\
    s_{34} &= \text{fraction of red a/c used for interdiction and close support} \\
    k_1 &= \text{expected number of red a/c destroyed on ground per blue a/c sortie} \\
    k_2 &= \text{expected number of red a/c destroyed in air per blue a/c assigned to defense} \\
    k_3 &= \text{expected number of successful blue ground missions per sorties} \\
    k_4 &= \text{expected number of red ground sorties cancelled/blue a/c on defense} \\
    k_5 &= \text{expected number of red airfield missions cancelled/blue a/c on defense} \\
    I_1 \text{ through } I_5 &= \text{have symmetrical definitions} \\
    c &= \text{expected fraction of blue planes lost per day per sortie due to AA and accidents} \\
    d &= \text{expected fraction of red planes lost per day due to AA and accidents} \\
    p &= \text{production rate of blue a/c per day} \\
    q &= \text{production rate of red a/c per day} \\
    T &= \text{time in days} \\
    G &= \text{total number of blue ground missions in excess of red missions}
\end{align*} \]

The quantities \( s_1, s_2, s_{34}, r_1, r_2, \) and \( r_{34} \) are used to specify the tactics of the blue and red forces respectively. The initial values of \( m \) and \( n \) are selected so as to agree with estimates of the sizes of air forces at the start of a campaign.
The numerical values of the coefficients were found largely from WWII experiences. A typical set of equations with numerical values are:

$$\dot{m} = 20 - 0.02(s_1 n - 0.2 r_2 m) - 0.01 s_2 n$$  \hspace{1cm} (4)$$
$$\dot{n} = 10 - 0.04 (r_1 m - 0.1 s_2 n) - 0.02 r_2 m$$  \hspace{1cm} (5)$$

$$G = \int_0^T (0.8 r_{34} m - 0.1 s_2 n - 0.4 s_{34} n + 0.2 r_2 m) \, dT$$  \hspace{1cm} (6)$$

Constraints

$$m \geq 0 \quad \sum s_i = 1$$
$$n \geq 0 \quad \sum r_i = 1$$
$$(s_1 n - 0.2 r_2 m) \geq 0$$
$$(r_1 m - 0.1 s_2 n) \geq 0$$
$$(0.8 r_{34} m - 0.1 s_2 n) \geq 0$$
$$(0.4 s_{34} n - 0.2 r_2 m) \geq 0$$

These equations give the blue forces twice the efficiency and twice the production rate of the red forces.

Fig. (8) of Appendix A indicates the values of G for a selected set of initial conditions. The tactics of each side was determined at the start of the campaign and held constant.

In the calculations of Fig. (9) Appendix A, the tactics were varied in a simple manner so as to obtain a somewhat more realistic representation of an air war. Here it was assumed that, when one air force was completely destroyed, the opposing commander released many of his aircraft from the air battle phase, using enough aircraft in the air battle to keep his opponent's air force equal to zero. The aircraft released performed ground support missions.

Figs. (8) and (9) indicated that the best tactic for either side depends upon the estimated duration of the conflict. It can be seen that, for a short war the best tactic is to devote most effort to close support, while for a long war it is best to concentrate upon the air battle.
Model No. 2

In Model No. 1 it was assumed that every aircraft dispatched on an air battle mission had a fixed and constant probability of destroying an enemy aircraft. It is probably a more precise description to assume that the probability of interception becomes less as the enemy aircraft become fewer in number and more difficult to locate.

In Model No. 2, it is assumed that the reconnaissance and warning are incomplete, and that the probability of interception by blue offensive is proportional to the number of blue air battle sorties times the total number of red aircraft. We then obtain for the equations:

Model No. 2

\[
\dot{m} = 20 - 2 \times 10^{-6} \left( s_{1n} \right) m \left[ 1 - \frac{0.6 r_2 \frac{m}{r_2 m + (s_1 + s_{34}) n}}{r_2 m + (s_1 + s_{34}) n} \right]
\]

(7)

\[
\dot{n} = 10 - 2 \times 10^{-6} \left( r_{1m} \right) n \left[ 1 - \frac{0.3 s_2 \frac{n}{s_2 n + (r_1 + r_{34}) m}}{s_2 n + (r_1 + r_{34}) m} \right]
\]

(8)

\[
G = \int_0^T \left\{ 0.8 r_2 m \left[ 1 - \frac{0.15 s_2 \frac{n}{s_2 n + (r_1 + r_{34}) m}}{s_2 n + (r_1 + r_{34}) m} \right] - 0.4 s_3 \frac{n}{s_3 n + (s_1 + s_{34}) n} \right\} dT
\]

(9)

Fig. (10) of Appendix B shows the plots of G as a function of time obtained from the REAC. The initial conditions used in each case are

\[
m_0 = 6,000 \text{ aircraft (blue)}
\]

\[
n_0 = 12,000 \text{ aircraft (red)}
\]

Each curve represents the values of G for a different set of tactics, as is indicated.

Fig. (11) of Appendix B shows the dependence of m and n with time for the initial conditions and tactics used in Fig. (9).

In Fig. (11), Appendix C, the initial number of aircraft remained at 6000 and
12,000 as before, but the tactics were modified so as to be simple functions of time. It was assumed that if blue gained an initial advantage in the air battle so that the red force was reduced to the same size as the blue force, the blue commander devoted the proper amount of his aircraft to the air battle to keep the two forces equal, and released the remainder for ground support missions. On the other hand if red won an initial advantage in the air battle, the red commander employed sufficient numbers of his aircraft to the air battle to obtain and maintain a four to one ratio in the size of the air forces. The rest of the red force was then devoted to ground support.

Comments on Model No. 2

The curves obtained from this second model show that once a commander has won the air battle, he can devote a part of this air force to maintain air superiority, and release the remainder for ground action. The model indicates that it is not profitable to strive for complete annihilation.

Once the air battle is lost, it is difficult to gain air superiority again.

Conclusions

Models No. 1 and No. 2 are not satisfactory for our purposes for the following reasons:

1. It is not possible to evaluate the relative effectiveness of interdiction and close tactical air support.

2. The model shows the difference between red and blue ground support missions, but not the number of support missions flown.

3. The dynamics of the war on the ground are ignored.

To overcome these difficulties, the model was extended to include additional factors. We shall now describe these changes.
II. AIR-GROUND BATTLE

Since a discussion of the air war without properly considering the ground phase proved to be inadequate, it was decided to incorporate two additional differential equations to describe the ground war, and to employ a new payoff function.

In this section we shall first derive a simple relation between the logistic capabilities of an area and the rate of increase of the air and ground forces; then discuss a simple abstract model of a theater. Several illustrative examples of differential equations are then used to describe the symmetric air-ground war in this theater. These equations are then applied to several examples of interest.

Logistic Capabilities

For simplicity, let us consider a hypothetical theater of operations in which the available transportation facilities remain fixed in capacity for an extended period of time. Reinforcements are drawn from a large pool of men in the rear areas. We introduce the following quantities, choosing the symbols so as to be consistent with subsequent notation:

\[ M = \text{number of ground divisions on one side at the front} \]

\[ b = \text{fraction of the total supply net that must be used to supply one division at the front} \]

\[ K = \text{reinforcement rate in divisions per day if all transport facilities are used to transport troops} \]

\[ T = \text{time in days} \]

\[ a = \text{number of casualties (in units of divisions per day of combat per division)} \]

We note first that the casualties in divisions per day in the entire theater for one side is equal to \( aM \).

The rate of reinforcements would be \( K \) divisions per day if all transport were devoted to moving troops. Not all of the transport can be used to carry troops since those already deployed at the front must be supplied. A fraction \( bM \) of the facilities must be used to supply the \( M \) front line divisions. The remaining fraction
Rate of Buildup in E.T.O.

- Theoretical Curve

- Number of Divisions from Historical Documents

- Days After Normandy Landing

- Number of Allied Divisions
of transport, (1 - 8 M), is available for moving K (1 - 8 M) divisions per day.

The rate of change of M is then obtained from the differential equation

$$\frac{dM}{dT} = -aM + K(1 - 8M)$$  \hspace{1cm} (10)

As can be proved by direct substitution, a solution of this differential equation is

$$M = (M_0 - \frac{K}{a + K_0})e^{-(a + K_0)T} + \frac{K}{a + K_0}$$  \hspace{1cm} (11)

where $M_0$ is the number of divisions at time $T = 0$.

Figure (1) contains a plot of the number of divisions vs time for the ETO in WW II as obtained from Ref. (1). These points lie close to the curve

$$M = -91.5e^{-\frac{T}{148}} + 97.5$$  \hspace{1cm} (12)

The supply problem in the ETO contained more complications than are included in our simple model. However, the number of divisions in the ETO show the predicted form for growth, and approaches a saturation value.

By comparing Eqs. (11) and (12), and by noting that in the ETO (Ref. 4) $a = 1/487$, we find, after some manipulation

$$K = 0.659$$
$$\delta = 1/140$$
$$M_0 = 6$$

Eq. (10) then takes the form

$$\frac{dM}{dT} = -\frac{1}{487}M + 0.659 \left(1 - \frac{M}{140}\right)$$  \hspace{1cm} (13)

We shall use these coefficients in a later section.

Theater of Operations

In this simple model, it is assumed that the air ground war takes place in a theater of operations that is devoid of unusual terrain features (Fig. 2). The front line boundary between the two antagonists is assumed to be a line of length $L(t)$. We shall concentrate our attention upon a part of this line, choosing a section of length such that the front can be considered approximately as a straight line.
over the region of interest. As one side advances or retreats, this line is assumed
to move back and forth, always remaining parallel to itself.

An adequate description of a complete theater would necessarily entail a simul-
taneous study of many small neighborhoods. However, in this first exploratory study
we shall consider only one region and assume that the others are behaving in a simi-
lar manner.

A cartesian coordinate system \((x, y)\) is introduced into this region in a manner
such that the origin lies in the interior of the area, and the \(y\) axis is parallel to
the front. The origin is placed so that at the start of the war (i.e., \(T = 0\)) the
front line is at \(x = 0\). As the war progresses the distance of advance or re-
treat \(x_t\) is indicated by the \(x\) coordinate of the front line, while the velocity of
advance is given by \(i\).

We shall designate the number of ground divisions of the blue and red forces as
\(M\) and \(N\) respectively and adopt the convention that blue tries to advance in the posi-
tive direction.

The blue and red forces have bases of supplies located along the line \((-x_B)\) and
\((x_R)\) respectively. The transportation nets available to blue and red are assumed
to be of equal capacity.

**Ground Forces**

In addition to air power, the red and blue forces possess ground armies. For
a detailed and accurate description of ground forces it would be necessary to in-
clude parameters representing infantry, armor, artillery, and other forces. How-
ever, for our present purposes this detail would excessively complicate our model.
At the expense of losing such information about the structure of ground forces we
introduce a weighted average division as a single parameter to describe the ground
force. A force composed of \(M\) divisions will have all branches of the service incor-
porated in it in a proportion suitable to describe a typical army group, composed
of \(M\) heterogeneous divisions.
Fig. 2 Schematic drawing of theater of operations.
Changes in Ground Force Strength

The changes in the strength of ground forces is assumed to depend upon the following three factors:

1. Rate of reinforcement, \( \frac{dM}{dT} \) or \( \frac{dN}{dT} \)

2. Casualties from ground action \( \frac{dM}{dT} \) or \( \frac{dN}{dT} \)

3. Casualties from air action \( \frac{dM}{dT} \) or \( \frac{dN}{dT} \)

The total rate of change of \( M \) or \( N \) is then given by the equations

\[
\frac{dM}{dT} = \frac{dM}{dT}_{\text{supply}} + \frac{dM}{dT}_{\text{ground}} + \frac{dM}{dT}_{\text{air}}
\]

\[
\frac{dN}{dT} = \frac{dN}{dT}_{\text{supply}} + \frac{dN}{dT}_{\text{ground}} + \frac{dN}{dT}_{\text{air}}
\]

We shall now describe each of these three in more detail.

Reinforcements

Each side reinforces his troops at a maximum rate consistent with his capabilities. In a previous section a relation between the rate of reinforcement and the supply capabilities was developed. This discussion led to the formula for rate of reinforcement.

\[
\frac{dM}{dT} = K (1 - 5 M)
\]  
\[ (14) \]

This formula was derived neglecting the interference with troop movements by enemy air action, the disadvantages of advancing faster than communications can be reconstructed, and neglecting changes in the length of supply lines as the front moves.

It is assumed in this model that these effects can be included by adding terms to Eq. (14), to form the following equations.

\[
\frac{dM}{dT} = F_1 (M, s, n, x, i)
\]

\[
\frac{dN}{dT} = F_2 (N, r, m, x, i)
\]

It is assumed that \( F_1 \) and \( F_2 \) can be expanded in a power series as follows:
\[
\begin{align*}
\left(\frac{dN_i}{dt}\right)_{\text{supply}} &= \sum K_{ijkl} M^i (s_3 n)^j (x)^k (\dot{x})^l \\
\left(\frac{dN_i}{dt}\right)_{\text{supply}}' &= \sum K_{ijkl}' M^i (r_3 m)^j (x)^k (\dot{x})^l
\end{align*}
\]  

\(15\)  \(16\)

where \(K_{ijkl}\) and \(K_{ijkl}'\) are coefficients that describe the effects of interdiction, lengthening of supply lines, and outdistancing supply facilities. The quantities \(s_3\) and \(r_3\) are the fraction of the air force devoted to interdiction. We note first that Eq. (14) contains one of many terms from Eq. (15).

With the present state of our knowledge we are justified in keeping only a few terms of the series, so that Eq. (15) and (16) can be written in the form

\[
\begin{align*}
\left(\frac{dN_i}{dt}\right)_{\text{supply}} &= K (1 - 5M - \epsilon s_3 n) (1 + \theta x) (1 + \nu \dot{x}) \\
\left(\frac{dN_i}{dt}\right)_{\text{supply}}' &= K' (1 - 5N - \epsilon' r_3 m) (1 + \theta' x) (1 + \nu' \dot{x})
\end{align*}
\]  

\(17\)  \(18\)

where \(\epsilon, \theta, \nu, \epsilon', \theta'\) and \(\nu'\) are constants to be evaluated empirically. We note that if \(s_3 = r_3 = x = \dot{x} = 0\), Eq. (17) and (18) reduce to Eq. (14), as is expected.

With our present knowledge it is not certain whether the various corrections should be all additive or multiplicative. In answering this question we shall use the following conventions in this document:

1. Two or more effects which interact with one another are included within the same parentheses. For example interdiction reduces the available transport by an amount \(K\epsilon s_3 n\). Of that remaining, an amount \(K\epsilon M\) must be given to supply the troops already deployed. The transport remaining \(K (1 - 5M - \epsilon s_3 n)\) can be devoted to reinforcement. In this manner, it was concluded that the \(5M\) and \(\epsilon s_3 n\) terms are additive.

2. If an effect is independent of all others, it is included as a multiplicative factor. For example lengthening the supply lines noticeably reduces the rate of reinforcement, but seems to have little effect upon reducing the total number eventually deployed. Hence this \((1 + \theta x)\) is included as a separate term.

These arguments are, of course, only qualitative in nature and should be em-
ployed with caution. It is hoped that further study will resolve these factors in a more satisfactory manner.

It should be explicitly stated that in our numerical calculations we have assumed \( v = v' = 0 \). It is believed that these parameters can be approximately evaluated for WW II, but this has not yet been done.

**Casualties from Ground Action**

The red and blue forces are assumed to have initially \( N_0 \) and \( M_0 \) divisions deployed along the front. At the start of hostilities \( (T = 0) \) these forces engage each other. A study of casualty ratios in WW II indicates that the casualties depend upon rate of advance, amount of close support of each side and other factors. We assume that the casualty rates due to ground fighting are functions which can be written in the following form:

\[
\left( \frac{dN_{\text{ground}}}{dT} \right) = -\sum_{i,j,k,\ell,p} A_{ijk\ell p}(s_{4}^{i}(r_{4}^{m})^{j}(s_{4}^{n})^{k} N_{\ell}^{\prime} N_{p}^{\prime} \right)
\]

where \( A_{ijk\ell p} \) and \( A'_{ijk\ell p} \) are constants that describe the dependence of casualty rates upon velocity of movement and psychological effect of friendly and psychological effect of enemy TA respectively, while \( s_{4} \) and \( r_{4} \) are the fractions of blue and red air forces devoted to close tactical air support.

It should be noticed that the assumption that blue's casualty rate depends upon the number of red divisions is an improvement over the more elementary assumptions of Eq. (10).

With the present state of our knowledge we are justified in keeping only the following terms;
\[
\frac{\text{d}N}{\text{d}t} = -(\alpha_0' + \alpha_1' \dot{x} + \alpha_2' \dot{x}^2 + \alpha_3' \dot{x}^3)(1 + \beta_1 r_4 m + \gamma_1 s_4 n) M \tag{22}
\]

The question of additive and multiplicative terms has been resolved in accordance with our convention.

The polynomial in \( \dot{x} \) is used to express the rate of change of casualties with respect to velocity. A cursory study of current and past conflicts indicates that the casualties depend in a very complicated manner upon the rate of advance. The general trend can be expressed in several general statements:

1. In a static situation, the number of casualties suffered by red and blue forces are approximately equal in a symmetric situation.

2. In a slow advance, in general, the attacker suffers heavier casualties than the defender.

3. In a rapid advance the defender suffers very heavy casualties (in the form of prisoners).

It is believed that a cubic polynomial in \( \dot{x} \) will approximately describe these three observations.

The factors involving \( m \) and \( n \) in Eqs. (21) and (22) represent an attempt to include psychological factors. In Ref. 7 a mathematical relation showing that blue artillery fire reduces blue's casualty rates and increases red's casualty rates is presented. Since there is considerable discrepancy among the references, the numerical values cannot be considered as firm. We assume here that the same formula applies to close support, with tons of aircraft payload substituted for tons of artillery shells.

Casualties from Air Action

Each side will suffer casualties due to air ground action. These casualties are assumed to take the form:
\[
\left(\frac{dM}{dt}\right)_{\text{air}} = - \sum_{j=0}^{\infty} \left[ \theta_j \left(s_3^2 n\right)^j + \psi_j \left(s_4^2 n\right)^j \right]
\]
(23)

\[
\left(\frac{dN}{dt}\right)_{\text{air}} = - \sum_{j=0}^{\infty} \left[ \theta'_j \left(r_3^2 m\right)^j + \psi'_j \left(r_4^2 m\right)^j \right]
\]
(24)

Here \(\theta_j, \psi_j, \theta'_j, \psi'_j\) are constants that describe the casualties per strike due to interdiction and close support.

As a first approximation we assume that close support sorties produce casualties proportional to the number of sorties flown. We neglect the casualties produced by interdiction. Eqs. (23) and (24) then become approximately

\[
\frac{dM}{dt} = - \psi_1 s_4 n
\]
(25)

\[
\frac{dN}{dt} = - \psi'_1 r_4 m
\]
(26)

General Form of Differential Equations

Combining Eqs. (15), (19), and (23) one obtains for \(M\)

\[
\frac{dM}{dt} = \left(\frac{dM}{dt}\right)_{\text{supply}} + \left(\frac{dM}{dt}\right)_{\text{ground}} + \left(\frac{dM}{dt}\right)_{\text{air}}
\]
(27)

The corresponding equation for \(N\) is

\[
\frac{dN}{dt} = \left(\frac{dN}{dt}\right)_{\text{supply}} + \left(\frac{dN}{dt}\right)_{\text{ground}} + \left(\frac{dN}{dt}\right)_{\text{air}}
\]
(28)

To a first approximation we obtain

\[
\frac{dM}{dt} = K \left(1 - 5M - s_3^2 n \right) \left(1 + \theta x \right) \left(1 + \psi z \right) - \psi_1 s_4 n - \left(\alpha_1^2 + \alpha_2^2 + \alpha_3^2\right)
\]

\[
\left(1 + \beta_3 s_4 n + \gamma_4 r_4 m\right) M
\]
(29)

\[
\frac{dN}{dt} = K' \left(1 - 5'M - r_3^2 m \right) \left(1 + \theta' x \right) \left(1 + \psi' z \right) - \psi'_1 r_4 m - \left(\alpha'_1^2 + \alpha'_2^2 + \alpha'_3^2\right)
\]

\[
\left(1 + \beta'_1 r_4 m + \gamma'_4 s_4 n\right) M
\]
(30)

These are the differential equations that describe the time changes in \(M\) and \(N\).

Rate of Advance

The rate of advance or retreat is assumed to depend upon a dimensionless func-
tion of \( M \) and \( N \).

\[
x = f (M, N)
\]

The function \( f (M, N) \) must have the following properties.

1. \( f (M, M) = 0 \)
2. \( f (M, N) = -f (N, M) \)
3. \( f (M, 0) = V_{\text{max}} \)

Here \( V_{\text{max}} \) is velocity of advance if an army moves unopposed.

A general function which possesses these properties is

\[
f = \sum_{p=0}^{\infty} V_p \left( \frac{M - N}{M + N} \right)^{2p + 1}
\]

With the present state of our knowledge we are justified in keeping only the first term

\[
\dot{x} = V_0 \left( \frac{M - N}{M + N} \right)
\]

The distance of advance is then found to be

\[
x = \int_{0}^{T} \dot{x} \, dt
\]

\[
x = V_0 \int_{0}^{T} \frac{M - N}{M + N} \, dt
\]

Air War

In Part I we developed a mathematical model of war between two air forces neglecting the war between ground forces. In this second chapter we are developing a model of a symmetric war involving both air and ground forces and have developed the equations to describe the ground forces.

We shall now incorporate two equations to describe the air phase of the war selecting two equations from Part I. In this exploratory study it is desirable to select the simplest equations that will satisfy our needs. The equations we
have used are:

\[
\frac{dm}{dT} = p - l_1 s_1 \text{ mm} \tag{35}
\]

\[
\frac{dn}{dT} = q - k_1 r_1 \text{ mm} \tag{36}
\]

As previously described, these equations include replacement and casualties to air battle.

Review

At the end of this somewhat lengthy discussion of the model of an air-ground war it is well to review the material which we have presented. This Part began with a discussion of the restrictions that supply capabilities placed upon size and rate of growth of the armed forces.

This was followed by a description of an abstract model of a theater of operations and the forces contained in it. Then mathematical equations were introduced to describe the interaction of two ground forces. Terms were included in these equations to describe the effects of reinforcements as well as casualties from air and ground action. An equation to describe the rate of advance or retreat was introduced to given the model a dynamic constraint and finally, two equations to describe the air phase were re-introduced.

The simplified forms of these equations, which we shall employ to calculate some interesting solutions, are presented again for completeness.

1. Air Equations

\[
\frac{dm}{dT} = p - l_1 s_1 \text{ mm} \tag{37}
\]

\[
\frac{dn}{dT} = q - k_1 r_1 \text{ mm} \tag{38}
\]

2. Ground Equations

\[
\frac{dM}{dT} = K (1 - 6M - 6s_2 n) (1 + \theta x) (1 + \nu x) - \Psi s_4 n
\]

\[- (d_0 + a_1 x + a_2 x^2 + a_3 x^3) (1 + \beta_4 s_4 n + r_4 n) N \tag{39}\]
\[
\frac{dN}{dT} = K'(1 - 8'N - C'r_m)(1 + \theta'x)(1 + v'\dot{x} - \psi' r_m) \\
- (1 + a'_1 \dot{x} + a'_2 \dot{x}^2 + a'_3 \dot{x}^3)(1 + \beta' r_m + \gamma' s_n) M \\
(40)
\]

3. Distance of Advance
\[
x = V \int_0^T \frac{M - N}{M + N} dT
(41)
\]

**Numerical Coefficients**

For a pilot study with Eqs. (37) through (41) the numerical values of the coefficients were estimated from a study of unclassified sources describing WW II (Refs. 1 to 7). The numerical values as found from these sources are contained in Eqs. (42), (43), (44), (45) and (46).

\[
m = 20 - 2 \times 10^{-5} a_{1 mm} 
(42)
\]

\[
\dot{n} = 10 - 4 \times 10^{-5} r_{1 mm} 
(43)
\]

\[
M = \frac{1}{400} \left[ 1 + \frac{\dot{x}}{50} - \left(\frac{\dot{x}}{25}\right)^3 \right] \left[ 1 + \frac{s_4}{10^4} + \frac{r_m}{2 \times 10^4} - \frac{r_{4 m}}{10^4} \right] N - \frac{s_{4 n}}{1.5 \times 10^4} + \frac{0.16x}{500}\left[ 1 - \frac{M}{200} - \frac{s_{3 n}}{10^4} \right] \left[ 1 - \frac{x}{600} \right]
(44)
\]

\[
N = \frac{1}{400} \left[ 1 - \frac{\dot{x}}{50} + \left(\frac{\dot{x}}{25}\right)^3 \right] \left[ 1 + \frac{r_m}{10^4} + \frac{r_m}{2 \times 10^4} - \frac{s_{4 n}}{10^4} \right] M - \frac{r_{4 m}}{1.5 \times 10^4} + \frac{0.84x}{500}\left[ 1 - \frac{N}{200} - \frac{r_{3 m}}{10^4} \right] \left[ 1 + \frac{x}{600} \right]
(45)
\]

\[
x = 25 \int_0^T \frac{M - N}{M + N} dT
(46)
\]

**Constraints**

1. \(0 \leq m \leq 10,000\)
2. \(0 \leq n \leq 20,000\)
3. \(0 \leq M \leq 200\)
4. \(0 \leq N \leq 200\)
5. \[ \left[ 1 + \frac{s_4 n}{10^4} + \frac{s_3 n}{2 \times 10^4} - \frac{r_1 n}{10^4} \right] \left[ 1 + \frac{r_1 m}{10^4} + \frac{r_2 m}{2 \times 10^4} - \frac{s_4 n}{10^4} \right]^2 \geq 0 \]

6. \[ \left[ 1 + \frac{r_1 m}{10^4} + \frac{r_2 m}{2 \times 10^4} - \frac{s_4 n}{10^4} \right]^2 \geq 0 \]

7. \[ \int \left[ \frac{1.16 + 0.9T}{500} \right] \left[ 1 - \frac{M}{200} - \frac{s_3 n}{10^4} \right] \left[ 1 - \frac{x}{600} \right] dT \leq 200 \]

8. \[ \int \left[ \frac{0.84 - 0.9T}{500} \right] \left[ 1 - \frac{N}{200} - \frac{r_2 m}{10^4} \right] \left[ 1 + \frac{x}{600} \right] dT \leq 200 \]

We note in passing that Eqs. 44 and 45 are linear in M and N. If we divide a theater into many sub-regions as proposed above, and if the two sides deploy their forces proportionately in each region, then the behavior of the forces in each sub-region is similar to that of the theater as a whole. Hence, in this restricted case we can use the equations to describe a complete theater of operations as well as any sub-region. The initial conditions can be varied to describe a particular campaign area.

1. The number of ground divisions available at T = 0
2. The number of aircraft available at T = 0
3. The tactics used by the air commander

These equations represent an attempt to describe the ETO in WW II. The constraints were determined from several sources. In order to formulate the problem for solution with REAC, constraints 1 through 4 were selected. Constraints 5 and 6 indicate extreme limits beyond which our linear approximations to complicated functions would no longer apply. Constraints 7 and 8 are dictated by assumptions as to the maximum number of divisions that one side can mobilize during an entire war.
European Theater of Operations

It is of interest to apply this model to European Theater of Operations beginning with the landings in Normandy and ending with the surrender of Germany. Since the coefficients were determined from information describing this theater, the model describes this war fairly satisfactorily. From the historical references (see Ref. 1 to 7) it was determined that at D-day the Allies had 6 divisions on the beaches and the Germans had 2. The number of divisions as a function of time for Allies and Axis armies is shown in Fig. (3).

It was also found that the allied air force flew about 400 sorties per day close support and about 400 sorties per day for interdiction. The effectiveness of the German Air Force was so small that it could be neglected.

Using this information, the course of the battle could be predicted from the model. The changes in the size of the armies was determined from the differential equations and the distance of advance, $x$, from the integral equation. Fig. (3), (4) compares the behavior of the model with historical information obtained from Refs. (1) to (6).

In determining the average distance of advance, from historical documents, an approximation was applied. The area, $A$, occupied at time $T$ was determined from maps in the above references. With the formula

$$x = \frac{\sqrt{2}}{\pi} A$$

the quantity $x$ was determined. The values of $x$ found in this way are contained in Fig. (4).

It is seen that the model describes the rate of advance in a satisfactory manner. The sizes of the forces found from the model do not precisely fit the data, but are fairly close.

It is felt that this model does satisfactorily describe the course of battle in the ETO.
Rate of Buildup in E.T.O.

- Allied Divisions From Model
- Axis Divisions From Model
- Allied Divisions From Historical Documents
- Axis Divisions From Historical Documents

Number of Divisions

Days After "D" Day

100 200 300 400
50 100 250 350
Hypothetical Campaigns

In order to study campaigns of the present or future, it is necessary to make a detailed study of the coefficients in the general equations (15) through (31). These studies would comprise the component parts of a general systems analysis.

We have not made a detailed study from a component point of view and so cannot properly consider future campaigns. We have made a deductive study of Normandy campaigns and determined the approximate coefficients for a WW II setting. These coefficients are contained in Eqs. (42), (43), (44), (45), and (46).

It is of some interest to apply these equations to a hypothetical situation to see what can be learned. We will consider a war between blue and red forces in Europe. At the start of hostilities each side has ground forces distributed along a line approximately corresponding to the Elbe River.

The red force is specified as consisting of a force of 50 divisions and 5000 aircraft. The red commander employs 1/3 of his aircraft in the air battle, 1/3 in interdiction, and 1/3 in close support.

The problem is to determine the composition of the minimum force blue must have at T = 0 in order to accomplish one of three missions.

1. Blue retreats 500 miles (to the channel), then makes a stand at the channel and maintains a foothold in Europe.
2. Blue retreats 200 miles (Rhine River) and makes a successful stand there.
3. Blue initially retreats an unspecified distance but returns to the Elbe within 50 days.

The blue air commander is allowed to choose the optimum strategy for his air force. The best strategies were found by a trial and error procedure.

Figure (5) displays the number of ground divisions and number of aircraft that blue must have to accomplish each of these missions.

It would be of interest to determine which of these solutions is the best. The question can be answered only by considering cost, casualties, manpower requirements, etc. We have not as yet made such a detailed study. An indication of the
Force Requirements in Europe

Red Force
50 Divisions
5000 Aircraft

1. Keep Foothold in Europe
2. Hold at Rhine
3. Return to Elbe in 50 days
manpower requirements can be obtained by assuming that the division slice is
50,000 men/division and the aircraft slice is 60 men/aircraft. Then the total
number of men, u, needed to accomplish each mission can be determined from the
relation

\[ u = (M) (50,000) + (m) (60) \]

Figure (6) contains a plot of number u as a function of M for each of the
three missions. The curves indicate that there is a unique smallest force for
each mission. It should be noticed that not all interesting regions have been
investigated.

It is also of interest to investigate the cost requirements in terms of
dollars. In calculating costs the following values for yearly plus one fourth
initial cost were used:

1. Cost of division as based upon a division slice of 50,000 men
   \[ \text{- - - 600 megabucks/year.} \]
2. Cost of one aircraft based upon an airplane slice \[ \text{- - - 0.884 megabucks/year.} \]

The cost in millions C to defend Europe is presented in Fig. (7) as a function
of the percentage of the money devoted to TA, \( C_A/C \), where

\[ C = 600 M + 0.884 m \]

\[ C_A = 0.884 m \]

These calculations and graphs by no means exhaust the possible types of in-
formation that can be obtained from the model nor the forms of presenting this
material.

Appendix D contains a few sample curves of \( x \) as a function of \( T \) as obtained
from the REAC. The initial conditions are selected so that the ground battle
initially is symmetric. Each side has the same effectiveness in the air, but
employs different tactics.

A detailed discussion of the capabilities and shortcomings of this model is
contained in Part III, however we note that the curves show an unrealistic oscill-
latory characteristic.
Number of Men Needed in Armed Forces in Theater of Operations at Start of Hostilities

- Keep Foothold in Europe
- Hold At Rhine
- Return To Elbe in 50 days

Red Forces have 50 Div. (Equiv.) and 5000 Aircraft

Thousands of Men in Air Plus Ground Forces

Ground Divisions (Allied)
Cost To Defend Europe With
Money Divided Between TAC
And Ground Forces

1. Return to Elbe in 50 Days
2. Hold at Rhine
3. Foothold in Europe
III. Improvements

Several modifications of the model would make it more realistic without exceeding the Reac computing capacity. Since most of the modifications are in the ground war phase of the model, we must take care to see that the tail does not wag the dog for we are more interested in air strategy, tactics, and equipment.

The more important modifications are

1) A higher order approximation of velocity
2) A command factor that permits the ground commander to keep men in reserve.
3) A means of taking into account highly defendable positions.
4) An improved equation for kill potential
5) The solution of a set of simultaneous, parallel ground war equations, and
6) The inclusion of several different type aircraft.

Many of the approximations of the ground war functions by a linear term seem adequate, but the velocity term should have a higher order approximation, or a non-linear representation. By letting

\[ \dot{x} = V_{\text{max}} \frac{M - N}{M + N} \]

we assume an advance is made whenever even the slightest superiority is achieved. This is not so; a commanding officer would require some minimum fire power advantage before ordering an offensive. Hence, it would probably be better to let \( \dot{x} \) be of the form

\[ \dot{x} = 0 \text{ for } \left| \frac{M - N}{M + N} \right| \leq W \]

\[ \dot{x} = \frac{V_{\text{max}}}{1 - W} \left( \frac{M - N}{M + N} - \frac{M - N}{M - N} W \right) \text{ for } \left| \frac{M - N}{M + N} \right| \geq W \]

as shown in Fig. 16.
Modified Form of Velocity Function
The model has assumed that once men and munitions have reached the battle area they go to the front. The ground commander should have the prerogative of keeping only a fraction $k$ of his strength in the battle lines with the remainder in reserve on essentially immediate call. By replacing $M$ and $N$ in Eqs. (19), and (20) by $k_1M$ and $k_2N$, the above modification can be readily accomplished.

Certain terrain features such as rivers or mountain ranges play important roles in a campaign. A natural barrier could be taken into account by increasing the defender's kill potential, reducing the attacker's kill potential, reducing $V_{\text{max}}$ across the barrier, and increasing $W$ (the required superiority for advance) for the attacker. Such modifications could be introduced into Reac solutions by means of input tables.

Making the ground force kill potential proportional to the number of divisions assumes aimed firing. Blind, area artillery fire kill potential, on the other hand, depends on the number of enemy troops in the area as well as the number of friendly troops. Since both types of firing take place, equations (21) and (22) should be of the form

$$\frac{dK}{dt}\text{\_ground} = - (\alpha_0 + \alpha_1 \dot{x} + \alpha_2 \dot{x}^2 + \alpha_3 \dot{x}^3)(1 + \beta_1 S_4 M + \gamma_1 r_4 m)(N + \xi KN)$$

$$\frac{dN}{dt}\text{\_ground} = - (\alpha_0' + \alpha_1' x + \alpha_2' x^2 + \alpha_3' x^3)(1 + \beta_1' r_4 m + \gamma_1' s_4 n)(M + \xi' KN).$$

The action of a break through could be taken into account if several sets of the ground warfare equations were solved simultaneously to take into account the holding action at certain sections of the front while the breakthrough is attempted at another. In order to do this on the REAC a greatly simplified model would be necessary.
Probably the most important modification of the air battle equations would be in the inclusion of the different types aircraft with specified efficiencies at the different type missions. This is a must if we hope to answer such questions as:

1) How do the economical operations and logistic support advantages of a multiple purpose aircraft compare with the superior performance of the special purpose aircraft?

2) How many different types of special purpose aircraft are required?

3) Is a special purpose ground support aircraft justifiable?

4) Must all TAC aircraft be able to defend themselves in the air or are escorts required?

5) What is the optimum TAC dynamic complement?
References


5. Jacob, Major General Sir Ian, "Defeat in the West", E. P. Dutton, 1948 (Unclassified).


7. Yarnold, K. W., Daly, Jean, "Part II - The Evaluation of Weapons with Special Reference to the Value of Artillery", ORO T-75, 7 May 1950 (Confidential).
SAME EQUATIONS AS TASK II

WHERE WAS A, ADJUSTED TO VALUE REQUIRED TO CONTINUE
WHERE WITH, BY ADJUST TO WASH ALREADY TO MILE 140
\[ \dot{m} = 20 - 2 \times 10^{-6}(a_h)n \left( 1 - 0.6 \frac{n_{2m}}{n_{2m} + (a_{2m})^2} \right) - 1 \times 10^{-6} (a_{2m})^2 n \quad \text{if} \quad n_0 = 6,000 \]

\[ n = 10 - \frac{4 \times 10^{-6}(a_h)n}{(1 - 0.3 \frac{n_{2m}}{n_{2m} + (a_{2m})^2})} - 2 \times 10^{-6}(a_{2m})^2 n \quad \text{if} \quad n_0 = 12,000 \]

\[ Q = \int_0^T \left\{ 0.82 \frac{n_m}{2m_n + (a_{2m})^2} \left( 1 - 0.15 \frac{n_{2m}}{n_{2m} + (a_{2m})^2} \right) - 0.42 \frac{n_m}{n_{2m} + (a_{2m})^2} \right\} dt \]
Equations Same As Figure II, Run #12-6

de = 0.01 (3, 0, 11) mL

e = 0.01 (3, 0, 11) mL
Fig. 14. Typical Solutions to Air Ground War Equations.

\[ M = 60 \quad N = 60 \]
\[ Mm = 9000 \quad m = 18,000 \]

\[ \begin{align*}
\Pi_1, \Pi_2, \Pi_3, \Pi_4, & \quad \eta_1, \eta_2, \eta_3, \eta_4, \\
0.75, & \quad 0.25, 0.15, 0.05, 0.03, 0.01, 0.00 \\
0.75, & \quad 0.25, 0.15, 0.05, 0.03, 0.01, 0.00 \\
\end{align*} \]
Fig. 15

Typical Solution to Air Ground War

\[ n_1, n_2, n_3, n_4, a_1, a_2, a_3, a_4 \]

1. 0 0 0 100 0.33 0.33 0.33
2. 0 0 0.5 0.5 0.33 0.33 0.33

\[ M = 60 \quad N = 60 \]

\[ m = 9000 \quad m = 18,000 \]