U. S. AIR FORCE

PROJECT RAND

RESEARCH MEMORANDUM

MINIMAX THEOREM FOR

UPPER AND LOWER SEMI-CONTINUOUS PAYOFFS

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RM-478

ASTIA Document Number ATI 210665

25 October 1950

Assigned to ________________________

The RAND Corporation

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We give here a proof of the minimax theorem for the game played over a pair of compact metric spaces A and B, in which the payoff is an upper or lower semi-continuous function on A x B. The theorem is a direct consequence of the minimax theorem for continuous payoffs.

It does not seem at the moment to be known whether such discontinuous payoffs fall under the Wald or Karlin criteria, that is, whether this is indeed any extension of the range of validity of the minimax type relationship.

1. Theorem: If M is an upper semi-continuous function over A x B, where A and B are compact metric spaces, then

$$\max_f \inf_g \int\int M(x,y)\,df(x)\,dg(y) = \inf_g \sup_f \int\int M(x,y)\,df(x)\,dg(y)$$

where f and g range over the regular measures defined on A and B, respectively, such that

$$f(A) = g(B) = 1.$$

Proof: Since M is an upper semi-continuous function on a compact metric space, $\exists$ continuous functions M_n defined on A x B $\ni$

$$M_n \geq M_{n+1},\ M_n(x,y) \rightarrow M(x,y) \text{ for each } (x,y).$$

Now by the minimax theorem for continuous payoffs we have f_n, g_n, v_n $\ni$

$$\int M_n(x,y)\,df_n(x) \geq v_n \geq \int M_n(x,y)\,dg_n(y).$$

But each set of strategies, i.e., measures, $\{f\}$, and $\{g\}$, are ω^* compact $[I, p. 4]$, hence $\exists \{n_i\}, f \ni$

$$\int \phi(x)\,df_{n_i}(x) \rightarrow \int \phi(x)\,df(x) \qquad \text{for each}$$

ϕ continuous on A, and

$$\int M_{n_i}(x,y)\,df_{n_i}(x) \geq v_{n_i} \geq \int M_{n_i}(x,y)\,dg_{n_i}(y).$$

Let us replace our sequence by this subsequence. We have

$$\int M_n(x,y)\,df_m \rightarrow \int M_n(x,y)\,df(x) \qquad \text{as } m \rightarrow \infty$$

and since $M_n \geq M_m$ for $m \geq n$

$$\int M_n(x,y)\,df_m(x) \geq \int M_m(x,y)\,df_m(x) \geq v_m$$

so that

$$\int M_n(x,y)\,df(x) \geq \overline{\lim}\ v_m .$$

Also, since the M_n decrease, by Lebesgue's Theorem

$$\int M(x,y)\,df(x) = \lim_n \int M_n(x,y)\,df(x) \geq \overline{\lim}\ v_m.$$

Moreover since for $\varepsilon > 0 \ \exists\, m_o \ni$

$$\left| v_{m_o} - \overline{\lim}\ v_m \right| < \varepsilon, \qquad \text{we have}$$

$$\overline{\lim}\, v_m > v_{m_o} - \varepsilon \geq \int M_{m_o}(x,y)\, dg_{m_o}(y) - \varepsilon$$

$$\geq \int M(x,y)\, dg_{m_o}(y) - \varepsilon .$$

Hence

$$\sup_f \inf_g \int\int M df dg \geq \inf_y \int M(x,y)\, df(x) \geq \overline{\lim}\, v_m$$

$$\geq \inf_g \sup_f \int\int M df dg - \varepsilon$$

or

$$\sup_f \inf_g = \overline{\lim}\, v_m = \inf_g \sup_f \leq \int M(x,y)\, df(x) ,$$

so that the game has a value, and one player an optimal strategy, whence we may write

$$\max_f \inf_g \int\int M df dg = \inf_g \sup_f \int\int M df dg.$$

The statement of the theorem for the lower semi-continuous payoff obviously need only be changed to min sup = sup inf., and the proof is of course the same.

2. The second player, in case M is upper semi-continuous, need not have an optimal strategy. To see this we give an example, and refer to [II] for notation. The game with payoff

$$M(x,y) = \begin{cases} x - \frac{1}{3} & x < y \\ \frac{1}{3} x + \frac{2}{3} x^2 & x = y \\ \frac{2}{3} - y & x > y \end{cases}$$

is precisely the game considered in [1] with $\alpha = \frac{2}{3}$, $\beta = -\frac{1}{3}$, $\gamma = \frac{2}{3}$, hence $\gamma - \alpha - \beta = \frac{1}{3}$. Hence player I has an optimal strategy $f^* = I_{\frac{1}{2}}$, and player II has none. Now we can alter the game so as to make the payoff upper semi-continuous without altering this fact. By setting

$$M'(x,y) = \begin{cases} x - \frac{1}{3} & x < y \\ \frac{1}{3}x + \frac{2}{3}x^2 & x = y > \sqrt{2} - 1 \\ \frac{2}{3} - x & x = y \leq \sqrt{2} - 1 \\ \frac{2}{3} - y & x > y \end{cases},$$

we find that $\int M' dI_{\frac{1}{2}} = K(y)$ is unchanged since $\sqrt{2} - 1 < \frac{1}{2}$. Moreover since we have increased the payoff to I, the value of the new game is increased. However a reference to [II] shows that the strategies g_ε there defined still yield $\sup_x \int M' dg_\varepsilon(y)$ arbitrarily close to the value of the original game, whence $I_{\frac{1}{2}}$ remains an optimal strategy for the first player. Now since

$$K(y) = \int M'(x,y)\, dI_{\frac{1}{2}}(x) > v \qquad \text{for } y \leq \frac{1}{2}$$

any optimal strategy for II avoids the region where M has been changed, hence would be an optimal strategy in the original game.

REFERENCES

[I] RM-433, Best Strategies for Continuous Games with a Continuous Payoff, I. Glicksberg.

[II] RM-474, Noisy Duel, One Bullet Each with Simultaneous Fire and Unequal Worths, I. Glicksberg.

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