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RESEARCH MEMORANDUM

n-PERSON GAMES — V: STABLE-SET SOLUTIONS INCLUDING AN ARBITRARY CLOSED COMPONENT

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RM-1005

19 December 1952

Assigned to ______________________

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Summary: A stable-set (von Neumann-Morgenstern) solution to a simple n-person game is constructed which includes an arbitrary closed subset of a certain (n-3)-dimensional convex region in the simplex of imputations, at a finite distance from the rest of the solution. This provides at one stroke a large fund of "pathological" examples against which conjectures on the behavior of stable-set solutions can be tested.

Define an n-person game Γ_n, $n > 3$, by the characteristic function:

$$v(N) = v(S_1) = v(S_2) = v(S_3) = 1,$$

$$v(S) = 0, \quad \text{all other } S \text{ in } N,$$

where N is the set $\{1,2,\cdots,n\}$ of all players, and S_i is $N - \{i\}$. (This is a non-constant-sum simple game.* A constant-sum game (also simple) having the same solutions can be obtained by adjoining another player to Γ_n.)

Let U denote the (n-3)-dimensional simplex of points $u = (u_4,\cdots,u_n)$ satisfying the inequalities

$$u_j \geq 0, \quad j=4,\cdots,n,$$

$$\sum_{j=4}^{n} u_j \leq 1.$$

Let C be an arbitrary, closed set in the interior of U. Let C' be

* A simple game may be defined as one whose characteristic function takes on only the values 0 and 1, or one strategically equivalent to such a game.

the union of C with the face of U defined by the equality

$$\sum_{j=4}^{n} u_j = 1.$$

We shall construct a stable-set solution to Γ_n consisting of a set congruent to C', and one other component.* The arbitrariness in the choice of C (for example, C may be a Cantor-type discontinuum) makes it easy to dispose of many conjectures concerning the regular behavior of stable sets.** The task of ascribing an operational meaning to every solution of a game is made correspondingly more difficult.

The cardinality of the set of known solutions to a game is not increased by the present example. Indeed, a continuum of solutions is already known for many games (including the game Γ_n); however, on the other hand, there are no more than c closed sets in a finite-dimensional simplex. Since stable sets are necessarily closed, there is no room left for an increase in cardinality.

The construction starts with an (n-2)-dimensional set K_{23} which dominates all of its complement in A with the exception of a set K_1, congruent to the "arbitrary" set C'. K_1, in turn, dominates part of K_{23}; but when this part is cut out, the resulting set, combined with K_1, is A-stable. The set K_{23} may be likened to a woodcut block, which prints the desired image of C on the boundary of the simplex A.

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* We shall prove A-stability (von Neumann's definition of solution). The G-stability of our solution will follow in virtue of Theorem 3 of RM-817

** We record here two surviving conjectures, still to be settled: (1) that no solution can consist of a denumerably infinite set of points; (2) [suggested by McKinsey] that no finite solution of an n-person game can have more than 2^n points.

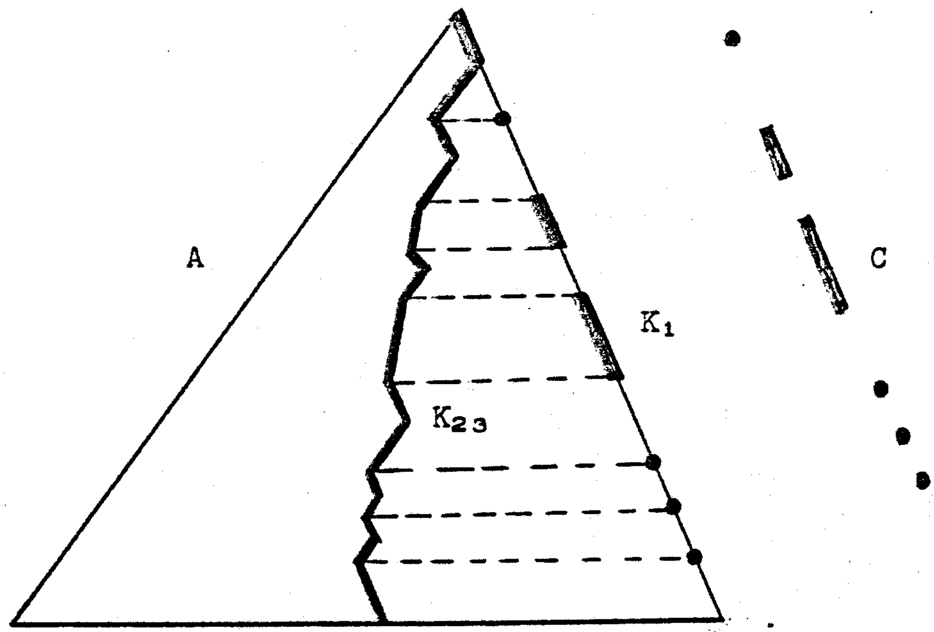

Figure 1

The actual construction follows. Define distance between points in U by

$$\rho(u,w) = \sum_{j=4}^{n} |u_j - w_j|;$$

and distance between points and closed sets in U by

$$\rho(u,B) = \min_{w \in B} \rho(u,w).$$

Let A, as usual, be the simplex of imputations α:

$$\alpha_i \geq 0, \quad i=1,2,\cdots,n,$$

$$\sum_{i=1}^{n} \alpha_i = 1.$$

Define four subsets of A as follows:

K_1: points of the form

$$(0, a, a, u_4, \cdots, u_n), \text{ with } u \text{ in } C';$$

K_2: points of the form

$$(a - 1/2\,\rho(u,C'), a + 1/2\,\rho(u,C'), 0, u_4, \cdots, u_n), \text{ with } u \text{ in } U;$$

K_3: points of the form

$$(a - 1/2\,\rho(u,C'), a + 1/2\,\rho(u,C'), 0, u_4, \cdots, u_n), \text{ with } u \text{ in } U;$$

K_{23}: points of the form

$$(a - 1/2\,\rho(u,C'), x, a + 1/2\,\rho(u,C') - x, u_4, \cdots, u_n),$$

with u in U.

The quantity a, in each case, is necessarily equal to $(1 - \sum u_j)/2$. On the other hand, the quantity x runs through all values between 0 and $a + 1/2\,\rho(u,C')$. Thus, K_{23} contains both K_2 and K_3 in its boundary. Since C was assumed interior to U, K_1 intersects K_{23} only in the set of points α with $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Thus, K_{23}, being closed, is at a finite distance from the set congruent to the "arbitrary" set C.

Define the set K:

$$K = K_1 \cup [K_{23} - \text{dom } K_1].$$

(The "dom" operator on sets is defined in RM-656, or see AMS-28, "Quota Solutions of n-person Games.") We shall proceed to prove that K is a solution—i.e., that

$$K = A - \text{dom } K.$$

For the sake of clarity, we break the proof up into a series of lemmata. "ϕ" will denote the empty set.

Lemma 1. $K_1 \cap \text{dom } K_1 = \phi$.

Lemma 2. $K_2 \cap \text{dom } K_1 = \phi$.

Lemma 3. $K_3 \cap \text{dom } K_1 = \phi$.

Lemma 4. $K_1 \cap \text{dom } K_{23} = \phi$.

Lemma 5. $K_{23} \cap \text{dom } K_{23} = \phi$.

Lemma 6. $K \cap \text{dom } K = \phi$.

<u>Proofs 1 — 6</u>: The first three are verified easily, since only S_1-domination need be considered. In Lemma 4, S_1-domination is out of the question. Suppose that a point α in K_{23} S_2-dominates the point $\beta = (0, b, b, w_4, \cdots, w_n)$, w being some point in U. Summing the components of β and α, we have

$$1 = 2b + \sum_{j=4}^{n} w_j = 2a + \sum_{j=4}^{n} u_j,$$

taking u and a as in the definition of K_{23}. Since $u > w$, we therefore have

$$\rho(u,w) = 2b - 2a.$$

Since $\alpha_3 > \beta_3$, we also have

$$a + 1/2\,\rho(u,c') - x > b.$$

Hence $\rho(u,C') > \rho(u,w)$. Hence w cannot be in C'. Hence β cannot be in K_1. Q.E.D. For Lemma 5, suppose that α and α' are two points in K_{23}, and that α' dominates α. Then $u' > u$ for the corresponding points in U, and

$$\rho(u,u') = 2a - 2a'.$$

Suppose first that S_1 is the effective set for the domination. Then, comparing the combined shares of players 2 and 3, we obtain

$$a' + 1/2\,\rho(u',C') > a + 1/2\,\rho(u,C').$$

Therefore

$$\rho(u',C') > \rho(u,u') + \rho(u,C').$$

Let $w \in C'$ be a point which minimizes $\rho(u,w)$. Then

$$\rho(u',w) \geq \rho(u',C') > \rho(u,u') + \rho(u,w),$$

contradicting the triangle inequality for ρ. Hence S_1 cannot be effective. The cases of S_2 and S_3 are argued analogously comparing player 1's shares. Q.E.D. Finally, Lemma 6 is proved as follows:

$$K \cap \text{dom } K = [K_1 \cup (K_{23} - \text{dom } K_1)] \cap [\text{dom } K_1 \cup \text{dom } (K_{23} - \text{dom } K_1)]$$

$$\subseteq [K_1 \cup (K_{23} - \text{dom } K_1)] \cap [\text{dom } K_1 \cup \text{dom } K_{23}].$$

Using the distributive law:

$$K \cap \text{dom } K \subseteq [K_1 \cap \text{dom } K_1] \cup [K_1 \cap \text{dom } K_{23}] \cup \phi \cup [(K_{23} - \text{dom } K_1) \cap \text{dom } K_{23}].$$

But all three terms are empty, by lemmata 1, 4, 5. Q.E.D.

Define the set H:

$$H = K_{23} \cap \text{dom } K_1,$$

the "hole" cut out of K_{23} by the dominion of K_1.

<u>Lemma 7</u>. For i = 1, 2, 3,

$$S_i \text{dom } H \subseteq \text{dom } K_i.$$

<u>Proof 7</u>. For i = 1, we have

$$\begin{aligned} S_1 \text{dom } [K_{23} \cap \text{dom } K_1] &\subseteq S_1 \text{dom } [A \cap \text{dom } K_1] \\ &\subseteq S_1 \text{dom } [S_1 \text{ dom } K_1] \\ &\subseteq S_1 \text{dom } K_1 \\ &\subseteq \text{dom } K_1, \end{aligned}$$

making use in the third line of the transitivity of S_1-domination, and in the second line of the fact that $A \cap \text{dom } K_1 = A \cap S_1 \text{dom } K_1$. For i = 2 or 3, we have in easy steps:

$$S_i \text{dom } [K_{23} \cap \text{dom } K_1] \subseteq S_i \text{dom } K_{23} = S_i \text{dom } K_i \subseteq \text{dom } K_i.$$

<u>Lemma 8</u>. dom K = dom $K_1 \cup$ dom K_{23}.

<u>Proof 8</u>. By lemmata 2 and 3 and the definition of K we see that K contains K_2 and K_3 as well as K_1. Hence, by lemma 7,

$$\text{dom } H \subseteq \text{dom } K.$$

But $K_1 \cup K_{23} = K \cup H$. Applying "dom" to both sides gives us

$$\text{dom } K_1 \cup \text{dom } K_{23} = \text{dom } K \cup \text{dom } H$$
$$= \text{dom } K. \quad \text{Q.E.D.}$$

The next lemma is crucial to the construction. It says that K_{23} dominates all of its complement $A - K_{23}$ except for the set $K_1 - K_{23}$ (congruent to the "arbitrary" set C).

Lemma 9. $K_1 \cup K_{23} \cup \text{dom } K_{23} \supseteq A$.

Proof 9. Fix a point $u \in U$, with $\sum u_j < 1$. Denote by A(u) the triangular (2-dimensional) cross-section of A:

A(u): points of the form

$$(\alpha_1, \alpha_2, \alpha_3, u_4, \cdots, u_n).$$

Define

$$K_{23}(u) = A(u) \cap K_{23};$$

this is a line segment parallel to the face $\alpha_1 = 0$ of A(u), and not more than half way to the opposite vertex (see figure). Define L(u) as the set of points in A(u) which would be dominated by $K_{23}(u)$ if the coalitions $\{1,2\}$, $\{1,3\}$, $\{2,3\}$ were effective.* We then have a situation analogous to the three-person constant-sum game.

A(u):

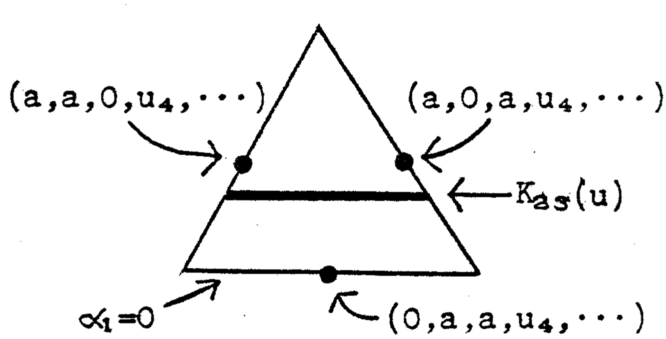

Figure 2

* Thus, $\beta \in L(u)$ if and only if $\beta \in A(u)$ and, for some $\alpha \in K_{23}(u)$, either (1) $\alpha_1 > \beta_1$ and $\alpha_2 > \beta_2$ or (2) $\alpha_1 > \beta_1$ and $\alpha_3 > \beta_3$ or (3) $\alpha_2 > \beta_2$ and $\alpha_3 > \beta_3$.

The sets $K_{23}(u)$ and $L(u)$ are disjoint. Let α be an arbitrary point in $A(u)$. We shall establish that

(i) if $\alpha \in K_{23}(u)$, then $\alpha \in K_{23}$,

(ii) if $\alpha \in L(u)$, then $\alpha \in \text{dom } K_{23}$,

(iii) if α is in neither $K_{23}(u)$ nor $L(u)$, then $\alpha \in K_1$.

The first is immediate. In case (ii), there is a point β in $K_{23}(u)$ which would dominate α but for the equalities $\beta_4 = \alpha_4, \cdots, \beta_n = \alpha_n$. By the continuity of the function ρ, it is clear from the definition of K_{23} (page 4) that there will be a point β' in $K_{23}(u')$ that dominates α, if u' is a vector near to, but majorizing, the vector u. That such a vector always exists follows from our constraint $\sum u_j < 1$. Hence, α is in dom K_{23}. Case (iii) is vacuous unless the points in $K_{23}(u)$ satisfy the equation

$$\gamma_1 = \gamma_2 + \gamma_3$$

(the line joining the midpoints of the triangle in the figure). But this implies that $\rho(u, C') = 0$ (see definition of K_{23}) and $\alpha = (0, a, a, u_4, \cdots, u_n)$ (see figure). Hence α is in K_1. The proof of the lemma is completed with the observation that the totality of cross-sections $A(u)$ considered comprise all the points of A with the exception of those α with $\alpha_1 = \alpha_2 = \alpha_3 = 0$. But these points are included in both K_1 and K_{23}.

<u>Lemma 10</u>. $K \cup \text{dom } K \supseteq A$.

<u>Proof 10</u>. By lemma 8:

$$K \cup \text{dom } K = K_1 \cup [K_{23} - \text{dom } K_1] \cup \text{dom } K_1 \cup \text{dom } K_{23}.$$

This is equal to $K_1 \cup K_{23} \cup \text{dom } K_1 \cup \text{dom } K_{23}$, which, by lemma 9, contains A.

Theorem. $K = A - \text{dom } K$.

Proof. By lemmata 6 and 10.

Remark. It is not difficult to show that the component of the solution lying within K_{23} is connected, though perhaps not simply connected. The rest of the solution, of course, has arbitrary connectivity properties.

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