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RESEARCH MEMORANDUM

NOTE ON DUELS WITH CONTINUOUS FIRING

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1. Statement of result

In many duel situations the firing may be regarded as a continuous process, with "kills" determined by a probability—density function proportional to the instantaneous rate of fire, the accuracy of fire, and a factor describing the vulnerability of the target. The most immediate example of this type is an engagement between aircraft using machine—guns.

The most outstanding difficulty in solving such duel games is the complex nature of the strategies. Even in the simplest cases, when the guns are "silent", the strategies generally form infinite-dimensional sets of functions. The prospect of having to consider mixed strategies — distribution functions defined over these sets — is not inviting. The purpose of the present note is to dispel that prospect. Under quite general conditions, mixed strategies never need be introduced into the problem. Specifically:

> In a continuous—fire, silent duel, every mixed strategy is dominated by a pure strategy, provided only that the duellists are permitted to vary their rate of fire along a continuous scale between zero and the maximum rate.

As might be expected, the result does not apply to elements of strategy other than actual firing plans (for example, target selection), which may appear in a particular model.

The assumption of a continuously variable rate of fire may be unnatural in many practical situations. On the other hand, mixed strategies may be equally hard to justify. Machine—guns which can alter their firing rate are not in general use, but neither are weapons with built—in adjustable randomizers suitable for generating mixed strategies. Moreover, a discrete variation in firing rate may often be easy to achieve, as for example on an aircraft carrying several, separately controlled machine-guns.

2. Proof

The proof is constructive. Given any mixed strategy, F, we exhibit a pure strategy R_F which gives better results. The intuitive relation between the two is simple and direct: the changing probabilities of fire appearing in F correspond to variations in the rate of fire as specified by R_F.

The instantaneous probability of kill, at time t, may be expressed by

$$R(t)\ A(t)\ dt,$$

where A(t) is the combined accuracy-vulnerability factor and R(t) is the rate of fire. The target will survive through time t with probability

$$P(R,t) = \exp\left\{- \int_0^t A(u)\ R(u)\ du\right\}. \tag{1}$$

After some obvious normalization, we may put the restrictions on R as follows:

$$\begin{aligned} &0 \leq R(t) \leq 1 && (\text{all } t, \quad 0 \leq t \leq 1) \\ &\int_0^1 R(t)dt = \beta && (\text{fixed } \beta, \ \beta < 1). \end{aligned} \tag{2}$$

If S is the set of all R satisfying (2), we may write a mixed strategy as F(R), where

$$\int_S dF(R) = 1.$$

The target, confronted with F, will survive through time t with probability

$$\phi(F,t) = \int_S \exp\left\{-\int_0^t A(u)\ R(u)du\right\} dF(R). \tag{3}$$

Given any F, we define

$$R_F(t) = \int_S R(t)\,dF(R).$$

Conditions (2) are evidently satisfied, therefore $R_F \in S$. Then

$$(4) \qquad P(R_F,\ t) = \exp\left\{-\int_0^t A(u) \int_S R(u)\ dF(R)du\right\}.$$

The boundedness of the functions permits us to reverse the order of integration in (4). A comparison of (3) and (4) then reveals

$$\phi(F,t) = \int_S \exp\psi\ (R,t)dF(R)$$

$$P(R_F,t) = \exp\int_S \psi\ (R,t)dF(R).$$

Since exponentiation is a convex functions, we have

$$(5) \qquad \phi(F,t) \geq P(R_F,t).$$

What we have been calling strategies up to now are actually contingent on the survival of their user. He may be killed or forced to curtail his fire at some time before t = 1. But the fact that (5) holds for all t makes R_F preferable to F independently of the defensive fire.

3. Remarks

1. It is generally assumed in duels that the value of a kill, in the ultimate pay—off, is independent of the time at which the kill is made. The foregoing result remains valid if the kill bounty is a non—increasing function of time. Tactically, of course, an earlier kill is always worth more than (or as much as) a later kill.

2. If the firing is not completely silent — i.e., if the players obtain information of their opponent's plans in time to change their own — then the strategies are more difficult to describe.

Generally speaking, a game with more information offers less scope for mixed strategies. Hence it seems likely that dominance of pure strategies holds in noisy as well as silent duels.

3. The constructive proof given above could be replaced by a shorter, direct proof. After defining an appropriate metric in S, it can be shown that S is convex and that P(R,t) is a convex function of R, from which the dominance of pure strategies follows immediately.