U. S. AIR FORCE

PROJECT RAND

RESEARCH MEMORANDUM

A THREE-MOVE GAME
WITH IMPERFECT COMMUNICATION

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<u>The game</u>

In this game, the y-player makes the first move, the x-player the second and third moves. The information function* is

$$i(1) = \Lambda$$
$$i(2) = \{1\}$$
$$i(3) = \{2\} .$$

Thus, when it comes to the third move, the x-player has "forgotten" his opponent's move itself, and remembers only his reaction to it. The second move is therefore important as a signal, over and above its effect in the pay-off function.

The moves themselves consist of choices as follows:

$$\left.\begin{array}{ll} 1. & \text{a real number } z_1 \\ 2. & \text{an integer } z_2 \\ 3. & \text{a real number } z_3 \end{array}\right\} \text{ all } 0 \leq z \leq 1 .$$

(Note that if a continuum of choices were permitted in the second move, too much information about the first move could be transmitted.)

The pay-off as a function of the move-choices is

$$H(z_1, z_2, z_3) = z_2/4 - |z_1 - z_3| .$$

* The definition and properties of the information function will be treated in a forthcoming paper by J. C. C. McKinsey.

We denote the strategies by

$$x = \langle f, g \rangle$$
$$y = \langle z_1 \rangle$$

where f maps the unit interval on the two points 0 and 1; g, the two points back into the interval. Then, as a function of the strategy-choices, the pay-off is

$$M(x, y) = f(z_1)/4 - |z_1 - g(f(z_1))| .$$

<u>Solution for the x-player</u>

The following hunches, <u>prima facie</u> reasonable, are justified by the result they lead to.

a) The function f involved in an optimal strategy will have just one discontinuity — that is, for example, it will be zero for z less than ζ, one for z greater than ζ.

b) The signal "1" will be used more extensively than "0" — that is, $\zeta < 1/2$.

c) The optimal mixed strategy will be an equal mixture of a pair of pure strategies symmetrical on the unit interval.

These assumptions dispose of all but a three-parameter family of strategies, as follows:

$$F^* = (I_{x_1} + I_{x_2})/2 ;$$

$$x_1 = \langle f_1, g_1 \rangle , \quad x_2 = \langle f_2, g_2 \rangle ;$$

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$$f_2(z) = f_1(1 - z), \quad g_2(z) = 1 - g_1(z);$$

$$f_1(z) = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \text{ if } \begin{Bmatrix} 0 \leq z \leq \zeta \\ \zeta < z < 1 \end{Bmatrix} \qquad (\zeta < 1/2);$$

$$g_1(0) = \alpha, \quad g_1(1) = \beta \qquad (0 \leq \alpha \leq \beta \leq 1).$$

Let

$$K^*(y) = \int M(x, y)dF^*(x) = \phi(z_1, \alpha, \beta, \zeta).$$

Then the problem is to find the values of α, β, ζ at which

$$\min_{z_1} \phi\ (z_1, \alpha, \beta, \zeta)$$

has its maximum. Since ϕ is piecewise linear, the calculation becomes a simple though lengthy enumeration of cases. The maximum turns out to have the value $\phi = 0$ and it is reached at (uniquely):

$$\alpha = 0, \quad \beta = 3/4, \quad \zeta = 1/4.$$

$\phi(z_1, 0, 3/4, 1/4)$ is identically zero in z_1. Thus the solution may be written:

$$F^*\big(f(z), g(z)\big) = \tfrac{1}{2} I_{I_{1/4}(z), 3z/4}\big(f(z), g(z)\big) + \tfrac{1}{2} I_{I_{1/4}(1-z), 1-3z/4}\big(f(z), g(z)\big).$$

Solution for the y-player

Because of the identity just mentioned, the x-solution gives no clue to the optimal y-strategy. However it is clear that the latter must involve more than two values of z_1. Thus we proceed on the hunch:

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d) There is an optimal y—strategy of the form:

$$G^* = \xi I_\mu + (1/2 - \xi) I_\nu + (1/2 - \xi) I_{1-\nu} + \xi I_{1-\mu};$$

where

$$0 < \xi < 1/2, \quad 0 \leq \mu < \nu \leq 1/2 .$$

As before, let

$$H^*(x) = \int M(x, y) dG^*(y) = \Theta(f, g, \xi, \mu, \nu).$$

The problem is then to find the values of ξ, μ, ν at which

$$\max_{f,g} \Theta(f, g, \xi, \mu, \nu)$$

reaches its minimum. There are sixteen signal functions f to choose among, to wit:

$$f(\mu) = 0, 1$$

$$f(\nu) = 0, 1$$

$$f(1 - \nu) = 0, 1$$

$$f(1 - \mu) = 0, 1.$$

Symmetry and dominance reduce these to three:

	μ	ν	$1 - \nu$	$1 - \mu$
f_1	0	0	1	1
f_2	0	1	1	1
f_3	1	1	1	1

In each case the best choice of $g(0)$ and $g(1)$ is easily found as a function of the three parameters. After another straightforward but tedious enumeration, the following result emerges:

$$\max_{f,g} \theta = 0 \quad \text{at} \quad \begin{Bmatrix} \mu = 0 \\ \nu = 1/4 \\ \xi = 1/4 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \mu = 0 \\ \nu = 3/8 \\ \xi = 1/3 \end{Bmatrix};$$

$$\max_{f,g} \theta > 0 \quad \text{everywhere else.}$$

Thus, a family of y-solutions is represented by

$$G^* = \left(\frac{1}{4} + \frac{t}{12}\right)\left[I_0 + I_1\right] + \left(\frac{1}{4} - \frac{t}{4}\right)\left[I_{\frac{1}{4}} + I_{\frac{3}{4}}\right] + \frac{t}{6}\left[I_{\frac{3}{8}} + I_{\frac{5}{8}}\right].$$

as t varies from zero to one.

There are some grounds for believing that all optimal step-functional strategies are covered in the above expression.

<u>A slight generalization</u>

Consider the family of games

$$M(x, y) = pf(z_1) - |z_1 - g(f(z_1))|, \quad (0 < p < 1).$$

It may be directly verified that

$$\begin{cases} \alpha = 0 \\ \beta = (2 + p)/3 \\ \zeta = (1 - p)/3 \end{cases}$$

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determines the only optimal F^* of the type described on page 3; and that

$$\begin{cases} \mu = 0 \\ \nu = (1 - p)/3 \\ \xi = 1/4 \end{cases} \qquad \begin{cases} \mu = 0 \\ \nu = (1 - p)/2 \\ \xi = 1/3 \end{cases}$$

determine optimal G^* of the type described on page 5. The value of the game is $(4p - 1)/6$. Thus the zero value obtained above was accidental.

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