A SYMMETRIC MARKET GAME

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SUMMARY

In Part I a family of solutions to the \((m+n)\)-person game

\[ v(S) = \min(|S \cap M|, |S \cap N|), \quad M \cap N = \emptyset, \]

is described, and various significant properties of the general solutions are obtained. In Part II (which is self-contained), the game is interpreted heuristically as a symmetrical market.
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1. Introduction

The game which we solve in this note is defined by the characteristic function

\[(1) \quad v(S) = \min(|S \cap M|, |S \cap N|), \quad M \cap N = \emptyset,\]

M and N being the sets of buyers and sellers, respectively. As the number of players of each type is unrestricted, the results obtained represent a new chain of outposts extending into the wilderness of games with large numbers of players. In Part I we give as complete an account of the stable-set solutions (in the sense of the von Neumann-Morgenstern theory) as we are presently able; that is, we exhibit a large and reasonably well-behaved family of solutions and obtain a number of strong restrictions on the nature of any other solutions that might exist. The techniques, and to a lesser extent the results, of this part we hope to extend to a wider class of games in subsequent notes.

Since an eventual aim of these investigations is to gain insight into the economic processes which inspired the definition of "market game,"\(^1\) we have added a second part, describing the game and its solutions in non-mathematical, non-game-theoretic

\(^1\)For a general description of market games, see reference [1].
language. The casual or uninitiated reader is welcome to skip directly to Part II, since it is short and self-contained. However, we do not intend it as either a summary or a popularization, but as an exercise in the art of translating theorems about n-person games into statements about the practical world. Essays in this vein have been all too rare since the publication of *Theory of Games and Economic Behavior*, and have lagged far behind the growth of the mathematical theory.
PART I

2. Notation and Basic Definitions

Let $m = |M|$, $n = |N|$. Vectors on $M \cup N$ will be written:

$$x = \langle x', x'' \rangle = (x'_1, \ldots, x'_m; x''_1, \ldots, x''_n).$$

Sums over the components of such vectors will be abbreviated:

$$x(S) \quad \text{for} \quad \sum_{S \in M} x'_\mu + \sum_{S \in N} x''_\nu.$$

The imputations of the game are the nonnegative vectors that satisfy $x(M \cup N) = g$, where $g$—called the "modulus"—is equal to $\min (m, n)$. The imputation space, a simplex of $m+n-1$ dimensions, is denoted by $A$. The "face" $A_S$ of $A$ is the set of imputations $x$ with $x(S) = g$; that is, the imputations that give nothing to players outside $S$.

We say that "$x$ dominates $y$ via $S$" if (i) $x - y$ is strictly positive on $S$, and if (ii) $x(S) \leq v(S)$. The dominion of a set $X$ of imputations, written "dom X," is the (open) set of all imputations dominated by elements of $X$. The set of undominated imputations is called the core of the game. A solution is defined to be any set $V$ that dominates precisely its complement: $V = A - \text{dom } V$. Every solution is closed, and contains the core; no solution contains another. The two properties $V \cap \text{dom } V = \emptyset$ and $V \cup \text{dom } V = A$, that combine to characterize a solution, will be referred to, respectively, as "internal" and "external" stability.
3. **Two Simple Cases**

Let us first dispose of the case \( m = n = g \). Let \( V \) be the set of imputations of the form

\[
(2) \quad z_p = (p, p, \ldots, p; 1-p, 1-p, \ldots, 1-p), \quad 0 \leq p \leq 1.
\]

We shall show that \( V \) is the core. For every \( S, p \), we have

\[
z_p(S) = p|S \cap M| + (1-p)|S \cap N|
\geq \min(|S \cap M|, |S \cap N|)
= v(S).
\]

Hence \( z_p \) is not dominated since condition (ii) could never be satisfied. On the other hand, any \( x \) not of form (2) must have \( x'_\mu + x''_\nu < 1 \) for some \( \mu, \nu \). Choose \( p \) between \( x'_\mu \) and 1 - \( x''_\nu \). Then \( x \) is dominated by \( z_p \) via \( S = \{\mu, \nu\} \). Hence \( V \) is precisely the core. Moreover, since \( V \) dominates its complement in \( A \), it is a solution, and the only one.

Geometrically, of course, this unique solution is the straight line joining the centroids of the two faces \( A_M \) and \( A_N \). In later cases we shall encounter more general curves running between \( A_M \) and \( A_N \), with only one end "anchored" at the centroid. Indeed, all known solutions are of this general type.

To illustrate, before continuing to the general case, consider the case \( m = 1, n \geq 2 \). This is a well-known "simple" game, whose winning coalitions are those consisting of the single buyer and any one or more sellers.\(^2\) Here \( g = 1 \), and the

\(^2\) See [2]. The game is the product of the one-person game \( B_1 \) and the \( n \)-person pseudogame \( P_n^\alpha \) in which every non-vacuous coalition wins.
core is the lone, undominated imputation \((1; 0,0,...,0)\). Unlike the preceding case, the core here fails to dominate the rest of \(A\); in fact, it dominates nothing at all. The solutions turn out to be monotonic curves running from the core to the opposite face of the simplex. Stated precisely, a solution is any set of points of the form

\[ z_p = (p; f_1(p),...,f_n(p)), \quad 0 \leq p \leq 1, \]

where the functions \(f_x\) are continuous, nonnegative, and nonincreasing, with \(\sum f_x(p) = 1-p\).

The verification that these are solutions is relatively easy, but the proof that there are no other solutions is not. (See Corollary 3, below.)

The special case \(n = 2\) is of course covered by von Neumann's complete analysis of the three-person game;³ we illustrate two typical solutions below. The shaded areas represent dominions

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³ See [3], especially Sections 60.3 (p. 550ff) and 47.5 (p. 409ff). His methods are easily extended to larger values of \(n\).
of typical points; they make it easy to see why the condition of monotonicity is necessary to preserve internal stability.

4. The General Case

We now take up the general case; \( m \) and \( n \) unrestricted. Let there be \( m+n \) monotonic functions, \( f_1'(p), \ldots, f_m'(p), \)
\( f_1''(p), \ldots, f_n''(p), \) nonnegative on \( 0 \leq p \leq 1, \) satisfying

\[
\sum_{M} f'_M(p) = pg, \quad \sum_{N} f''_N(p) = (1-p)g.
\]

(That they are continuous follows from the other conditions.) Let \( V_f \) denote the set of imputations of the form \( < f'(p), f''(p) >, \) \( 0 \leq p \leq 1. \) Geometrically, \( V_f \) is an arbitrary monotonic curve joining the two faces \( A_M \) and \( A_N. \)

Let \( E \) denote the subset of \( A \) defined by the \( m+n \) inequalities:

\[
x'_\mu + x''_\nu \leq 1, \quad \text{all } \mu, \nu.
\]

Geometrically, \( E \) is a nonempty, closed, convex set which intersects both \( A_M \) and \( A_N; \) one of them in a single point. \( E \) contains at least one \( V_f \) curve, namely the chord joining the centroids of \( A_M \) and \( A_N. \)

THEOREM 1. Every curve \( V_f \) contained in \( E \) is a solution of the game.

Remark. The special cases already mentioned are both included in this result. Specifically, if \( m = n \) then \( E \) is the core, and is itself a solution of the form \( V_f. \) Also, if \( m = 1 \) then \( E = A, \) and all monotonic curves \( V_f \) are solutions.
Proof. For external stability, choose any \( x \) in \( A \). Define \( p_1 \) to be the largest \( p \) such that
\[
(5) \quad f'_{\mu}(p) \leq x
\]
holds for all \( \mu \), and \( p_2 \) to be the smallest \( p \) such that
\[
(6) \quad f''_{\mu}(p) \leq x
\]
holds for all \( \mu \). The argument splits up into two cases.

A: Suppose \( p_1 < p_2 \). Let \( p_1 < p^* < p_2 \). Then for some \( \mu^* \), \( \nu^* \) we will have
\[
f'_{\mu^*}(p^*) > x_{\mu^*} \quad \text{and} \quad f''_{\nu^*}(p^*) > x_{\nu^*}.
\]
With the aid of (4) we see that the imputation \( z_{p^*} \) dominates \( x \) via \( \{\mu^*, \nu^*\} \). B: Suppose \( p_1 \geq p_2 \). Inserting \( p_1 \) and \( p_2 \) for \( p \) in (5) and (6) respectively, and adding all \( m+n \) inequalities, we obtain
\[
p_1 g + (1-p_2) g \leq x(M \cup N) = g,
\]
using (3). This can only be true if \( p_1 = p_2 \) and if equality prevails throughout (5) and (6). But then \( x = \langle f'(p_1), f''(p_2) \rangle = z_{p_1} = z_{p_2} \). This completes the proof of external stability, since we have shown that every imputation is either in \( V_f \) or in its dominion. As for internal stability, the contrary directions of the two groups of functions \( \{f'_\mu\} \) and \( \{f''_\mu\} \) ensure that there is no domination within \( V_f \), since all domination in this game occurs via sets which contain both a \( \mu \) and a \( \nu \). This completes the proof of Theorem 1.

* In fact, the only coalitions which we shall have to consider for purposes of domination are those consisting of one buyer and one seller, since the others are not "vital" (in the sense of Gillies [4], p.12).
We remarked earlier that $E$ touches either $A_M$ or $A_N$ in a single point, thus pinning down one end of the solutions of Theorem 1. The reason for this is apparent if we observe that the core of the game is precisely the imputation \((1, \ldots, 1; 0, \ldots, 0)\) if $m < n$, \((0, \ldots, 0; 1, \ldots, 1)\) if $m > n$, and recall that every solution must contain the core. In the case $m = n$, as we saw in Section 3, the core comprises the entire segment spanned by those two imputations, and gives rise to the unique solution.

It would be pleasant to be able to state that the "$E$" condition of Theorem 1 is necessary as well as sufficient. Unfortunately this is not the case. We give below two examples of monotonic curves for the case $m = 2$, $n = 3$. Both curves go outside $E$; one is a solution, one is not. The formulation of precise conditions under which monotonic curves are solutions appears to be a difficult matter.\(^5\)

Let $V_1$ and $V_2$ be the polygons with vertices:

\[
\begin{align*}
(1, 1; 0, 0, 0) & \quad (1, 1; 0, 0, 0) \\
(3/4, 3/4; 1/6, 1/6, 1/6) & \quad (2/3, 2/3; 1/3, 1/6, 1/6) \\
(3/4, 1/4; 1/2, 1/4, 1/4) & \quad (2/3, 1/3; 2/3, 1/6, 1/6) \\
(1/4, 1/4; 1/2, 1/2, 1/2) & \quad (1/3, 1/3; 2/3, 1/3, 1/3) \\
(0, 0; 2/3, 2/3, 2/3) & \quad (0, 0; 1, 1/2, 1/2)
\end{align*}
\]

connected in the order given. (The third vertex of each is outside $E$.) Then $V_1$ is a solution of the game (we omit the proof, which is not difficult), but $V_2$ is not, since it leaves undominated.

\(^5\) It can be shown that if $E'$ is the union of $E$ and any monotonic curve which goes outside $E$, then $E'$ contains a monotonic curve which is not a solution.
the imputation \((1/3, 2/3; 1/3, 1/3, 1/3)\), among others. Moreover, \(V_2\) cannot be extended to a solution.

It is also natural to ask whether there are ever any solutions which are not monotonic curves. In the next section we give some results tending to show that there are none, but the question is not completely settled except for the two simple cases already discussed, namely \(m = n\) and \(g = 1\).

5. **Further Results**

Two imputations \(x\) and \(y\) will be said to be **skew** if either \(x' \geq y'\) and \(x'' \leq y''\), or \(x' \leq y'\) and \(x'' \geq y''\). A set of imputations is skew if and only if every two-element subset is skew. It is clear that domination cannot occur among elements of a skew set. A monotonic curve, in the sense of the preceding section, is an obvious example of a skew set.

**THEOREM 2.** If a solution is skew, then it is a continuous, monotonic curve connecting \(A_M\) and \(A_N\).

Proof. Let \(V\) be a skew solution, and consider the mapping \(x \rightarrow x(M)/g\) from \(V\) into the unit interval. The skewness ensures that this mapping is 1–1; let \(B\) denote the image of \(V\) and let \(f\) denote the inverse mapping of \(B\) onto \(V\). If \(B\) comprises the whole unit interval, then it is easy to show that \(V\) is a continuous curve of the required type. Suppose then that \(B\) is not the whole interval. Since \(B\) is closed and not void, we can find an open interval \((p_1, p_2)\) not intersecting \(B\), such that either \((i)\) \(p_1\) and \(p_2\) are in \(B\); \((ii)\) \(p_1 = 0\),
and is not in B, and \( p_2 \) is in B; or (iii) \( p_1 \) is in B and \( p_2 = 1 \), and is not in B. In case (i) the imputation in the center of the gap: \( y = (f(p_1) + f(p_2))/2 \), is not in \( V \), and not dominated by \( V \), contrary to the assumption that \( V \) was a solution. In case (ii) a similar conclusion holds for the imputation \( z \) defined by \( z' = 0, z'' = f''(p_2) + p_2e'' \), where \( e'' \) is any non-negative vector on \( N \) with \( e''(N) = 1 \). Case (iii) is like case (ii). This completes the proof.

All known solutions are skew. Some light on the nature of possible non-skew solutions is given by Theorem 3, which we shall prove with the aid of the following lemma.

**LEMMA.** Let \( V \) be any solution and let \( x \) and \( y \) be any elements of \( V \). Define the vectors \( a \) and \( b \) by \( a_i = \max(x_i, y_i) \), \( b_i = \min(x_i, y_i) \), where \( i \) ranges over \( M \cup N \). Then the vectors \( < a', b'' > \) and \( < b', a'' > \) are imputations, and belong to \( V \).

(We note that this lemma is trivial unless \( x \) and \( y \) are not skew.)

**Proof.** Our first object is to prove that \( a(M) + b(N) \geq g \). We therefore suppose the contrary, and select a strictly positive vector \( c \) with \( c(M \cup N) = g - a(M) - b(N) \). Then the vector \( w = w(c) = < a', b'' > + c \) is an imputation. By construction, any imputation dominating \( w \) also dominates either \( x \) or \( y \). Hence, \( w \) is in \( V \). If \( w_{\mu*}' \) and \( w_{\mu*}'' \) are minimum components of \( w' \) and \( w'' \), respectively, then clearly

\[
w(M \cup N) \geq mw_{\mu*}' + nw_{\mu*}'' .\]

Hence we have
\[ g \geq g(w^1_{\mu^*} + w^2_{\nu^*}), \]
showing that \( w^1_{\mu^*} + w^2_{\nu^*} \leq 1 \), and making \( w \) available for domination via \( \{\mu^*, \nu^*\} \). Now start again with a strictly positive \( c^* \), with \( c^*(M \cup N) = c(M \cup N) \), and such that \( c^*_{\mu^*} < c'_{\mu^*} \) and \( c^*_{\nu^*} < c'_{\nu^*} \). (This is possible except in the trivial case \( m = n = 1 \).) Define \( w^* = w(c^*) \). Then \( w^* \) is in \( V \) for the same reason as \( w \). But \( w \) dominates \( w^* \) via \( \{\mu^*, \nu^*\} \), although both are in \( V \). This is the desired contradiction. Therefore we have established that
\[ (7) \quad a(M) + b(N) \geq g. \]

Similarly,
\[ (8) \quad b(M) + a(N) \geq g. \]
Adding (7) and (8) and making use of the relation \( a + b = x + y \), we obtain the expression \( 2g \geq 2g \). Therefore, equality must hold in both (7) and (8), and we see that \( \langle a', b'' \rangle \) and \( \langle b', a'' \rangle \) are imputations. Moreover, they are both in \( V \), since anything which dominates either one of them also dominates at least one of \( x, y \).

**COROLLARY 1.** If \( x \) and \( y \) are any two elements of a solution then
\[ (9) \quad \sum_{\mu} |x^1_{\mu} - y^1_{\mu}| = \sum_{\nu} |x^2_{\nu} - y^2_{\nu}|. \]

Proof. Observe that \( |x^1_1 - y^1_1| = a^1_1 - b^1_1 \). Thus (9) becomes \( a(M) - b(M) = a(N) - b(N) \), which is an immediate consequence of \( a(M) + b(N) = b(M) + a(N) = g \), established above.
COROLLARY 2. If $x$ and $y$ are any two elements of a solution and if $x' \geq y'$, then $x'' \leq y''$ and the two points are skew.

Proof. This is obvious from the fact that if $a' = x'$ then $b'' = x''$.

THEOREM 3. If two elements of a solution are not skew, then neither of them is in $E$.

Proof. By Corollary 2 above, non-skewness of a pair $x, y$ of elements of a solution entails that there are players $\mu_1, \mu_2, \nu_1, \nu_2$, such that

$$x'_{\mu_1} < y'_{\mu_1} \quad \text{and} \quad x'_{\mu_2} > y'_{\mu_2},$$

and

$$x''_{\nu_1} < y''_{\nu_1} \quad \text{and} \quad x''_{\nu_2} > y''_{\nu_2}.$$ 

But the only way to avoid having $x$ and $y$ dominate each other is to have $x'_{\mu_2} + x''_{\nu_2} > 1$ and $y'_{\mu_1} + y''_{\nu_1} > 1$. Therefore, neither $x$ nor $y$ is in $E$.

THEOREM 4. All solutions contained in $E$ are monotonic curves connecting $A_M$ and $A_N$.

Proof. By Theorems 2 and 3.

COROLLARY 3. If $g = 1$ then all solutions are given by Theorem 1.

Proof. By Theorem 4, since in this case $E = A$. 
It is easily verified that the \( V_r \) solutions of Theorem 1 sweep out the entire set \( E \). We now describe a larger set, in which all solutions must lie. Let \( F \) be the set of imputations \( x \) such that

\[
\begin{align*}
\max_{\alpha} x_{\alpha}^1 + \min_{\beta} x_{\beta}^2 & \leq 1, \\
\min_{\alpha} x_{\alpha}^1 + \max_{\beta} x_{\beta}^2 & \leq 1.
\end{align*}
\]

Clearly \( F \) contains \( E \).

**THEOREM 5.** Every solution is contained in \( F \).

Proof. Let \( V \) be any solution, \( x \) any element of \( V \), and \( \mu^* \) any player such that \( x_{\mu^*}^1 = \max_{\alpha} x_{\alpha}^1 \). Suppose that

\[
x_{\mu^*}^1 + x_{\mu^*}^2 > 1 \quad \text{for all } \mu.
\]

The case \( x_{\mu^*}^1 = 0 \) leads to the immediate absurdity: \( x(N) > n \geq g \). If \( x_{\mu^*}^1 \) is positive we can find an imputation \( z \) which majorizes \( x \) in all components except \( x_{\mu^*}^1 \), and so close to \( x \) that the inequalities (10) hold for \( z \) as well as \( x \). There must be some pair \( \mu_1, \nu_1 \) such that \( z_{\mu_1}^1 + z_{\nu_1}^2 \leq 1 \), for otherwise we could add up \( g \) inequalities of the form \( z_{\mu}^1 + z_{\nu}^2 > 1 \), involving \( 2g \) distinct players, and obtain the absurdity \( z(S) > g \). Hence \( z \) dominates \( x \) via \( \mu_1, \nu_1 \). Hence \( z \) is not in \( V \). Hence some \( y \) in \( V \) dominates \( z \). In view of (10), this latter domination must be via a coalition not containing \( \mu^* \). Hence \( y \) dominates \( x \) as well, since \( x \) is majorized by \( z \) except for \( x_{\mu^*}^1 \). This contradiction refutes the hypothesis (10), and establishes \( \max_{\alpha} x_{\alpha}^1 + \min_{\beta} x_{\beta}^2 \leq 1 \). By symmetry \( \min_{\alpha} x_{\alpha}^1 + \max_{\beta} x_{\beta}^2 \leq 1 \). This completes the proof.
COROLLARY 4. No component of any imputation of any solution exceeds 1.

In attempting to prove (or disprove) that all solutions are monotonic curves joining \( A_M \) and \( A_N \), we find it convenient to speak in terms of the "price level" \( p = x(M)/g \) associated with each imputation \( x \). (Compare the definition of \( V_p \), also the proof of Theorem 2.) One would like to show that for every level, in a given solution, there is one and only one imputation. Since distinct points on the same level are necessarily non-skew, Theorem 3 tells us that multiple points can occur only outside \( E \). At present, we are unable to say more on the question of multiplicity.

Similarly, on the question of vacant levels, we are not able to prove that they can never occur, but only that at least one of the endpoints of any gap is outside \( E \). This result is stated precisely in the following theorem, in which we have had to distinguish between gaps containing an extreme value, \( p = 0 \) or 1, and gaps in the middle of the price range of a solution.

THEOREM 7. (a) If a solution \( V \) contains an element \( u \) with \( u(M) > 0 \), such that \( x(M) < u(M) \) holds for no \( x \) in \( V \), then \( u \) is outside \( E \).

(b) If a solution \( V \) contains an element \( w \) with \( w(M) < g \), such that \( w(M) < x(M) \) holds for no \( x \) in \( V \), then \( w \) is outside \( E \).

(c) If a solution \( V \) contains elements \( u \) and \( w \) with \( w(M) < u(M) \) such that \( w(M) < x(M) < u(M) \) holds for no \( x \) in \( V \), then at least one of \( u, w \) is outside \( E \).
Proof. (a) Pick $x$ in the gap and skew to $u$: $u' \geq x' = 0'$ (say) and $u'' \leq x''$. Since $x$ is not in $V$, it is dominated by some $z$ in $V$ via some $\{\mu, \nu\}$. Hence $z''_\nu > u''_\nu$. But $z(N) \leq u(N)$ by hypothesis, so that $z$ and $u$ cannot be skew. By Theorem 3, then, $z$ and $u$ are outside $E$.

(b) Pick $x$ in the gap and skew to $w$: $w' \leq x'$, $w'' \geq x'' = 0''$ (say), and proceed as in part (a).

(c) If $u$ and $w$ are not skew, then both are outside $E$, by Theorem 3. If $u$ and $w$ are skew, then we can choose $x$ in the gap and skew to both—say $x = (u+w)/2$—giving us $u' \geq x' \geq w'$ and $u'' \leq x'' \leq w''$. Since $x$ is not in $V$, it is dominated by some $z$ in $V$ via some $\{\mu, \nu\}$. Hence $z'_\mu > w'_\mu$ and $z''_\nu > u''_\nu$. By hypothesis, either $z(M) \leq w(M)$ or $z(N) \leq u(N)$. Hence $z$ is not skew to both $u$ and $w$. By Theorem 3, then, either $z$ and $u$, or $z$ and $w$, are outside $E$.

6. Symmetric Solutions

The definition of the game (1) does not distinguish among the individual buyers, or the individual sellers. Hence there are at least $m! \cdot n!$ automorphisms of the game, obtained by permuting the players, and twice that number if $m = n$. Obviously the set of solutions must be equally symmetric, although (as we have already seen) there may in general be many non-symmetric solutions. Even the symmetric solutions (if any) might be expected a priori to contain imputations that are not symmetric.\(^6\)

\(^6\) For example, in the completely symmetric 3-person simple majority game, the only symmetric solution consists of three non-symmetric imputations, while the only symmetric imputation occurs in three non-symmetric solutions.
However, the next theorem shows that this is not the case. In fact, when \( m \neq n \) the only symmetric solution is precisely the set of symmetric imputations.

THEOREM 8. The only solution which possesses the full symmetry of the game is the set of imputations of the form

\[(r, r, \ldots, r; s, s, \ldots, s)\]

—that is, the straight line joining the centroids of \( A_M \) and \( A_N \).

Proof.\(^7\) Let \( V \) be any symmetric solution and let \( x \) be a hypothetical non-symmetric point of \( V \). Without loss of generality, we may assume that the \( x'_\mu \) are not all equal. Let \( r \) be the average of the \( x'_\mu \) and \( s \) the average of the \( x''_\nu \). For some \( \nu_1, \nu_1' \) we will have \( x'_{\nu_1} < r \) and \( x''_{\nu_1} > s \). Both inequalities will be strict if we replace \( r \) by \( r' = r - \varepsilon/m \) and \( s \) by \( s' = s + \varepsilon/n \), with \( \varepsilon \) chosen suitably small and positive. The vector

\[y = (r', r', \ldots, r'; s', s', \ldots, s')\]

is a symmetric imputation, and dominates \( x \) via \( \{\nu_1, \nu_1'\} \).

Hence \( y \) is not in \( V \). Hence there is some imputation in \( V \) which dominates \( y \), via (say) \( \{\nu_2, \nu_2'\} \). By the symmetry of \( V \) and of \( y \), there will also be an imputation which dominates \( y \) via \( \{\nu_1, \nu_1'\} \). But this one will also dominate \( x \) via \( \{\nu_1, \nu_1'\} \)—a contradiction. We conclude that there are no non-symmetric points in \( V \). The rest of the proof is obvious.

\(^7\) It would be a simple matter to verify the theorem directly for monotone-curve solutions, i.e., for all known solutions. However, the proof must also cover the possibility of other types of solutions.
PART II

1. Description of the Market

There are in the market two kinds of traders: "buyers" and "sellers." A seller's aim is to dispose of a single item in his possession; for example, a house or an automobile. An external outlet is available to take care of any items that are not disposed of within the market, at a definite, fixed price. This establishes a basis for figuring each seller's profit.

In similar fashion, each buyer wants to acquire just one item; in other words, his demand is completely inelastic. External sources of supply, charging fixed prices, are also available, giving us the basis for determining buyers' profits as well. Thus, there is a definite profit on each transaction in the market—the difference between the two external prices—which is split up between the buyer and seller in a manner depending on the transmission price. In the present, symmetric model, we are assuming that these profits are all equal no matter who trades with whom.

However, we envisage that the traders may make side payments among themselves, in one form or another, while dickering over who shall sell to whom at what price. Thus the prices arrived at, in themselves, will not adequately summarize the outcome of the trading. By the outcome, we shall mean the final net profit that each trader realizes, based on the external prices. The price agreements, special arrangements, coalition
tactics and other bargaining maneuvers, that may have preceded this outcome, will not be considered in what follows, unless incidentally or by implication.

2. Description of the Solutions

Even though the traders of each type are indistinguishable (essentially) in the initial conditions, there is no good reason for assuming that their outcomes will also be indistinguishable. We may imagine differences due to imperfect or irregular communication, disparities in bargaining tactics, "irrational" habits or biases, or even chance. Disregarding the operation of such influences, we merely search for those outcomes which are stable, when compared with other, contemplated outcomes. There is no single, stable outcome—a "one-point solution"—dominating all others. But we can find a set of outcomes which are mutually compatible while excluding the rest; in fact, there will usually be many such sets. These sets of outcomes are what we shall call "solutions."

In the present market, each solution contains one free parameter, whose value cannot be determined from the initial conditions alone. (It presumably depends in a complex way on the details of bargaining, communication, habit, etc.) The parameter can be interpreted as a "price level," giving the relative status of the two camps: buyers and sellers. At one extreme the buyers pay top prices while the sellers take all the profits. At the other extreme, the positions are reversed.
When it is the minority group dividing the total profits, all members receive equal shares; when it is the majority group, the shares may be uneven, and differ from solution to solution, since the total profit is not enough to go around if each person gets the maximum.

The fact that every solution includes these extreme cases does not imply that they will ever occur in practice. It means merely that the extremes can never be excluded a priori from consideration by the market. In a particular solution we have a specification of how the profits would be distributed among the winners, in the event that cut-throat competition in the other camp (say) brings about an extreme case.

What is more important, a particular solution also specifies the distribution of profits associated with every intermediate value of the price-level parameter. It is the elucidation of this property, possessed by all known solutions of the market, that is the central feature of the mathematical section of the present paper. Going into more detail, we can describe a number of other characteristic properties of the solutions:

(1) The maximum number of transactions always take place. That is, the external market is used as little as possible.

(2) No trader's net profit ever exceeds the total profit of one transaction, even after side payments.
(3) A change in price level which increases the total profit to one camp will never reduce the profit of any member of that camp, though it may not increase every such individual's profit.

(4) No buyer's profit can exceed the profit of one transaction, less the profit of the least successful seller; and similarly for sellers' profits. Thus, disproportionately high gains are not stable.

3. An example

Suppose, for a numerical example, that there are 40 sellers and 50 buyers, and a uniform per-item profit of $100. Cut out any 10 of the buyers. Then there will be a solution in which the remaining buyers each make a profit of P, while the sellers get $100 - P apiece. Here P is undetermined in the range $0 to $100. Thus we have strict equality of pricing among the active traders in the market, and complete exclusion of the inactive traders. This illustrates a general rule: when the maximum number of traders is excluded, the remaining traders of each type must share equally in the profits. As a special case, when there are just as many buyers as sellers then the solution is unique, and all sellers get treated alike, and all buyers.

Since no side-payments were needed to account for the outcomes described, the foregoing example might be considered the typical solution for the "private-party" used-car market,
under the hypothesis of a uniform $100 markup by commercial dealers. In this market, excluded buyers do not usually receive money compensation for "staying out." However, from another point of view, we might want to say that each buyer has a certain probability of contacting a seller and making a deal, even if buyers outnumber sellers. It would then be expected profit that we should consider, not the actual profit. For example, if we make all the probabilities equal, we obtain a completely symmetrical solution. Using the same numbers we would find each seller making an expected (and actual) profit of $100 - P, and each buyer making an expected profit of $\frac{4}{3}P$, with P varying as before.

If the probabilities are not equal, we obtain a less symmetrical solution, with buyers' profits in our numerical example equal to $p_1P$. Here the probabilities $p_1$ would have an average value of $\frac{4}{3}$. More generally, the $p_1$ could be functions of P, so that the chance of a particular buyer making a deal within the private market would depend on the extent to which the price level favors buyers over sellers.

In this manner, quite elaborate "natural" modes of compensation can enter the solution without involving cash side-payments, or conscious collusion of any sort. In fact, all known solutions to the symmetrical markets of this paper can be similarly explained by the mechanism of a variable "probability of contact" for each trader, depending only on the value of the price-level parameter.
REFERENCES


