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RESEARCH MEMORANDUM

GAMES WITH MANY MOVES

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The purpose of this note is to summarize and expound the results of three papers on information in extensive games (*), particularly the third, as they apply to the problem of simplifying the solution of multi-move games. The special terminology of these papers (necessary for the formal development of the theory but likely to be misleading or obscure in the present context) is avoided where possible. Thus the term "pattern of information" is used, but only where the intuitive meaning agrees with the precise definitions of (*).

A (2-person, 0-sum) game with many moves has frequently been found to break up into a sequence of "inner games", the outcome of each setting the parameters of the next. Of especial importance is the decomposition into "normal" games--inner games that are in essentially the normalized form, so that the strategies and the move-choices are the same. Helmer and Blackwell demonstrated exhaustively that all three-move games with perfect communication admit such a decomposition; but more general three-movers, as well as perfect-communication four-movers, are known which are without normal decompositions. (**)

Decomposition or other simplification of the solution is sometimes made possible by some special property of the pay-off function (e.g., convexity). Here we are only concerned with those reductions that result from the formal structure of the information pattern, independently of the pay-off function.

(*) McKinsey [2] ; Quine [3], [redacted]
(**) Blackwell and Helmer [1] ; Shapley [4] .

The pay-off of the N-mover is given by a function

$$H(x_1, x_2, \ldots, x_N)$$

of the choices of the two players, the moves of the first (second) player composing a subset $A \cap (\bar{A})$ of the set of integers from 1 to N:

$$A \cap \bar{A} = \Lambda \;, \quad A \cup \bar{A} = \left\{1, 2, \ldots, N\right\} .$$

We distinguish between the <u>moves</u>, 1, 2, ..., N, and the <u>choices</u> $x_1, x_2, \ldots, x_N$. The domains from which these choices are drawn are assumed to be fixed and not dependent on previous choices. Their precise nature will not concern us here -- but we will assume that the min-max theorem holds wherever we need it.

In general the <u>solution</u> may always be expressed in a form such as the following exemplar:

(1)

$$\begin{cases} \max\limits_F \min\limits_G \iint H \Big\{ x_1, q_2(x_1), p_3(x_1), p_4\left[q_2(x_1), p_3(x_1)\right], q_5\left[x_1, q_2(x_1), p_3(x_1)\right], \\ p_6(q_2(x_1), p_4\left[q_2(x_1), p_3(x_1)\right]) \Big\} \, dF(x_1, p_3, p_4, p_6) \, dG(q_2, q_5). \end{cases}$$

It is important to distinguish the <u>functions</u> q_2, p_3, etc., from their <u>values</u> $q_2(x_1)$, $p_3(x_1)$, etc. The information pattern on which this example is based is evidently:

move	made by	knowing only
1	I	---
2	II	x_1
3	I	x_1
4	I	x_2, x_3
5	II	x_1, x_2, x_3
6	I	x_2, x_4.

F and G are distributions over the indicated mixed sets of choices and functions of choices. In practice such distribution functions are usually quite unmanageable.

A <u>solution of the first kind</u> is said to exist if there is an expression of the form:

$$\max_{x_1} \min_{x_2} \min_{x_3} \cdots \max_{x_N} H(x_1, \ldots, x_N)$$

which for any function H gives the value of the game. A game of perfect information (for example) has this type of solution.

A <u>solution of the second kind</u> exists if an expression of the form typified by

$$\max_{x_1} \left\{ \min_{G} \max_{F} \iint \left[\max_{F'} \min_{G'} \iint H(x_1, \ldots, x_6) dF'(x_4, x_6) dG'(x_5) \right] dF(x_3) dG(x_2) \right\}$$

exists giving for any pay-off function H the value of the game. It is convenient to write the above as

$$\max_{1} \ \underset{2 \quad 3}{\text{minimax}} \ \underset{4 \quad 5 \quad 6}{\text{maximinimax}} \ H(x_1, \ldots, x_6),$$

which may in turn be shortened to

$$(2) \qquad \underset{1}{\text{mix}}\ \underset{2,3}{\text{mix}}\ \underset{4,5,6}{\text{mix}}\ H(x_1, y_2, x_3, x_4, y_5, x_6)$$

with the y's distinguishing the set $\bar{A}$ of the second player's moves. Each "mix" operator corresponds to a normal "inner game".

THEOREM. The pattern of information admits a solution of the first kind if and only if for every pair of moves m, n, belonging to the first and second players respectively, either

(i) player I knows y_n when it is his turn to make the m^{th} move

or

(ii) player II knows x_m when it is his turn to make the n^{th} move.

THEOREM. The pattern of information admits a solution of the second kind if and only if

(a) Each player's knowledge of his opponent's past choices is a non-decreasing function of the move number (i.e., -- no forgetting of opponent's choices);

(b) This function increases, if at all, by the maximum amount (i.e., -- whenever a player learns of any of his opponent's choices, he learns of all that have been made up to that time);

(c) Such points of maximum increase occur simultaneously for the two players (i.e., -- whenever one player is brought up to date on his opponent's actions the other player is too). (*)

(*) More precisely (since some points of no increase will also be points of maximal increase), each point of increase for one player is simultaneous with a point of maximal increase for the other. The time scale is of course quantized, with one time-point available between each pair of moves for adding or expunging items of information.

<u>The state of a player's information</u> ("memory", "communication") <u>regarding his own previous choices is immaterial</u> in these theorems. (**)

The second theorem has been stated, for convenience, in terms of a definite chronological order of moves. It should be recalled (***) that a given information pattern does not in general determine a unique temporal order. Under one order the conditions (a) - (c) may be fulfilled, while under another, equally permissible, order they may not. In applying the theorem all permissible orders must be considered. (Any order is permissible if it does not imply knowledge of future events.)

For example, consider the three-move pattern:

move	made by	knowing only
1	I	(nothing)
2	II	x_1
3	II	(nothing)

With respect to the order (123), condition (a) is not satisfied. Other permissible orders are (132) and (312), the only restriction being that move 2 follow move 1. Both of these orders meet all the conditions of the second theorem. Hence, the solution may be written

$$v = \underset{1\quad 3}{\text{maximin}}\ \underset{2}{\min}\ H(x_1, y_2, y_3) = \underset{1,3}{\text{mix}}\ \underset{2}{\text{mix}}\ H(x_1, y_2, y_3),$$

or

$$v = \underset{3\quad 1}{\text{minimax}}\ \underset{2}{\min}\ H(x_1, y_2, y_3) = \underset{3,1}{\text{mix}}\ \underset{2}{\text{mix}}\ H(x_1, y_2, y_3).$$

(**) See Quine's ([3]) notion of "deflated pattern". Only one level of depletion is involved here.
(***) McKinsey ([2]).

Proofs of theorems equivalent to the two given here will be given elsewhere.

The rule for obtaining the actual solution once the conditions of the second theorem are met is fairly obvious. The points of simultaneous, maximal increase in information specified in (c) cut the sequence of N moves into a number of segments. The moves of each segment should be "mixed" separately, and the "mix" operators should then be arranged, from left to right, according to the temporal order of the corresponding segments.

In view of the informal way in which (in this note) the "mix" operator was defined, it might be well to emphasize two of its characteristic properties. First, in consequence of the Min-max Theorem, a "mix" may be taken apart at the joints and reassembled in any order without affecting the value of the expression. Second, in the definitional expansion of a "mix", never more than two distribution functions are used.

It is possible to proceed from expressions such as (1) to the corresponding normal simplifications such as (2) by formal manipulation of the operators, integrals, functionals, etc. Such methods are of use when no solution of the second kind exists, and only partial simplification is possible. The exposition of this topic is deferred. In particular, the definition and characterization of solution of the n^{th} kind, for $n \geq 3$, we leave to the ingenuity and enterprise of future researchers.

[1] David Blackwell and Olaf Helmer, Continuous Three-move Games - RM-124.

[2] J. C. C. McKinsey, Notes on Games in Extensive Form - RM-157.

[3] W. V. Quine, Notes on Information Patterns in Game Theory - RM-216.

[4] L. S. Shapley, A Three Move Game with Imperfect Communication - RM-137.

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