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RM-3530-PR
FEBRUARY 1963

ON APPROXIMATING LINEAR ARRAY FACTORS

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PREPARED FOR:
UNITED STATES AIR FORCE PROJECT RAND

The RAND Corporation
SANTA MONICA • CALIFORNIA

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PREFACE

The study reported in this Memorandum is a product of RAND's continuing interest in electronically scanned radars. It is a contribution to the theoretical understanding of certain problems in the design of such radars.

SUMMARY

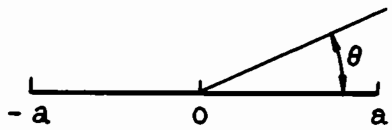
This Memorandum considers least-mean-square comparison of directivity patterns of line sources having different illumination distributions. In particular, it considers approximation of a given directive pattern by means of discrete arrays of equally excited, unequally spaced elements. The given array factor may come from an ideal array that is continuous, discrete or mixed. The common equal-area approximation to the illumination distribution is shown to be equivalent to least-mean-square approximation with weighting proportional to the inverse square of the usual normalized pattern argument. An illustrative example is worked and the ideal and approximate array factors are shown. It is suggested that iterative solutions for other weight functions of low-pass type could start from the equal-area result.

CONTENTS

LIST OF FIGURES

I. LEAST MEAN SQUARE APPROXIMATION OF DIRECTIVITY FACTORS

Let $-a \le z \le a$ be the aperture of a linear array, as sketched below. Let $I(z)$ be the (cumulative) distribution function and $dI(z)/dz \ge 0$



the density function, which corresponds to the magnitude of the aperture excitation. We allow $I(z)$ to consist of a finite number of steps plus an absolutely continuous component. In addition, the aperture excitation may have a phase proportional to z, so that the main lobe points in the direction θ_0 . Then the array factor produced has voltage pattern[1]

$$p(\theta) = \int_{-a}^{a} e^{2\pi i z(\cos\theta - \cos\theta_0)/\lambda}\, dI(z) \qquad (1)$$

where the direction θ is measured from the line along which the excitation is produced. We shall suppose $I(z)$ so scaled that $p(\theta_0) = 1$. Putting $z = at$ and $x = 2a(\cos\theta - \cos\theta_0)/\lambda$, we may write Eq. (1) in the normalized form

$$p(x) = \int_{-1}^{1} e^{\pi i x t}\, dI(t) \qquad (2)$$

A particular case of Eq. (2) which will interest us occurs when the distribution I(t) is a sum of the sort given by

$$NJ(t) = \sum_{k=1}^{N} S(t - t_k) \qquad (3)$$

where S(t) is the unit step and $-1 \le t_1 < t_2 < \cdots < t_N \le 1$. The array factor corresponding to Eq. (3) is

$$q(x) = \frac{1}{N} \sum_{k=1}^{N} e^{i\pi t_k x} \qquad (4)$$

Recently there has been considerable interest in the subject of discrete linear arrays of equally excited, unequally spaced radiators[(2-5)] with patterns and illuminations given by Eqs. (4) and (3) respectively. One problem receiving attention may be formulated as follows. We are given a model distribution, I(t), which is often continuous, has a known array factor, p(x), and has some useful shape (for example, all of the visible sidelobes of p(x) may have the same conveniently low amplitude). We wish to approximate this p(x) as well as possible by a sum as given by Eq. (4). The corresponding approximate aperture distribution is given by Eq. (3), and our problem is to locate the jumps, t_k, optimally.

Since one is usually interested in power, it seems reasonable to choose as a measure of pattern agreement a weighted mean square difference such as

$$e_w = \int_{-\infty}^{\infty} |p(x) - q(x)|^2 \, w(x) \, dx \qquad (5)$$

where the sidelobe weight function $w(x) \geq 0$ represents our opinion about the relative importance of agreement at various distances off the main beam ($x = 0$). The range of x over which $w(x)$ is appreciable will generally depend on the range of scanning, the range of wavelength and the actual aperture. For example, if agreement is equally important at all points of the interval $-x_0 \leq x \leq x_0$ and of no interest outside, then we would take $w(x) = S(x + x_0) - S(x - x_0)$. Another $w(x)$ we shall consider is $w(x) = 1/x^2$, which indicates steadily waning interest in agreement the farther we move off the main beam. It is easy to check that $p(x) - q(x) = O(x^2)$ for $|x|$ near 0; hence $[p(x) - q(x)]^2/x^2 = O(x^2)$ also, and the contribution to Eq. (5) from the region near the main lobe will be small despite the behavior of $1/x^2$ near the origin.

II. MINIMIZATION OF ERROR EXPRESSION

One can go through the motions of setting the N partials of e_w with respect to the parameters t_k equal to zero, obtaining the N simultaneous transcendental equations

$$\operatorname{Im} \int_{-\infty}^{\infty} [p(x) - q(x)] \, w(x) \, x e^{-i\pi t_k x} \, dx = 0 \tag{6}$$

If the integrals can be evaluated, the resulting simultaneous equations for the t_k will still defy closed form solution for almost any $w(x)$. However, particular numerical cases can be solved iteratively by the Newton-Raphson method of replacing each transcendental equation by an approximate linear equation in the differentials of the unknowns. If a good enough first approximation is available so that only one or two iterations are necessary, then this process may be practical even though it is as difficult as inverting N^{th} order matrices. We shall have a few more remarks about this after examining the special weight $w(x) = 1/x^2$ more closely.

III. MINIMIZATION FOR WEIGHT x^{-2}

We consider a $q(x)$ which is a least mean square approximation to a model pattern $p(x)$, with sidelobe weighting function $w(x) = 1/x^2$. We have, after integrating by parts

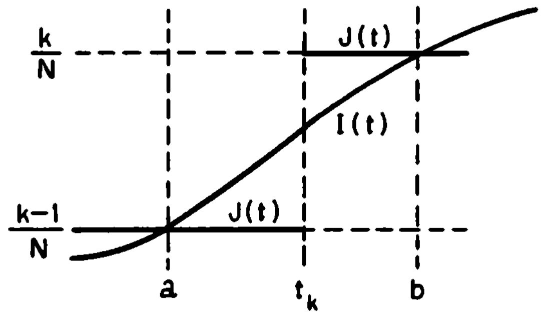
$$p(x) - q(x) = \int_{-\infty}^{\infty} e^{\pi i x t}\, d[I(t)-J(t)] = -i\pi \int_{-\infty}^{\infty} x e^{\pi i x t}\, [I(t)-J(t)]\, dt$$

Dividing by x and applying Parseval's theorem, we get

$$\int_{-\infty}^{\infty} \frac{|p(x)-q(x)|^2}{x^2}\, dx = \pi^2 \int_{-\infty}^{\infty} [I(t)-J(t)]^2\, dt \qquad (7)$$

Thus, a least mean square approximation to the array factor $p(x)$, with weight function $w(x) = 1/x^2$, is equivalent to making $J(t)$ a least mean square approximation to $I(t)$.

Now let us restrict attention to the case when $J(t)$ has the form of a sum of steps of equal height, Eq. (3). It is obvious that an optimum choice of $J(t)$, in the sense of minimizing Eq. (7), will be one for which $I(t)$ passes through each step. In other words, if a step of the optimum $J(t)$ occurs at t_k, then we have $\frac{k-1}{N} = J(t_k^-) \le I(t_k) \le J(t_k^+) = \frac{k}{N}$. If we let $I(a) = \frac{k-1}{N}$ and $I(b) = \frac{k}{N}$, we now know that the optimum t_k is in the interval $a \le t_k \le b$ as sketched on the following page:



Thus, to make a least mean square fit of the step function, J(t), to the model distribution, I(t), we need only locate each t_k within its own interval so that the contribution

$$\int_a^b [I(t)-J(t)]^2 \, dt = \int_a^{t_k} [I(t) - \frac{k-1}{N}]^2 \, dt + \int_{t_k}^b [\frac{k}{N} - I(t)]^2 \, dt$$

is minimum for each interval separately. Differentiating with respect to t_k and solving

$$[I(t_k) - \frac{k-1}{N}]^2 - [\frac{k}{N} - I(t_k)]^2 = 0$$

gives

$$I(t_k) = \frac{2k-1}{2N}$$

This says to choose each step of J(t) so that I(t) crosses the middle of the step. Notice that this analysis also applies to the case when

I(t) itself contains a finite number of steps.

In terms of the illumination density function, dI(t)/dt, this solution amounts to dividing the total "area" of the illumination density into 2N equal sub-areas and then placing an element at every other division point. This scheme has been used before,[2,3] with good results. However, although it has seemed obvious to many people that this equal-area approximation (as we shall call it) should give good results, it has not up to now been clear in precisely what sense, if any, this approximation was optimal. Our result supplies an answer to this question.

It also suggests that the equal area approximation should be a good first trial in any iterative scheme, based on Eq. (6), for obtaining optimum mean square approximations for other weight functions of "low pass" type. In particular, it should be a good first approximation when the weight function is $S(x + x_0) - S(x - x_0)$.

IV. AN APPLICATION

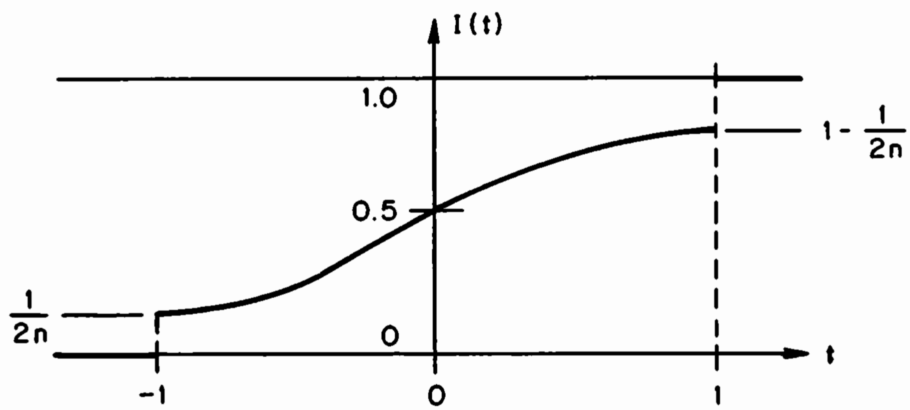
We give an example which may be of interest in its own right. Suppose we would like to produce an array of equally excited, unequally spaced radiators whose array factor has uniform sidelobes (in the visible region) with some convenient level. An array factor that behaves in this way is Taylor's "ideal space factor." For the side-lobe voltage ratio $\eta = \cosh b$, the basic relations are[4]

$$p(x) = \cos\sqrt{\pi^2 x^2 - b^2}$$

and

$$\eta \frac{dI(t)}{dt} = \begin{cases} \frac{b}{2} \quad \frac{I_1(b\sqrt{1-t^2})}{\sqrt{1-t^2}} + \frac{1}{2}\,\delta(t-1) + \frac{1}{2}\,\delta(t+1) & \text{for } |t| \le 1^+ \\ \\ 0 \quad \text{for} \quad |t| > 1 \end{cases}$$

The distribution function, I(t), looks like the sketch below.



We have taken $\bar{n}$ = 10 (20 db design sidelobe level) and calculated (equal area) approximate step distributions for several values of N (number of elements). We expect, because of the tapered weighting, $1/x^2$, that the agreement will deteriorate steadily as we move toward the farther out sidelobes.

The ideal illumination density itself appears in Fig. 1, along with the boundaries for equal areas and the resulting element locations when this illumination is approximated by an array of 20 equally excited radiators.

Table 1 lists the element locations for six different approximations to this illumination distribution. The locations may be off about two in the last decimal place.

Table 1

ELEMENT LOCATIONS, TAYLOR IDEAL ILLUMINATION, 20-db SIDELOBES

Element Number	Number of Elements					
	12	14	16	18	20	24
1	.069	.059	.051	.045	.040	.033
2	.214	.182	.159	.141	.126	.105
3	.367	.311	.270	.239	.214	.177
4	.537	.449	.387	.340	.304	.251
5	.742	.605	.514	.449	.399	.327
6	.948	.794	.659	.568	.501	.407
7		.974	.836	.704	.613	.492
8			.990	.871	.742	.584
9				.998	.900	.686
10					1.000	.802
11						.948
12						1.000

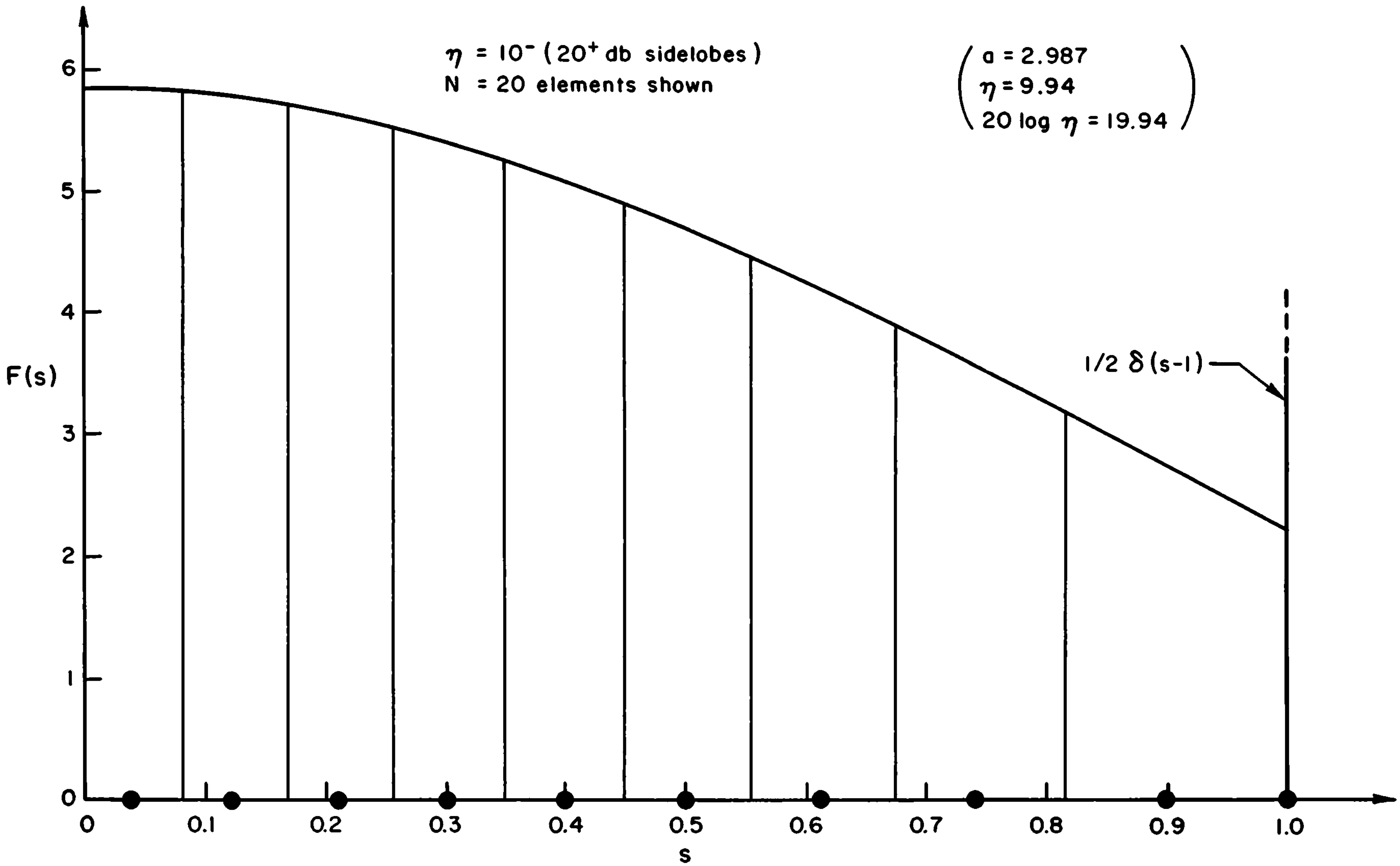


Fig. 1 — Illumination factor and elements

The resulting approximations to the ideal array factor are shown in Figs. 2 through 7. The dotted curve is the ideal directivity factor, while the solid curve is the directivity of the unequally spaced discrete approximation. The abscissa is

$$u = \frac{4a(\cos \theta - \cos \theta_o)}{(N - 1)\lambda}$$

Its significance is perhaps most easily grasped by considering the "standard" case when the average inter-element spacing is $\lambda/2$ and the beam is aimed broadside ($\theta_o = \pi/2$). In that case the range of angles from broadside to the line of the array, $\pi/2 \geq \theta \geq 0$, corresponds to the range $0 \leq u \leq 1$. Thus the range out to $u = 1$ may be considered the ordinary visible range of the array. For the example given the approximation is fairly good out to this point, deteriorating with increasing u, as expected from the weight function $1/x^2$.

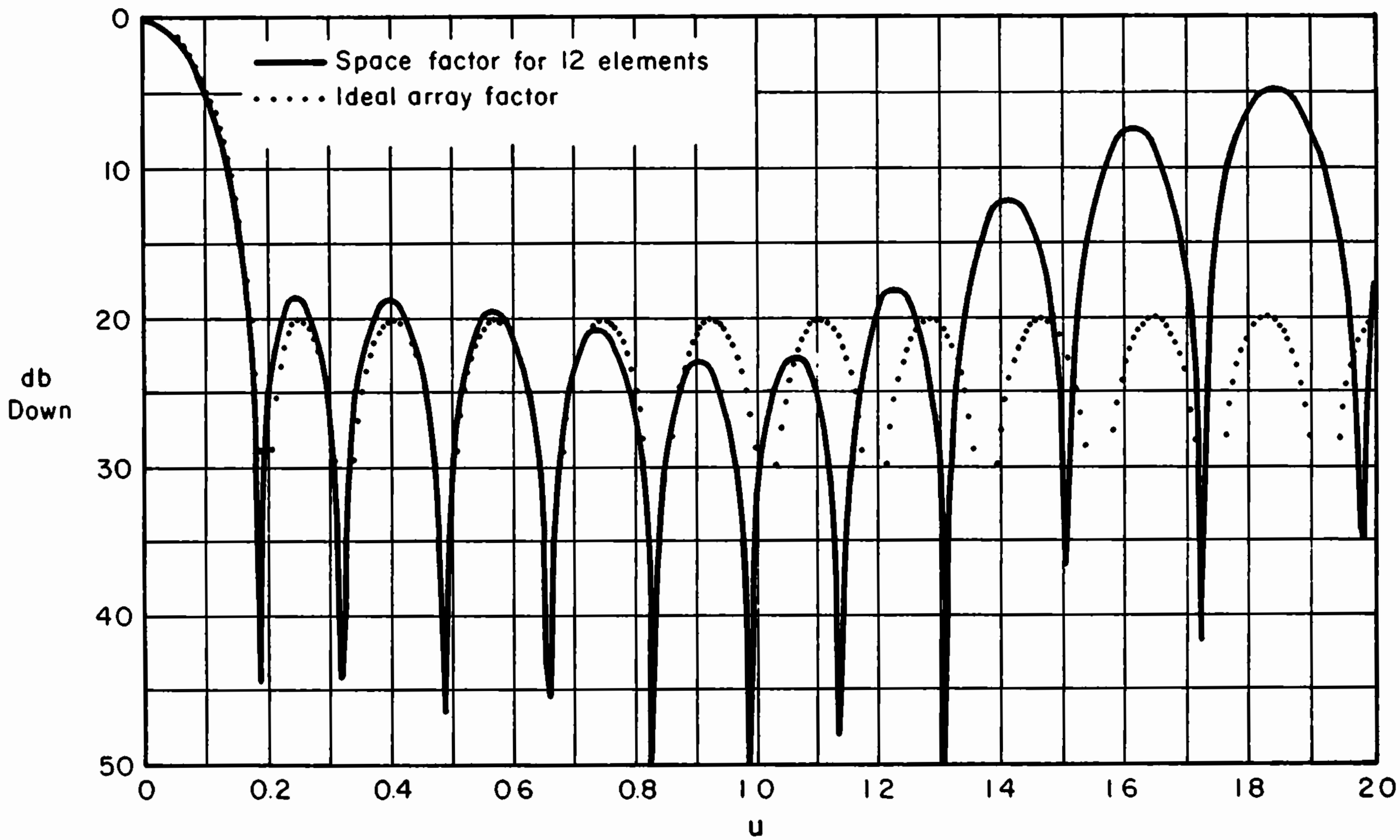


Fig. 2 — Approximate and ideal space factors

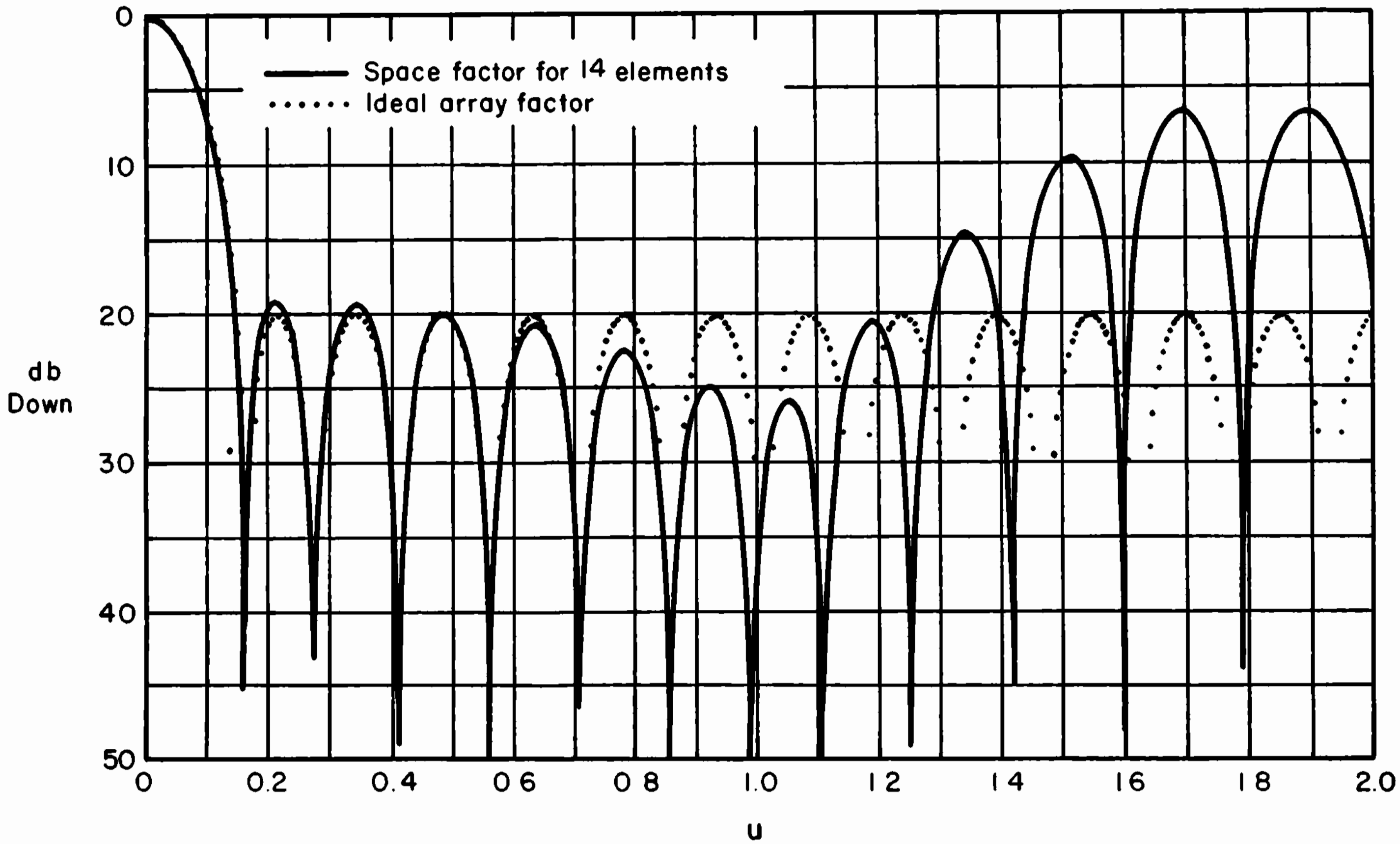


Fig. 3 — Approximate and ideal space factors

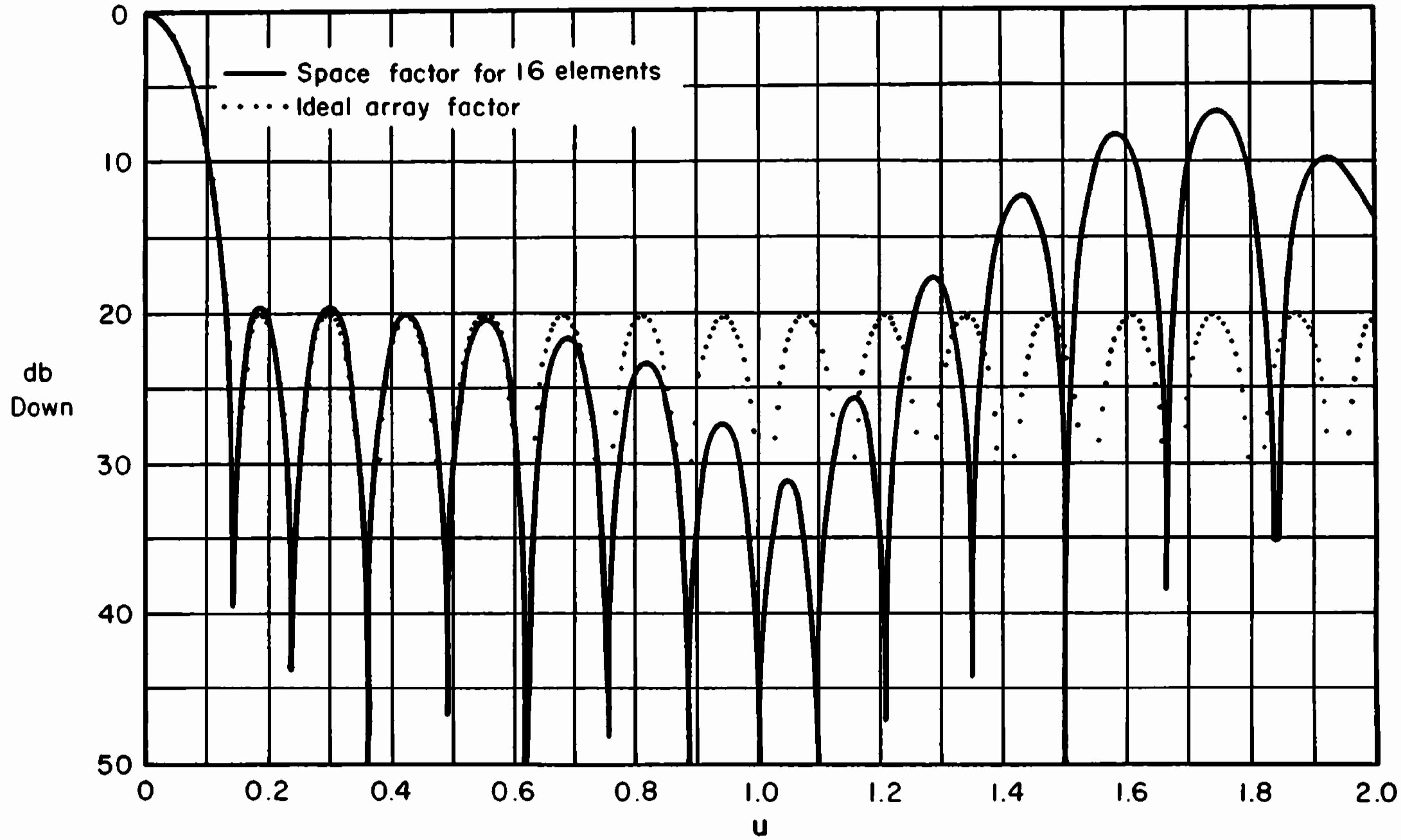


Fig. 4 — Approximate and ideal space factors

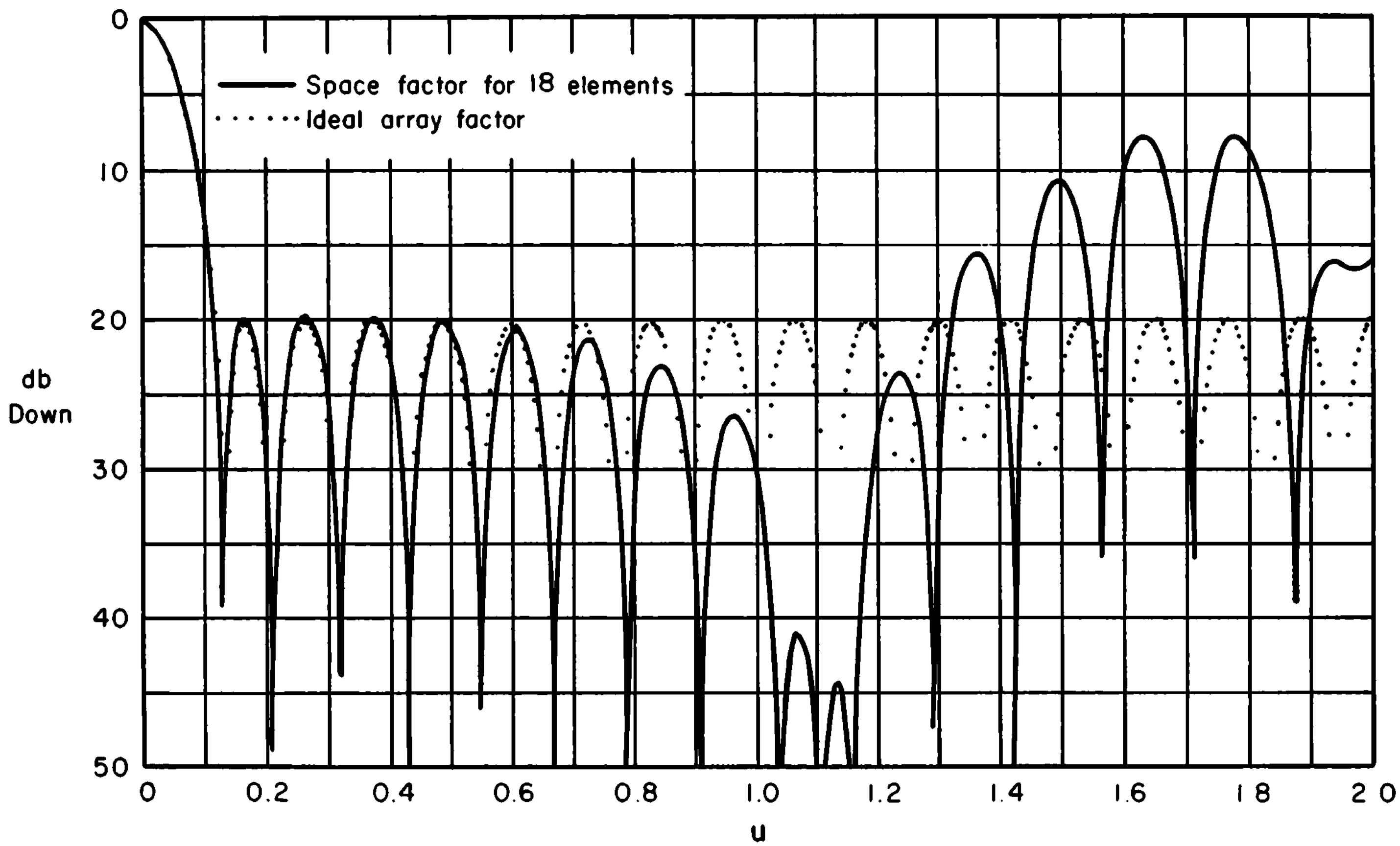


Fig. 5 — Approximate and ideal space factors

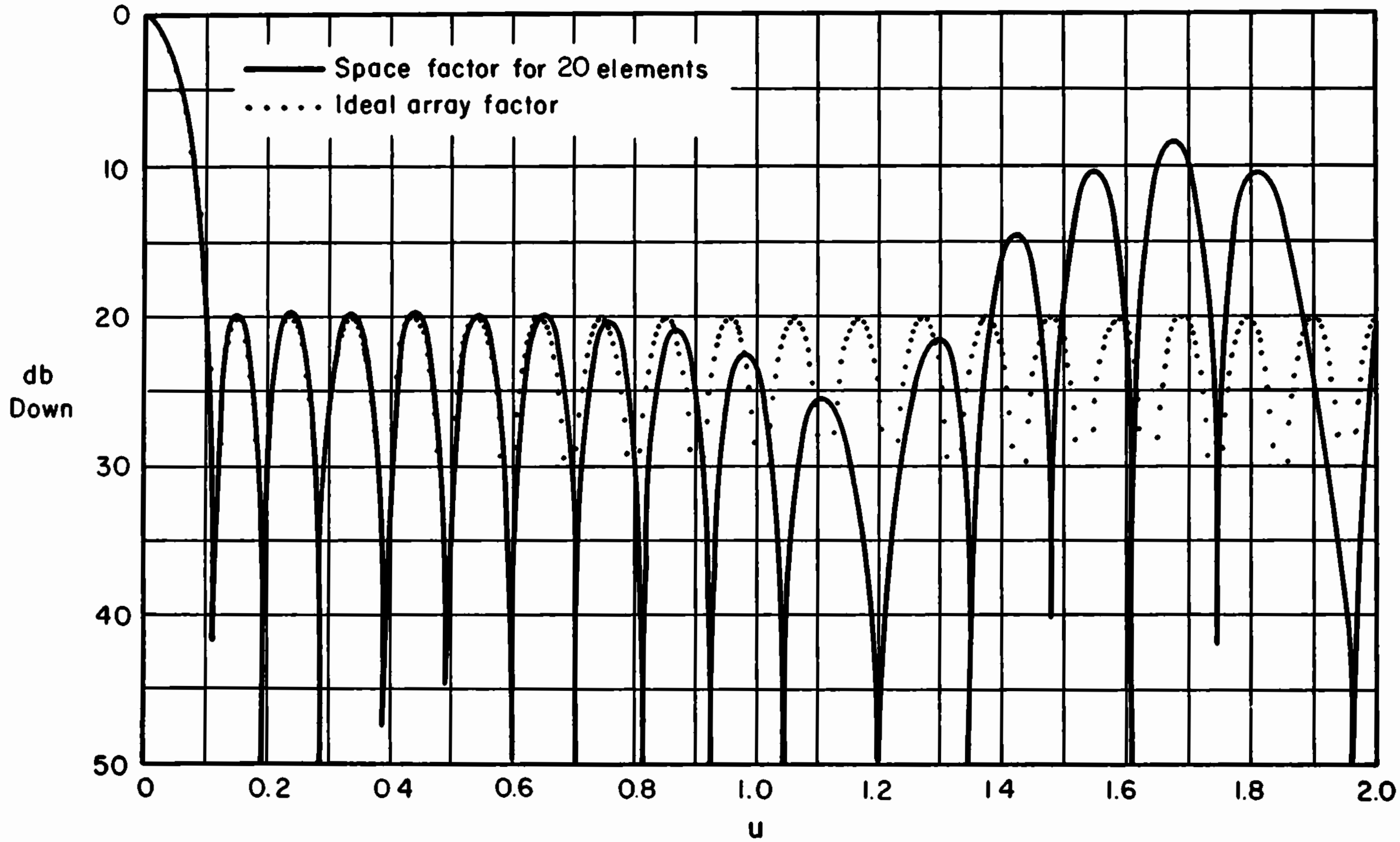


Fig. 6 — Approximate and ideal space factors

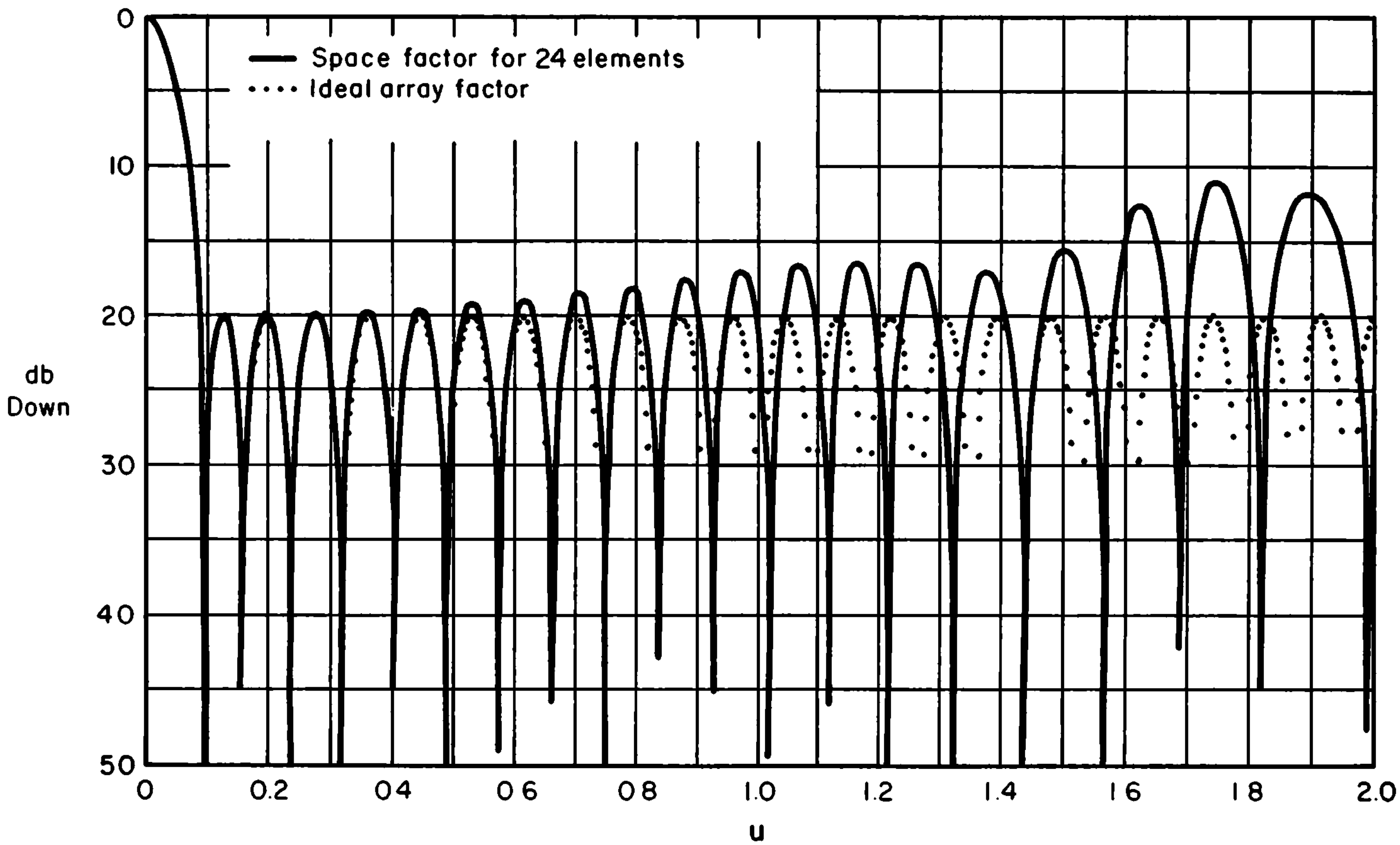


Fig. 7 — Approximate and ideal space factors

V. CONCLUSIONS

Discrete arrays of equally excited, unequally spaced elements have been used to approximate desirable patterns by assigning elements to equal increments of the corresponding illumination distribution. By examining this approximation from the viewpoint of least mean square pattern fitting, we have discovered the sense in which such a fit is optimum: it is optimum for the sidelobe weighting function $1/x^2$.

We suggest that approximations for other weights, for example, uniform over an interval and zero beyond, should be obtainable fairly quickly by the Newton-Raphson method if the solution for $1/x^2$ weighting is taken as a first approximation.

Some illustrative cases are worked for the $1/x^2$ case and the fit of ideal and approximate space factors is shown.

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