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COMPOUND SIMPLE GAMES II: SOME GENERAL COMPOSITION THEOREMS

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PREFACE

The mathematical research presented in this series of Memoranda is concerned with solutions for certain types of games of strategy. The theory of games is of great importance because of its applicability to a variety of conflict situations--economic, political, and military.

The first Memorandum of this series was RM-3192-PR, Compound Simple Games I: Solutions of Sums and Products.

SUMMARY

This is an investigation of the solutions of the type of compound game that is formed when the players of a given, m-person game are replaced by m committees, each committee having its own special voting rule or other method of reaching a decision. Relationships are established between the solutions of the compound game, on the one hand, and the solutions of the original game and the m committee games, on the other.

CONTENTS

COMPOUND SIMPLE GAMES II: SOME GENERAL COMPOSITION THEOREMS

1. INTRODUCTION

A compound simple game is made up of a number of simple games, called components, linked together by another simple game, called the quotient. In effect, the compound--an expanded version of the quotient--is to be played by committees, which are in turn "played" (controlled) by their individual members through the formation of winning coalitions within each committee. In our present account we shall assume that the committee memberships do not interlock.*

A familiar example of a compound structure of this type is the method of electing a president under the federal consitution. The "quotient" is the Electoral College (a 51-person weighted majority game), and the "components" are the electorates of the fifty states and the District of Columbia (simple majority games of various sizes).

When there are only two components, the possibilities

*Compound simple games were introduced by the author in 1953 (unpublished lecture notes). Gurk and Isbell mention them in [3], p. 259, calling them "committee compositions," and give a special case of our present Theorem 1. Isbell uses them again in [4], as an aid in proving the existence of certain symmetric n-person simple games for different values of n. In his definition (but not in his application), he permits the player sets of the components to overlap.

for the quotient are very limited. There are in fact just four cases, two of them trivial: either the compound is the "sum" or "product" of its components, or else it is a game equivalent to one of the components alone, with the other players appearing only as dummies. In Part I of this series [11], we investigated in some detail the solutions of sums and products from the standpoint of the von Neumann-Morgenstern theory [12].

When there are more than two components, the possible compound structures become more numerous.* The quotients (which were not explicitly introduced in Part I) correspond to monotonic Boolean functions of as many variables as there are components. From this point of view, sums and products stand revealed as very special cases. In the wider context of the present paper, we shall attempt to unify and extend some of the results of Part I.

Specifically, our fundamental result (Theorem 1, in Sec. 3) states that if the total proceeds of a compound game are divided first among the components according to a solution of the quotient, and after that among the players in each component according to a solution of that component, then the resulting set of allocations will constitute a solution of the compound game. Some illustrative examples are discussed in Secs. 2 and 4.

*For example, there are 179 five-person simple games that might serve as quotients, not counting symmetric repetitions or games with dummies.

The problem of aggregation. An important question in the application of game theory to economics and the social sciences is the extent to which it is permissible to treat firms, committees, political parties, labor unions, etc., as though they were individual players. Behind every game model played by such aggregates, there lies another, more detailed model: a compound game of which the original is the quotient. Given any solution concept, it is legitimate to ask how well it stands up under aggregation--or disaggregation. To what extent are the theoretical predictions sensitive to the particular level of refinement adopted in the model?*

From the verbal description we have given of Theorem 1, it is evident that the solutions in the von Neumann-Morgenstern theory do stand up, at least under disaggregation. The solutions of the coarse model (the quotient), with internal details added for the components, become solutions of the refined model (the compound game). The situation with respect to aggregation is more complex, since we are not dealing with a theory in which solutions are unique. Two things could conceivably go wrong. The refined

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*The concept of "power index," for instance, is not even approximately invariant under aggregation. The author has found that the multimillion-person compound game of "Presidential Election" (see above) distributes power among the states quite differently from the 51-person quotient game of "Electoral College," evaluated in [5]. The larger game gives substantially heavier weight to the more populous states.

solution might reduce to a nonsolution of the coarse model, or it might have such a structure that it cannot be reduced at all. These possibilities are discussed in Secs. 5 and 6 of this paper.

In Sec. 5 we ask whether it is possible for a compound game to have a solution with the same internal compound structure that would arise from Theorem 1, but based on elements that are not necessarily solutions of the respective component and quotient games. In other words, can nonsolutions replace solutions in the composition process? We suspect that the answer is no, but cannot yet prove it. What we can show is that any such nonsolutions must belong to a certain narrowly restricted class of "generalized" solutions, and the whole matter hinges on proving whether or not the generalization is empty.

In Sec. 6 we inquire into the existence of compound-game solutions that do not possess the corresponding compound structure. Such solutions would not be susceptible to aggregation in a straightforward way. In Part I we found that sums do not have such noncompound solutions, while products (usually) do. By a direct transfer of our results for products, we are able to show it is the sums that are exceptional; "most" compound games, it appears, have noncompound solutions, along with the compound solutions provided by Theorem 1. Heuristically, this means that possibilities for stable social configurations may exist that are undetectable in an aggregated model.

2. COMPOUND SIMPLE GAMES

We recall from Part I that the notation $\Gamma(P, \mathcal{W})$ denotes the simple game with players P and winning coalitions $\mathcal{W}$. The elements of $\mathcal{W}$ are nonvoid subsets of P, and $\mathcal{W}$ itself is nonvoid and contains all the supersets of each of its elements.

Let there be given m component games:

$$G_i = \Gamma(P_i, \mathcal{W}_i), \qquad i = 1, 2, \ldots, m,$$

together with an m-person quotient game:*

*We write $\overline{12\ldots m}$ instead of $\{1, 2, \ldots, m\}$ in the interest of typographical compactness.

$$Q = \Gamma(\overline{12\ldots m}, \mathcal{U}).$$

The sets P_i are assumed disjoint, and their union will be denoted by P. The players of Q are identified with the integers 1, 2, ..., m; they are, of course, of an entirely different species from the elements of P. For $S \subseteq P$ we define $K(S)$ to be the set of players of Q that are "controlled" by S, thus:

$$K(S) = \text{set}\{i \mid S \cap P_i \in \mathcal{W}_i\}.$$

Then the <u>compound game</u>

$$G = Q[G_1, \ldots, G_m] = \Gamma(P, \mathcal{W})$$

is defined by the condition

$$S \in \mathcal{W} \Longleftrightarrow K(S) \in \mathcal{U}.$$

Thus, a coalition wins in the compound if and only if <u>it includes winning coalitions from enough of the components to make up a winning coalition in the quotient</u>.

Another way to write this definition is

$$\mathcal{W} = \bigcup_{R \in \mathcal{U}} \bigcap_{i \in R} \mathcal{W}_i^+,$$

where $\mathcal{W}_i^+$ consists of all subsets of P that are supersets of elements of $\mathcal{W}_i$. In other words, $\Gamma(P, \mathcal{W}_i^+)$ is the game $\Gamma(P_i, \mathcal{W}_i)$ with the members of $P - P_i$ thrown in as dummies.

Elementary properties of the definition. The symbol B_n will denote the n-person "pure bargaining game," in which the only winning coalition is the set of all players. B_n^* will denote the dual of B_n, i.e., the n-person game in which only the empty coalition fails to win.* M_n will denote the n-person "simple majority game," in which every coalition of more than $n/2$ players wins, and all others lose; it is constant-sum if and only if n is odd. (The games B_n, B_n^* are non-constant-sum except for $B_1 = B_1^* = M_1$.)

To begin with, we have some trivial identities involving B_1:

$$G = B_1[G] = G[B_1, \ldots, B_1].$$

We shall not call G "compound" unless it can be decomposed in some less trivial way. Sums and products may be defined as follows:

*Duality in this sense interchanges winning coalitions and complements of losing coalitions. "Self-dual" is equivalent to "constant-sum." The dual of a compound is found by dualizing the components and the quotient. (See [10].)

$$G_1 \oplus G_2 = B_2^*[G_1, G_2],$$

$$G_1 \otimes G_2 = B_2[G_1, G_2].$$

A sum of three games corresponds to the quotient B_3^*, etc. Polynomials in $\oplus$ and $\otimes$ can be expressed in our present notation by substituting B_1 into the polynomial to determine the quotient, for example:

$$G_1 \otimes (G_2 \oplus G_3) = (B_1 \otimes B_2^*)[G_1, G_2, G_3].$$

In fact, iterated compound structures can always be collapsed to a single level, i.e., a quotient--possibly compound--operating on irreducible components.

<u>Some examples</u>. The smallest compounds that are not sums or products are the four-person game:

(1) $$M_3[B_1, B_1, B_2],$$

and its dual:

(2) $$M_3[B_1, B_1, B_2^*].$$

The smallest constant-sum compound is the following five-person game:

(3) $M_3[B_1, B_1, M_3]$.

These three examples all happen to be weighted majority games.* Note that the inessential game B_1, which has no internal structure at all, is nevertheless a nontrivial building block, in contrast with its role in the von Neumann-Morgenstern decomposition theory (see [11], p. 357 ff.).

The first constant-sum compound that has only essential components is the following highly symmetric nine-person game:

(4) $M_3[M_3, M_3, M_3]$.

This has 27 minimal winning coalitions, each containing four players; it is not a weighted majority game. Examples (1) - (3) have 3, 5, and 7 minimal winning coalitions, respectively. The general formula for this number is

$$|\mathcal{W}^m| = \sum_{R \in \mathcal{U}^m} \prod_{i \in R} |\mathcal{W}_i^m|.$$

This bears an obvious resemblance to our second definition

*The vote symbols are [4; 2, 2, 1, 1], [3; 2, 2, 1, 1], and [4; 2, 2, 1, 1, 1], respectively. (See [10].)

of $\mathcal{W}$ on page 6.

We shall discuss the solutions of these examples in Section 4.

3. THE MAIN THEOREM

We shall use A to denote the imputation simplex of the compound game $G = Q[G_1, \ldots, G_m]$ with player-set $P = \bigcup_1^m P_i$, and we shall use A_i to denote the face of A on which all members of $P - P_i$ receive zero. The faces A_i will be identified in the natural way with the imputation spaces of the respective components G_i. The imputation space of Q will be denoted by $\mathfrak{A}$.

Elements of A will be denoted by latin letters x, y, etc., sometimes with subscripts indicating the face of A to which they belong. (The coordinates of these "latin" vectors will not be referred to individually.) Elements of $\mathfrak{A}$ will be denoted by greek letters α, β etc., and their coordinates will be indicated by subscripts in the usual way.

Let $\mathfrak{X}$, $X_1, \ldots, X_m$ be subsets of $\mathfrak{A}$, $A_1, \ldots, A_m$, respectively. Then the expression

$$\mathfrak{X}[X_1, \ldots, X_m]$$

will denote the set of all $x \in A$ of the form

$$x = \sum_{i=1}^{m} \alpha_i x_i, \quad \alpha \in \mathfrak{X}, \quad x_i \in X_i, \ i = 1, \ldots, m.$$

Such a set will be called <u>compound</u>.

<u>Theorem 1</u>. <u>Let</u> $X_1, \ldots, X_m$ <u>be solutions of the simple games</u> $G_1, \ldots, G_m$, <u>respectively, and let</u> $\mathcal{X}$ <u>be a solution of the m-person simple game</u> Q. <u>Then the compound set</u> $X = \mathcal{X}[X_1, \ldots, X_m]$ <u>solves the compound game</u> $G = Q[G_1, \ldots, G_m]$.

A special case of this theorem, in which all solutions are "simple" (i.e., allot to each player at most one non-zero payoff), was noted by Gurk and Isbell ([3], page 258). Two other special cases, corresponding to $Q = B_2$ and $Q = B_2^*$, appear as Theorems 1 and 2 of Part I.

<u>Proof of Theorem 1</u>. (A) External stability. Consider an imputation y in $A - X$. It can be represented in the form

$$y = \sum_{i=1}^{m} \beta_i y_i,$$

where $\beta \in \mathfrak{N}$ and $y_i \in A_i$, $i = 1, \ldots, m$. The object is to find a vector $x \in X$ that dominates y. We shall first construct a vector $z = \sum \gamma_i z_i$, not necessarily in X, which dominates y, and then we shall modify z to obtain the desired vector x. There are two cases that must be distinguished.

<u>Case 1</u>: $\beta \in \mathcal{X}$. Since $y \notin X$, there must be an i^*

such that $\beta_{i^*} > 0$ and $y_{i^*} \notin X_{i^*}$. To simplify notation, suppose $i^* = 1$. Find a $z_1 \in X_1$ such that $y_1 \in \mathrm{dom}_1 z_1$, and let $S_1 \in \mathcal{W}_1$ denote the effective set for the domination. For a sufficiently small $\epsilon > 0$, we then have

$$(\beta_1 - \epsilon)z_1 - \beta_1 y_1 > 0 \quad \text{on} \quad S_1.$$

Let $\gamma_1 = \beta_1 - \epsilon$, and for $i = 2, \ldots, m$ define

$$\begin{cases} \gamma_i = \beta_i + \frac{\epsilon}{m-1} \\ \\ z_i = \frac{1}{\gamma_i} (\beta_i y_i + (\gamma_i - \beta_i)u_i), \end{cases}$$

where the u_i are arbitrary interior vectors in the corresponding A_i. It is clear that $\gamma \in \mathfrak{N}$ and $z_i \in A_i$, $i = 1, \ldots, m$. This completes the construction of z.

It can easily be shown that z dominates y, but we have no assurance that $z \in X$. We therefore make certain adjustments. To start, take $x_1 = z_1$. For each $i > 1$, if $z_i \notin X_i$, find a dominating $x_i \in X_i$ with $x_i - z_i$ positive on some $S_i \in \mathcal{W}_i$; but if $z_i \in X_i$, simply take $x_i = z_i$ and $S_i = P_i$. In similar fashion, if $\gamma \notin \mathcal{X}$, find a dominating $\alpha \in \mathcal{X}$ with $\alpha - \gamma$ positive on some $T \in \mathcal{U}$; but if $\gamma \in \mathcal{X}$, simply take $\alpha = \gamma$ and $T = \overline{12\ldots m}$. Now we can assemble the pieces, by defining

$$x = \sum_{1}^{m} \alpha_i x_i \quad \text{and} \quad S = \bigcup_{i \in T} S_i .$$

By construction, $x \in X$ and $S \in \mathcal{W}$. To show that x dominates y, we shall verify that $x - y$ is positive on S. Applying the various definitions, we see that the chain of inequalities

$$\alpha_i x_i \geq \alpha_i z_i \geq \gamma_i z_i > \beta_i y_i$$

is valid on S_i , for every $i \in T$. Hence $y \in \text{dom } X$, as required.

<u>Case 2</u>: $\beta \notin \mathcal{X}$. Find a dominating $\alpha \in \mathcal{X}$ with $\alpha - \beta$ positive on some $T \in \mathcal{U}$. Define

$$z_i = \begin{cases} \frac{1}{\alpha_i} (\beta_i y_i + (\alpha_i - \beta_i) u_i) & \text{for } i \in T, \\ \\ u_i & \text{for } i \notin T, \end{cases}$$

where the u_i are arbitrary interior vectors of the corresponding A_i. Then $z_i \in A_i$ for every i. If $z_i \notin X_i$, find a dominating $x_i \in X_i$ with $x_i - z_i$ positive on some $S_i \in \mathcal{W}_i$; but if $z_i \in X_i$, simply take $x_i = z_i$ and $S_i = P_i$. Defining x and S as in Case 1, we see that $x - y$ is again positive on S, by virtue of the inequalities

$$\alpha_i x_i \geq \alpha_i z_i > \beta_i y_i,$$

which are valid on S_i for all $i \in T$. Since $x \in X$ and $S \in \mathcal{W}$, this means that $y \in \text{dom } X$ as before. This completes part (A) of the proof.

(B) Internal stability. We must show that $y \in \text{dom } x$ never occurs for $x, y \in X$. Suppose it did occur. Then there would exist $\alpha, \beta \in \mathcal{X}$, $T \in \mathcal{U}$, $x_i, y_i \in X_i$, and $S_i \in \mathcal{W}_i$, such that $\alpha_i x_i - \beta_i y_i$ is positive on S_i for all $i \in T$. We must distinguish two cases.

<u>Case 1</u>: $\alpha_i > \beta_i$ for all $i \in T$. Then $\beta \in \text{dom } \alpha$, contradicting the internal stability of the quotient solution $\mathcal{X}$.

<u>Case 2</u>: $\alpha_i \leq \beta_i$ for at least one $i \in T$. Then we have

$$\beta_i x_i \geq \alpha_i x_i > \beta_i y_i \quad \text{on} \quad S_i.$$

Dividing through by the positive number β_i, we see that this entails $y_i \in \text{dom}_i x_i$, contradicting the internal stability of X_i. This completes the proof of the theorem.

4. EXAMPLES

Let us apply Theorem 1 to the examples formulated at the end of Sec. 2.

For the two four-person games, we obtain the solutions illustrated in Figures 1 and 2. The labels (a), ..., (d) are meant to correspond to the four categories of solutions of M_3, as listed in Part I, p. 31. In every case but (a) there is a variable parameter, whose range is indicated by the arrow outside the simplex. In the second game another parameter enters through the possible variation in the solution of B_2^*.

The first game has no other solutions, compound or otherwise. In the second game the straight lines in cases (b) and (c) remain solutions if they are distorted into monotonic curves, giving us a simple instance of solutions that are not compound sets. The second game has no other solutions.

Noteworthy among the compound solutions of (3) and (4) are the finite sets generated by the finite solutions of M_3 and B_1. Thus, the five-person game $M_3[B_1, B_1, M_3]$ has a solution consisting of the seven imputations $(\frac{1}{2}; \frac{1}{2}; 0, 0, 0)$, $(\frac{1}{2}; 0; \frac{1}{4}, \frac{1}{4}, 0)$, $(\frac{1}{2}; 0; \frac{1}{4}, 0, \frac{1}{4})$, $(\frac{1}{2}; 0; 0, \frac{1}{4}, \frac{1}{4})$, $(0; \frac{1}{2}; \frac{1}{4}, \frac{1}{4}, 0)$, $(0; \frac{1}{2}; \frac{1}{4}, 0, \frac{1}{4})$, and $(0; \frac{1}{2}; 0, \frac{1}{4}, \frac{1}{4})$. Similarly, the nine-person game $M_3[M_3, M_3, M_3]$ has a solution consisting of the 27 imputations of the form $(\frac{1}{4}, \frac{1}{4}, 0; \frac{1}{4}, \frac{1}{4}, 0; 0, 0, 0)$.

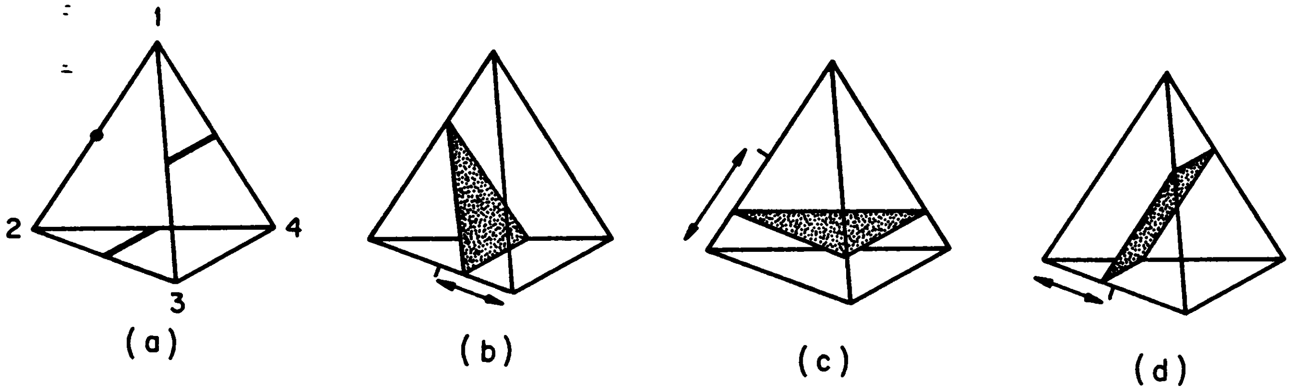

Fig. 1 — Solutions of $M_3[B_1, B_1, B_2]$

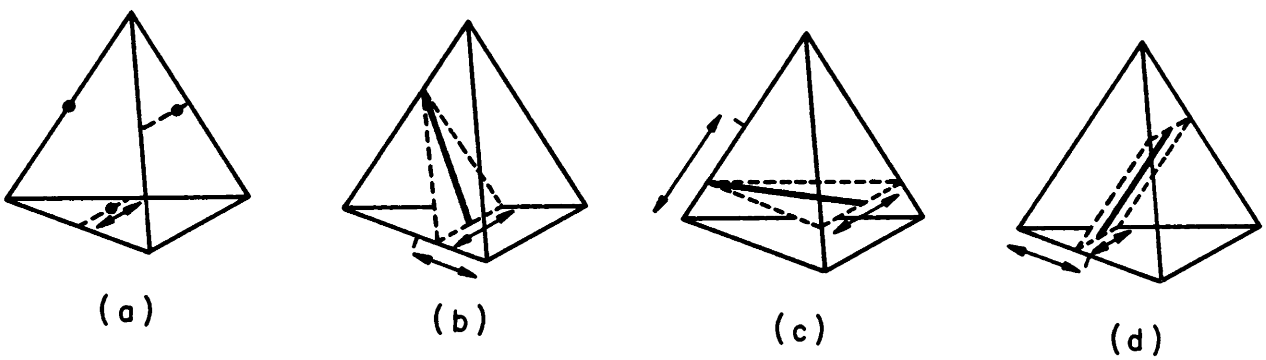

Fig. 2 — Solutions of $M_3[B_1, B_1, B_2{}^*]$

These are both main simple solutions, in the sense of [12], p. 444, and are instances of the "composition" principle described in [3], loc. cit.

We have not attempted to determine all solutions of these two games. By the method of winning fractions,* it is easily shown that they have discriminatory solutions corresponding to the solutions of $B_1 \otimes M_3$ and $B_2 \otimes M_3$, respectively, which are known by [8] to include solutions with arbitrary closed components.** Thus we are assured in both cases of a great variety of solutions not derivable from Theorem 1.

5. PROPERTIES OF COMPOUND SOLUTIONS

We consider the following question: If a solution to $Q[G_1, \ldots, G_m]$ is the compound set $\mathcal{X}[X_1, \ldots, X_m]$, then are $X_1, \ldots, X_m$, and $\mathcal{X}$ necessarily solutions of the games $G_1, \ldots, G_m$, and Q, respectively? This question bears on the problem of aggregation in the application of game theory (see the Introduction). We shall devote some effort in this section to a search for the answer.

In a similar investigation, but dealing with the embedding problem rather than the aggregation problem, von Neumann and Morgenstern found that their "decomposable"

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*See [9]; also Sec. 5 <u>infra</u>.

**See the discussion of $B_1 \otimes M_3$ in Part 1, Sec. 5.

games can be solved by fitting together solutions of the components, in a rough analogy to our Theorem 1. They also found, however, that there are generally other solutions as well, which have the same decomposable structure but decompose into sets that are not solutions of the components in the ordinary sense. This discovery led them to a useful and significant generalization of the notion of solution (see [12], pp. 361 ff.).

Our present endeavor promises similar fruit. Although we are not yet able to give a final answer to the question posed above, the obstacles that confront us serve to point the way to another generalization of the solution concept, which may likewise prove to have important applications. Before discussing this new idea, we shall clear the ground with a preliminary result that sharply limits the possibilities.

A partial converse to Theorem 1. Given a nonempty compound set $X = \mathscr{X}[X_1, \ldots, X_m]$, we shall say that X_i (or i) is relevant if α_i does not vanish identically on $\mathscr{X}$. It is clear that $\mathscr{X}$ and all relevant X_i are uniquely determined when we know X. On the other hand, the nonrelevant X_i, if any, can be chosen quite arbitrarily, since their only function is to fill out the notational form; any nonempty subset of the corresponding A_i will do.

Theorem 2. _Let_ $X = \mathcal{X}[X_1, \ldots, X_m]$ _be a compound solution of the compound game_ $G = Q[G_1, \ldots, G_m]$. _Then either it is the case that_

(a) $\mathcal{X}$ _is a solution of_ Q _and each relevant_ X_i _is a solution of the corresponding_ G_i,

or the following four statements are all true:

(b) $\mathcal{X}$ _is internally stable for_ Q;

(c) $\mathcal{X}$ _is externally unstable for_ Q;

(d) _each relevant_ X_i _is externally stable for its_ G_i;

(e) _some relevant_ X_i _is internally totally unstable for its_ G_i, _in the sense that_ $X_i \subseteq \mathrm{dom}_i X_i$.

This theorem generalizes Theorem 4 of Part I, which in effect asserts (a) for the case $Q = B_2$, $\mathcal{X} = \mathcal{V}$. We should remark at once that we have found no instances where (a) does not hold, despite much searching. Note that (b) - (e) imply that both $\mathcal{X}$ and at least one of the X_i are nonsolutions. We therefore have the following corollary, which depends of course on Theorem 1 as well as Theorem 2.

Corollary. _Let_ $G = Q[G_1, \ldots, G_m]$ _be a compound game, and let_ $X = \mathcal{X}[X_1, \ldots, X_m]$ _be a compound set, similarly indexed. Then any two of the following statements imply the third_:

(i) X _solves_ G;

(ii) $\mathcal{X}$ _solves_ Q, and

(iii) _every relevant_ X_i _solves its_ G_i.

It remains an open question whether (i) alone implies (ii) and (iii).

Proof of Theorem 2. The theorem can be broken down into the following four propositions:*

(A) $\mathcal{X}$ is internally stable.

(B) Each relevant X_i is externally stable.

(C) $\mathcal{X}$ is externally stable, provided no relevant X_i is internally totally unstable.

(D) Each relevant X_i is internally stable, provided $\mathcal{X}$ is externally stable.

We shall take up these propositions in order. First we establish a useful lemma.

Lemma. If X_i is relevant, then there exist $x_i \in X_i$ and $S_i \in \mathcal{W}_i$ such that $x_i > 0$ on S_i.

Suppose no such x_i, S_i exist. Then no member of P_i is essential to any domination by any $x \in \mathcal{X}$. Hence X solves not only G, but also the weaker game $G' = \Gamma(P, \mathcal{W}')$, defined by

$$S \in \mathcal{W}' \Longleftrightarrow S - P_i \in \mathcal{W}.$$

*To see this, observe that if the provisos in (C) and (D) are not violated, then we have (a). But violation of either proviso implies violation of the other, and we have (b) - (e).

In this game all members of P_i are dummies. But dummies always get zero in a solution ([12], p. 398); hence X_i is not relevant. Q.E.D.

(A) Internal stability of $\mathcal{X}$. Take $\alpha, \beta \in \mathcal{X}$, and suppose $\alpha > \beta$ on some $R \in \mathcal{U}$. Then $\alpha > 0$ on R. Using the lemma, for each $i \in R$ choose $x_i \in X_i$ such that $x_i > 0$ on some $S_i \in \mathcal{W}_i$. For $i \notin R$, choose $x_i \in X_i$ arbitrarily. Then

$$\alpha_i x_i > \beta_i x_i \quad \text{on} \quad S_i, \quad \text{for all} \quad i \in R.$$

Hence

$$\sum \alpha_i x_i > \sum \beta_i x_i \quad \text{on} \quad \bigcup_R S_i \in \mathcal{W}.$$

This violates the internal stability of X. Hence $\mathcal{X}$ is internally stable.

(B) External stability of relevant X_i. Without loss of generality, suppose X_1 relevant. Take $\alpha \in \mathcal{X}$ such that $\alpha_1 > 0$. Let y_1 be an arbitrary element of $A_1 - X_1$. Using the lemma, choose $x_1 \in X_1$ such that $x_1 > 0$ on some $S_1 \in \mathcal{W}_1$. For $i \neq 1$, choose $x_i \in X_i$ arbitrarily. Define

$$x = \sum \alpha_i x_i$$

and

$$y = \alpha_1 y_1 + \sum_{i \neq 1} \alpha_i x_i .$$

Then $x \in X$ and $y \notin X$. By the external stability of X, there is a $z \in X$ that dominates y via some winning set $T = \bigcup_R T_i$, where $R \in \mathcal{U}$ and $T_i \in \mathcal{W}_i$ for all $i \in R$. Let us represent z as follows:

$$z = \sum \gamma_i z_i, \quad \text{with } \gamma \in \mathfrak{X} \text{ and all } z_i \in X_i .$$

Also define

$$w = \gamma_1 x_1 + \sum_{i \neq 1} \gamma_i z_i ;$$

this is also in X. Now z dominates y, but must not dominate x. Hence $1 \in R$ and

$$\gamma_1 z_1 > \alpha_1 y_1 \quad \text{on} \quad T_1 .$$

There are two possibilities. If $\gamma_1 > \alpha_1$, then w would dominate x via the winning set $(T - T_1) \cup S_1$, in violation of the internal stability of X. Thus $\gamma_1 \leq \alpha_1$, and we have $z_1 > y_1$ on T_1, so that $y_1 \in \mathrm{dom}_1\, z_1$. But z_1 is in X_1, and y_1 was arbitrary in $A_1 - X_1$; hence X_1 is externally stable.

(C) External stability of $\mathfrak{X}$, provided no relevant X_i is internally totally unstable. Let β be arbitrary in $\mathfrak{A} - \mathfrak{X}$. Using the proviso, choose $y_i \in A_i - \mathrm{dom}_i X_i$ for each relevant i; for other values of i, if any, let y_i be any element of A_i. Define

$$y = \sum \beta_i y_i .$$

Then $y \notin X$, and there is an $x \in X$ that dominates y via some winning set $S = \bigcup_R S_i$, where $R \in \mathcal{U}$ and $S_i \in \mathcal{W}_i$ for all $i \in R$. Let us represent x as follows:

$$x = \sum \alpha_i x_i , \quad \text{with } \alpha \in \mathfrak{X} \text{ and all } x_i \in X_i .$$

There are two possibilities to consider. If $\alpha > \beta$ on R then we have $\beta \in \mathrm{dom}\mathfrak{X}$, as required. But if $\alpha_i \leqq \beta_i$ for some $i \in R$, then

$$\alpha_i x_i > \beta_i y_i \geqq \alpha_i y_i \quad \text{on } S_i .$$

This inequality implies that $x_i > y_i$ on S_i, giving us $y \in \mathrm{dom}_i X_i$. But it also implies that $\alpha_i > 0$, making i relevant and giving us $y \notin \mathrm{dom}_i X_i$, by definition of y. This contradiction establishes the result.

(D) Internal stability of each relevant X_i provided $\mathfrak{X}$ is externally stable. Suppose X_1 (say) is

relevant but internally unstable. Choose $\alpha \in \mathfrak{X}$ such that $\alpha_1 > 0$, and choose $x_1, y_1 \in X_1$ such that $x_1 > y_1$ on some $S_1 \in \mathcal{W}_1$. Take $\epsilon > 0$ so small that

$$(\alpha_1 - \epsilon)x_1 > \alpha_1 y_1 \quad \text{on} \quad S_1,$$

and define $\beta \in \mathfrak{M}$ by

$$\beta_1 = \alpha_1 - \epsilon$$

$$\beta_i = \alpha_i + \epsilon', \quad \text{for} \quad i \neq 1,$$

where $\epsilon' = \epsilon/(m-1)$. Note that $\beta > 0$. There are two possibilities. Suppose first that $\beta \notin \mathfrak{X}$. Then, using the proviso, we may choose $\gamma \in \mathfrak{X}$ such that $\gamma > \beta$ on some $R \in \mathcal{U}$. Suppose instead that $\beta \in \mathfrak{X}$. Then define $\gamma = \beta$ and $R = \overline{12\ldots m}$. In either case, we have $\gamma \geq \beta$ on R. Also, we have $\gamma > 0$ on R; hence, by the lemma, for each $i \in R$ there is an $x_i \in X_i$ that is positive on some $S_i \in \mathcal{W}_i$. (Note that R may or may not contain 1; if it does, we shall use the x_1, S_1 previously specified, which have the stated property.) For any remaining values of i, choose $x_i \in X_i$ arbitrarily. Now define

$$x = \sum \gamma_i x_i$$

and

$$y = \alpha_1 y_1 + \sum_{i \neq 1} \alpha_i x_i .$$

Both x and y are in X. For every $i \in R$ we have

$$\gamma_i x_i \geq \beta_i x_i = \begin{cases} (\alpha_1 - \epsilon) x_1 > \alpha_1 y_1 & \text{on } S_1 \quad (\text{if } i = 1), \\ (\alpha_i + \epsilon') x_i > \alpha_i x_i & \text{on } S_i \quad (\text{if } i \neq 1). \end{cases}$$

It follows that $x > y$ on the winning set $\bigcup_R S_i$, violating the internal stability of X. Hence X_1 is internally stable. This completes the proof of the theorem.

<u>The generalized solution concept</u>. Let us pursue the implications of the possible failure of (a) in Theorem 2. We shall define a relation called <u>ρ-domination</u> between imputations of an m-person simple game, where ρ is either a positive scalar or a positive m-vector. In the first (scalar) case

$$\alpha \ \rho\text{-dominates} \ \beta \iff$$
$$\rho\alpha > \beta \ \text{ on some winning coalition.}$$

In the second (m-vector) case,

$$\alpha \ \rho\text{-dominates} \ \beta \iff$$
$$\rho_i \alpha_i > \beta_i \ \text{ for all } i \text{ in some winning coalition.}$$

A <u>ρ-solution</u> is a set of imputations that ρ-dominates precisely its complement.* Thus, the classical solutions are the 1-solutions.

What makes this solution concept so interesting, in addition to the application to compound games, is the empirical observation that for $\rho < 1$ there always seems to be a unique ρ-solution, which (at least in the scalar case) depends continuously on ρ and converges to a classical solution as $\rho \longrightarrow 1$. These matters will be discussed in another paper; here we shall just describe without proof the relationship between ρ-solutions and compound simple games.

A ρ-solution will be called <u>exact</u> if it weakly ρ-dominates itself, where weak ρ-domination is defined as above, but with "$\geq$" in place of "$>$". The classical solutions are automatically exact, since any imputation weakly 1-dominates itself. In fact, any ρ-solution with $\rho \geq 1$ is exact--but note that there are no ρ-solutions at all if $\rho > 1$.** A nontrivial example of an exact ρ-solution is provided by the game B_2, with $\rho = (1, 1/r)$, $r < 1$. Every ρ-solution of this game is a denumerable set, of the form $\{(1 - a, a), (1 - ar, ar), (1 - ar^2, ar^2), \ldots, \ldots, (1, 0)\}$, where a is an arbitrary parameter satisfying $r < a \leq 1$.

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*The definitions are easily extended to nonsimple games.

**Here and elsewhere "1" denotes "(1, ..., 1)" if ρ is a vector.

For $\rho < 1$ we have discovered no exact ρ-solutions; our experience has been that ρ-solutions with $\rho < 1$ invariably contain open sets, a property incompatible with exactness.

We can now state the result for compound games.

<u>Theorem 3</u>. <u>The set</u> $\mathfrak{X}[X_1, \ldots, X_m]$ <u>is a (classical) solution of</u> $Q[G_1, \ldots, G_m]$ <u>if and only if, for some</u> $\rho = (\rho_1, \ldots, \rho_m)$, $\mathfrak{X}$ <u>is an exact ρ-solution of</u> Q <u>and each relevant</u> X_i <u>is an exact</u> $(\frac{1}{\rho_i})$-<u>solution of its corresponding</u> G_i.

The proof will be given elsewhere. In view of this theorem and of the example of an exact solution for B_2 given above, the question with which we opened this section has been narrowed down to the question of whether there exists, for any simple game, an exact ρ-solution with $\rho < 1$ (ρ a scalar). Indeed, if there is a game G having such a solution, then the compound $B_2[G, B_1]$ will have a "classical" solution that is compound, but not composed of (classical) solutions. On the other hand, if no such game exists, then (a) in Theorem 2 always holds.

6. NONCOMPOUND SOLUTIONS

In Part I we found ways to construct noncompound solutions to product games, using the idea of monotonic (or semi-monotonic) variation of the component solutions. By a simple trick, we can extend these techniques to general compound games, and produce, in most cases, a

wealth of noncompound solutions.

Theorem 4. *Let* G *denote the compound game* $Q[G_1, \ldots, G_m]$, *and let* R *be a minimal winning coalition of* Q. *Then any solution of the product game*

$$H = \bigotimes_{i \in R} G_i$$

is also a solution of G, *it being understood that the players of* G *that are not players of* H *receive zero*.

The proof follows easily from the observation that, in any game, if the players outside a given winning coalition are replaced by dummies, then the solutions of the resulting, weaker game are precisely those solutions of the original game that exclude the players outside that coalition. (For a proof, see [9].) Thus, in the present case, the transition from G to H amounts to replacing Q by the pure bargaining game on R, and makes all players outside $\bigcup_R P_i$ dummies, but does not introduce any new solutions.

Theorem 4, together with Theorem 5 of Part I, produces only a few of the curvilinear (i.e., noncompound) solutions of $M_3[B_1, B_1, B_2^*]$ (see p. 15), namely, those that exclude one of the first two players. The other noncompound solutions, which lie in the interior cross-sections of Figs. 2b and 2c, can be

derived by a process of "inflation," (see Gillies [2], p. 335 ff). The exclusive solutions of larger compound games will often be similarly susceptible to inflation, though we cannot hope in general to obtain all solutions by this method.

It was shown in Part I (Theorem 3) that sums do not have noncompound solutions. If we attempt to apply Theorem 4 to a sum (i.e., $Q = B_m^*$), we obtain $|R| = 1$, and hence a trivial product H. Another case in which the solutions are necessarily of compound form is typified by our example $M_3[B_1, B_1, B_2]$ (Fig. 1). Here H is trivial in a different sense, every factor--and hence H itself--being a pure bargaining game. Generalizing from these cases, we are led to conjecture that every solution of a given compound game is a compound set if and only if (in our standard notation) for each $R \in \mathcal{U}^m$, either $|R| = 1$, or $|\mathcal{W}_i^m| = 1$ for all $i \in R$. However, we have not yet succeeded in establishing either the necessity or sufficiency of these conditions.

7. EXTENSION TO NONSIMPLE COMPOUNDS

In this section we generalize the definition of compound game. Let w be a set-function defined on the subsets of $\overline{12...m}$, satisfying the conditions

$$w(\emptyset) = 0 \quad \text{and} \quad w(\overline{12...m}) = g \geq 0$$

and interpreted as the characteristic function of a game Γ_w, not necessarily simple. The number g will be called the "modulus" of the game. Superadditivity is not required.

Given the simple components $G_1, \ldots, G_m$, we define the generalized compound

$$\Gamma_v = \Gamma_w[G_1, \ldots, G_m]$$

by means of the relation

$$v(S) = w(K(S)), \quad \text{all} \quad S \subseteq P,$$

where $K(S)$ is defined as before (p. 6). We see that v is a set-function of the same general type as w, and has the same modulus.

It is clear that this includes our previous definition, if we agree that the characteristic-function values of winning and losing coalitions in a simple game are 1 and 0, respectively. The interpretation of these generalized compounds, in terms of "committees" playing the quotient game Γ_w, is apparently straightforward. There are, however, some unexpected complications, which will be discussed below under "Invariant Compounds."

Modification of the imputation space. In working with generalized compounds, the most suitable imputation space for Γ_w proves to be the set of all nonnegative m-tuples

with sum g. This simplex will be denoted by $\mathfrak{U}(g)$. (Thus, $\mathfrak{U}(1) = \mathfrak{U}$.) This differs from the classical imputation space whenever it happens that not all $w(\bar{i}) = 0$. Nevertheless, we may define "effectivity" and "domination" in the usual way:

$$\mathcal{E}(\alpha) = \text{set}\{R \mid \textstyle\sum_R \alpha_i \leq w(R)\};$$

$$\alpha \succ\!\!\rightarrow \beta \iff \alpha > \beta \text{ on some } R \in \mathcal{E}(\alpha).$$

(Note that domination via one-element sets is now possible.) The solutions in this theory will be called $\mathfrak{U}(g)$-stable sets, to distinguish them from the solutions based on the classical imputation space. Even when the two imputation spaces are different, the two classes of solutions are very closely related, and usually overlap. The relationship is basically the same as that between "A-stable" and "E-stable" sets (see [7] or [1]).

Similarly, we shall use the notion of "A(g)-stable set" for the game Γ_v, where $A(g)$ denotes the simplex of nonnegative vectors on P with sum g. Note that $A(g)$ is more likely to conform to the classical imputation simplex than $\mathfrak{U}(g)$, since v usually comes out in zero-normalized form. Indeed, it takes a dictator in some G_i, coupled with a nonzero $w(\bar{i})$, to produce an exception.

This modification of the classical solution concept

is intended primarily as a means to an end--a technical device useful in generating classical solutions to our generalized compounds. It is analogous to the generalization introduced by von Neumann and Morgenstern ([12],pp. 363-364), in which they dissociate the modulus of the game (sum of payoffs in an imputation) from the characteristic function. Here we have dissociated the individual imputational minima from the characteristic function.*

The main theorem for nonsimple quotients. We recall that A_S denotes the set of $x \in A$ in which all players outside X get zero.

Theorem 5. For each $i = 1, \ldots, m$, let X_i be a solution of the simple game $G_i = \Gamma(P_i, \mathcal{W}_i)$, such that

$$X_i \subseteq \bigcup_{S \in \mathcal{W}_i^m} A_S .$$

Let $\mathfrak{X}$ be an $\mathfrak{n}(g)$-stable set of the general m-person game

*Actually, we have been using the present variant, without fanfare, throughout our study of simple games. It has been a distinction without a difference, however, except in the case of improper games with a dictator (i.e., games of the form $B_1 \oplus G$), where the classical imputation space would be trivial or even empty. Our divergence from the standard definition arose naturally as a result of the fact that, in the development of the theory of simple games per se, the difference between "winning" and "losing" is qualitative, not quantitative.

Thus, in the present extension to nonsimple quotients, the numerical values assumed by v and w should be viewed as refinements of the primitive notions of "winning" and "losing," rather than as measurements on a von Neumann-Morgenstern utility scale. At least, this view should be adopted if the new solutions are regarded as anything more than technical devices.

Γ_w, <u>where</u> $g = w(\overline{12...m})$. <u>Then the compound set</u> $X = \mathfrak{X}[X_1, \ldots, X_m]$ <u>is an</u> $A(g)$-<u>stable set of the generalized compound game</u> $\Gamma_w[G_1, \ldots, G_m]$.

Let us re-emphasize that the $A(g)$-stable sets are the same as the classical solutions, except for those cases in which some G_i has a one-person winning coalition and the corresponding $w(\overline{i})$ is nonzero.

The special condition imposed on the solutions X_i in Theorem 5 can be restated as follows: For every imputation in X_i, there is a <u>minimal</u> winning coalition of G_i that gets all the profit. Solutions with this property exist for all simple games; indeed, A_S itself is a solution of G_i for each $S \in \mathcal{W}_i^m$. All "main simple" solutions* also satisfy the condition. On the other hand (for example), the nonexclusive discriminatory solutions of M_3 do not. Using the latter, Owen [6] gives an example to show that the special condition on the X_i cannot be dropped.

The proof of Theorem 1 (Sec. 3, pp. 11-14) applies with very little change to Theorem 5. In part (A)(external stability), we must only change $\mathcal{U}$, $\mathcal{W}$, $\mathcal{V}$, and A, wherever these terms appear in the text, to $\mathcal{E}(\alpha)$, $\mathcal{E}(x)$, $\mathcal{V}(g)$, and $A(g)$, respectively. As for part (B)(internal stability), suppose that $x \leftarrow\!\!\!\rightarrow y$ for some $x, y \in X$. Then $x \leftarrow\!\!\!\rightarrow y$ via some "vital" set, i.e., a set $S \in \mathcal{E}(x)$ of the form

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*See [12], p. 444; [3], p. 254.

$$S = \bigcup_{i \in T} S_i, \quad \text{with} \quad S_i \in \mathcal{W}_i^m, \quad i \in T,$$

and we have $\alpha_i x_i > \beta_i y_i$ on S_i, for all $i \in T$, where $x = \sum \alpha_i x_i$, $y = \sum \beta_i y_i$, $\alpha, \beta \in \mathfrak{X}$, and $x_i, y_i \in X_i$, for all $i \in \overline{12...m}$. Moreover, the sum of all components of x with index in S is not greater than $v(S)$. Since $x_i > 0$ on S_i, $i \in T$, we have $x_i \in A_{S_i}$ by our special condition on X_i. This means that the sum of all components of x with index in S is equal to $\sum_T \alpha_i$, which in turn implies that $T \in \mathcal{E}(\alpha)$, since $v(S) = w(T)$. We can now proceed as in the proof of Theorem 1:

<u>Case 1</u>: $\alpha_i > \beta_i$, all $i \in T$. Then $\alpha \mapsto \beta$ via the effective set T, contradicting the internal stability of $\mathfrak{X}$.

<u>Case 2</u>: $\alpha_i \leq \beta_i$, some $i \in T$. Then we have, for that i,

$$\beta_i x_i \geqq \alpha_i x_i > \beta_i y_i \quad \text{on} \quad S_i.$$

Dividing through by β_i, we see that $x_i \mapsto y_i$ in the component game G_i, violating the internal stability of X_i. This completes the proof.

Note that "Case 2" does not make use of our special condition on the X_i, which was only needed to assure $T \in \mathcal{E}(\alpha)$.

<u>Some applications</u>. There is already an application of Theorem 5 with just three players. Consider $\Gamma_w[B_1, B_2]$, where $w(\overline{1}) = 0$, $w(\overline{2}) = a$, $w(\overline{12}) = g$, $0 < a < g$. The characteristic function of the compound is given by $v(\overline{23}) = a$, $v(\overline{123}) = g$, and $v(S) = 0$, $S \neq \overline{23}, \overline{123}$. Thus, $A(g)$ is classical, but $\mathfrak{A}(g)$ is not. The unique $\mathfrak{A}(g)$-stable set of Γ_w is the interval with endpoints $(0, g)$ and $(g-a, a)$. The resulting solution for Γ_v, which is unique, is the shaded trapezoid shown in Fig. 3.

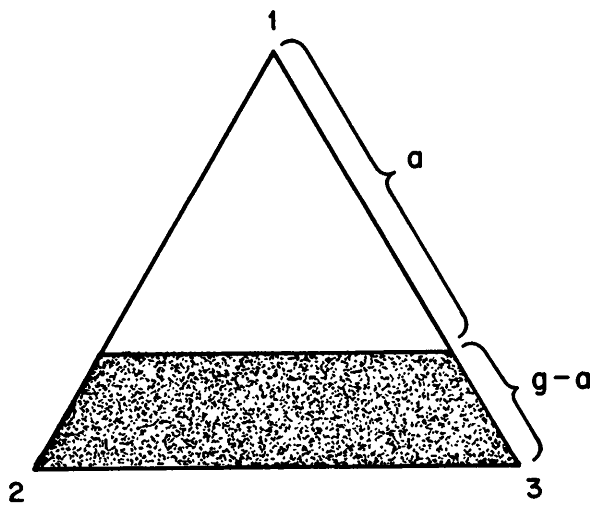

Fig. 3 — Solution of $\Gamma_w[B_1, B_2]$

For a more general application, let w be any positive additive function on the subsets of $\overline{12...m}$. Then the compound $\Gamma_w[G_1, \ldots, G_m]$ is just the "composition" of the games $G_1, \ldots, G_m$ (see [12], Chapter IX), provided that there are no dictators among the G_i and provided that each G_i is expressed by a characteristic function in "$0, w(\overline{i})$-normalized" form. The quotient, being inessential, has a unique $\mathfrak{N}(g)$-stable set, consisting of the single imputation $\alpha \mid \alpha_i = w(\overline{i})$. Since "Case 1" of the preceding proof cannot occur, we do not have to impose the special restriction on the solutions of the G_i. We therefore obtain the elementary but important class of solutions, first found by von Neumann and Morgenstern, in which no transfer of wealth takes place among the components ([12], p. 361).

A third application is to the compounding of "simple" solutions, as described in [3], p. 259.

<u>Invariant compounds</u>. The reader may have been puzzled by the form that our generalization has taken, feeling that it could have been formulated more in harmony with the classical, characteristic-function-based theory. In particular, starting from the "committee" idea, he may have felt entitled to insist on the usual invariance property, namely, that two compounds with the same components should be strategically equivalent whenever the quotients are strategically equivalent. Such a postulate, of course,

would conflict with the definitions that we have adopted above, in which 0 plays a special role.

An invariant theory of generalized compounds is indeed feasible, and for some purposes it may be preferable.* The counterpart to Theorem 5 goes through without difficulty (see the Corollary below), but there are other drawbacks. For one thing, we no longer have a true generalization of our original simple-quotient theory, since the latter is not invariant under strategic equivalence. The reason, again, lies in the qualitative distinction between winning and losing.

To illustrate, consider the two-person game D_1 in which 1 is a dictator and 2 a losing dummy. In the classical sense, both players are dummies, and the game is strategically equivalent to any other inessential two-person game--in particular, to the game D_2, in which 2 is the dictator, 1 the losing dummy. However, when used as quotients, these "equivalent" games behave quite differently. In fact, $D_1[G_1, G_2]$ is just the game G_1 with some dummies added, while $D_2[G_1, G_2]$ is the game G_2 with some dummies added.

Given simple games $G_1, \ldots, G_m$ and a general m-person

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*The authors of [3] do not clearly commit themselves for or against invariance, in their brief discussion of nonsimple quotients (p. 259). Under 0,1-normalization, which they adopt in their paper, the two methods of compounding are practically indistinguishable.

game Γ_w, let us define the <u>invariant compound</u>

$$\Gamma_v = \Gamma_w[G_1, \ldots, G_m]_{inv}$$

by the rule:

$$v(S) = w(K(S)) - \sum_{K(S)} w(\overline{i}), \quad \text{all} \quad S \subseteq P.$$

In this case we must assume that w satisfies

$$w(\emptyset) = 0 \quad \text{and} \quad w(\overline{12\ldots m}) \geq \sum_{i=1}^{m} w(\overline{i})$$

(compare p. 29). The classical imputation spaces can now be used for both Γ_w and Γ_v; our second condition on w ensures that they are not empty. Unfortunately, the same condition serves to exclude, among others, the important quotients B_m^*; hence sums cannot be formed in the invariant theory.

Note that in the invariant compound the characteristic function v is automatically 0-normalized, and is not affected by the addition of an arbitrary additive function to w. The "committees" therefore do not in general play the game Γ_w, but a strategically equivalent game. The reason for this bit of arbitrariness is the lack of any better way of assigning the individual minima of the

committee members.*

The contrast between the two kinds of generalized compound can be illustrated by means of the two-person quotient game Γ_n, where $n(S) = |S|$. This is strategically equivalent to D_1 and D_2, discussed above. We observe that $\Gamma_n[G_1, G_2]_{inv}$ is just the null game $v(S) = 0$. On the other hand, $\Gamma_n[G_1, G_2]$ is the von Neumann-Morgenstern composition of G_1 and G_2.

We do not wish to overstate the disadvantages of the "invariant" definition, since they are relatively minor. The decisive point against it, for our present purpose (i.e., obtaining an extension of Theorem 1), is simply the fact that the class of compounds generated by our first definition includes the class of invariant compounds. This is easily seen. Given w, let w_0 be defined by

$$w_0(S) = w(S) - \sum_S w(\bar{i}), \quad \text{all} \quad S \subseteq \overline{12\ldots m}.$$

Then we have

$$\Gamma_w[G_1, \ldots, G_m]_{inv} = \Gamma_{w_0}[G_1, \ldots, G_m]_{inv} = \Gamma_{w_0}[G_1, \ldots, G_m].$$

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*An alternative definition, which maintains the original payoff levels but has little else to recommend it, is the following:

$$v(S) = w(K(S)) - \sum_{K(S)} w(\bar{i}) + \sum_{i=1}^{m} (|S \cap P_i|/|P_i|) w(\bar{i}).$$

In view of this identity, the following corollary to Theorem 5 is immediate:

Corollary. *Let X_i, G_i be as in Theorem 5, and let $\mathfrak{X}$ be a solution of Γ_w. Then the set of all vectors* x *of the form*

$$x = \sum_{i=1}^{m} (\alpha_i - w(\bar{i}))x_i, \quad \alpha \in \mathfrak{X}, \quad x_i \in X_i \text{ all } i \in \overline{12\ldots m}$$

is a solution of $\Gamma_w[G_1, \ldots, G_m]_{inv}$.

Nonsimple components. Encouraged by the contact established with the composition theory of [12] (see page 36 above), we might proceed to search for a further generalization, in which neither the quotient nor the components would be simple games. For the record, we present one such generalization that we have found. It differs from the generalization proposed by Owen [6].

Let us require that all characteristic functions be nonnegative, and make the usual assumption about nonoverlapping player sets. Then we define $\Gamma_w[\Gamma_{v_1}, \ldots, \Gamma_{v_m}]$ to be the game Γ_v with characteristic function

$$v(S) = \sum_{\underline{R} \subset \overline{12\ldots m}} \; \sum_{\underline{T} \subset R} (-1)^{r-t} \, w(T) \min_{i \in R} v_i(S \cap P_i),$$

where we have written r for $|R|$, t for $|T|$. This definition has the following properties:

(a) If the components are simple games (with winning coalitions awarded 1, losing coalitions 0), then Γ_v is the <u>generalized compound</u>, as already defined.

(b) If w is additive, then Γ_v is the <u>von Neumann-Morgenstern</u> composition of the games with characteristic functions $w(\overline{i})v_i$, $i = 1, \ldots, m$.

(c) If Γ_w is the pure bargaining game B_m (i.e., $w(S) = 0$ except for $w(\overline{12\ldots m}) = 1$), then Γ_v is the <u>generalized product</u> of the components, defined by $v(S) = \min_i v_i(S \cap P_i)$.

(d) If Γ_w is B_m^* (i.e., $w(S) \equiv 1$ except for $w(\emptyset) = 0$), then Γ_v is the <u>generalized sum</u> of the components defined by $v(S) = \max_i v_i(S \cap P_i)$.

(e) Define the <u>dual</u> of an arbitrary function v on the subsets of P by the relation $v^*(S) = v(P) - v(P - S)$. Then the characteristic functions of $\Gamma_w[\Gamma_{v_1}, \ldots, \Gamma_{v_m}]$ and $\Gamma_{w^*}[\Gamma_{v_1^*}, \ldots, \Gamma_{v_m^*}]$ are duals.

We emphasize that it is only the game structure that is generalized under this proposed definition. We have not obtained any structure theorems for the solutions of these games.

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