MEMORANDUM
RM-4601-PR
JUNE 1965

ON BALANCED SETS AND CORES

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PREPARED FOR:
UNITED STATES AIR FORCE PROJECT RAND

The RAND Corporation
SANTA MONICA • CALIFORNIA

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This research is sponsored by the United States Air Force under Project RAND—Contract No. AF 49(638)-700—monitored by the Directorate of Operational Requirements and Development Plans, Deputy Chief of Staff, Research and Development, Hq USAF. Views or conclusions contained in this Memorandum should not be interpreted as representing the official opinion or policy of the United States Air Force.

The RAND Corporation
1700 MAIN ST • SANTA MONICA • CALIFORNIA • 90406

Published by The RAND Corporation

PREFACE

This Memorandum is one of a series of mathematical notes on topics in the theory of multi-person games. The mathematical idea introduced and developed here (that of a "balanced set") promises to be of general interest in the study of finite sets, zero-one matrices, and nonadditive set functions.

SUMMARY

In this note we establish a direct correspondence between the balanced sets of coalitions of a multi-person game and the conditions that determine whether the game has a core. A <u>balanced set</u> is a collection of subsets of a finite set that can be so weighted as to cover the whole set uniformly. The <u>core</u> of a game is the set of outcomes that cannot be blocked by any coalition.

CONTENTS

ON BALANCED SETS AND CORES

1. BALANCED SETS

Let $S_1, \ldots, S_p$ be distinct, nonempty, proper subsets of a finite set N. The set

$$\mathcal{S} = \{S_1, \ldots, S_p\}$$

is said to be balanced if there exist positive coefficients $\gamma_1, \ldots, \gamma_p$ such that

$$\sum_{j \mid i \in S_j} \gamma_j = 1, \qquad \text{all } i \in N.$$

Coefficients satisfying (1) are called weights for $\mathcal{S}$. If the weights are all equal to 1, then $\mathcal{S}$ is a partition of N; thus balanced sets may be regarded as generalized partitions.

A minimal balanced set is one that includes no other balanced set. It is easily seen that a minimal balanced set has a unique set of weights. Indeed, if γ and δ are distinct weight vectors for a balanced set $\mathcal{S}$, then any linear combination of the form $t\gamma + (1 - t)\delta$ will satisfy (1), and for some value of t we would have $t\gamma + (1 - t)\delta \geq 0$ but not > 0, showing that a proper subset of $\mathcal{S}$ is balanced.

Uniqueness of the weights in turn implies that a minimal balanced set can have no more than n elements, where n is the number of elements of N. The example of partitions shows that we cannot assert p = n.

The notion of balanced set seems to be of general interest in the study of finite sets, zero-one matrices, and nonadditive set functions. We shall present here an application of the theory of n-person games in characteristic-function form — i.e., to nonadditive set functions interpreted in a special way.

2. WEAK GAMES

In the following, a game is a function v from the subsets of a finite set (in general nonadditive) to the reals, such that $v(0) = 0$. The core of the game v may be defined as the set of all additive functions x such that

$$(1) \qquad x(S) \geq v(S), \quad \text{all} \quad S \subset N,$$

and

$$(2) \qquad x(N) = v(N).$$

Games having nonempty cores will be called weak.

The set of all weak games describes a closed convex polyhedral cone W in the linear space of dimension $2^n - 1$ with coordinates $v(S)$, $0 \subset S \subseteq N$. It is easily verified that this cone has full dimension. We shall be interested in discovering the boundary faces of W, that is, in finding a system of linear inequalities that will tell whether or not a given game has a nonempty core.

An important category, from the viewpoint of the intended interpretation, is the class of proper games, which are just the superadditive set functions:

$$(3) \qquad v(S) + v(T) \leq v(S \cup T), \quad \text{all} \quad S, T \subseteq N \ \text{with} \ S \cap T \neq 0.$$

The proper games, and the weak proper games as well, also form closed convex polyhedral cones of dimension $2^n - 1$, contained in W.

3. EXAMPLE: THE CASE n = 3

Let $N = \overline{123}$, and let v be given by*

$$(4) \qquad \begin{cases} v(\overline{1}) = v(\overline{2}) = v(\overline{3}) = 0; \\ v(\overline{12}) = a, \quad v(\overline{13}) = b, \quad v(\overline{23}) = c; \\ v(\overline{123}) = 1. \end{cases}$$

How does the core of this game depend on the parameters, a, b, c?

Applying (1), (2), we see that x is in the core if and only if

$$(5) \qquad \begin{cases} x(\overline{1}) \geq 0, \quad x(\overline{2}) \geq 0, \quad x(\overline{3}) \geq 0; \\ x(\overline{1}) + x(\overline{2}) \geq a, \quad x(\overline{1}) + x(\overline{3}) \geq b, \quad x(\overline{2}) + x(\overline{3}) \geq c; \\ x(\overline{1}) + x(\overline{2}) + x(\overline{3}) = 1. \end{cases}$$

Adding the middle group of inequalities, we obtain

$$2x(\overline{1}) + 2x(\overline{2}) + 2x(\overline{3}) \geq a + b + c.$$

Hence the condition

$$(6) \qquad a + b + c \leq 2$$

is necessary for the existence of a core. Similarly, it is necessary that

*We write $\overline{12}$ for $\{1, 2\}$, etc.

(7) $a \leq 1,\ b \leq 1,\ c \leq 1.$

Conditions (6) and (7) prove to be sufficient as well. Indeed, if they are satisfied we can take x in (5) as follows:

$$x(\overline{1}) = 1 - c,$$
$$x(\overline{2}) = \min(c,\ 1 - b),$$
$$x(\overline{3}) = \max(0,\ b + c - 1).$$

Conditions (7) are clearly implied by superadditivity (3), and hence are unnecessary if the game is known to be proper. Since every proper three-person game (with the trivial exception $v \equiv 0$) can be brought into the form of (4) by adding an additive function and multiplying by a positive scalar (operations that do not affect the existence of a core) we see that the whole question of the weakness of proper three-person games can be reduced to a single condition — the general form of (6):

(8) $$v(\overline{12}) + v(\overline{13}) + v(\overline{23}) \leq 2v(\overline{123}).$$

For $n > 3$ we shall find that matters are not so simple.

4. BALANCED FORMS

Let us call a linear form $L(v) = \Sigma\alpha_j v(S_j)$ <u>balanced</u> if it vanishes identically for additive functions v. Similarly, let us call an inequality (or equation) between two such forms <u>balanced</u> if their difference is balanced. Thus,

$$\sum_{j=1}^{p} \gamma_j v(S_j) \leqq \sum_{k=1}^{q} \delta_k v(T_k)$$

is balanced if for each $i \in N$, the sum of the γ_j such that $i \in S_j$ is equal to the sum of the δ_k such that $i \in T_k$. (Compare Sec. 1.)

<u>LEMMA 1</u>. Any linear inequality $L(v) \leq M(v)$ satisfied by all weak games v is necessarily balanced.

<u>Proof</u>. If a is an arbitrary additive function, then an inspection of (1), (2) reveals that x is in the core of v if and only if $x + a$ is in the core of $v + a$. Thus, v is weak if and only if $v + a$ is weak for all additive a. It only remains to point out that any <u>un</u>balanced inequality $L(v) \leq M(v)$ can be made false by adding a suitable additive function to v.

<u>COROLLARY 1</u>. The set of weak games can be defined by a system of simultaneous balanced inequalities.

COROLLARY 2. The set of weak proper games can be defined by a system of simultaneous balanced inequalities

These follow from the fact that a closed convex cone is the intersection of the closed half-spaces that contain it, plus the fact that (3) is balanced.

THEOREM 1. The game v has a nonempty core if and only if it satisfies all balanced inequalities of the form.

(9) $$\gamma_1 v(S_1) + \dots + \gamma_p v(S_p) \leqq v(N), \quad \text{all} \quad \gamma_j > 0.$$

Proof. Let v be a weak game. Then reducing the value of $v(S)$ for any $S \neq N$ will preserve weakness. This means that $v(S)$ must occur on the "less" side of any defining inequality in which it occurs at all (assuming positive coefficients). Hence, all defining inequalities can be put into the form of (9). By Lemma 1, they must be balanced.

It remains to show that all balanced inequalities (9) are satisfied by all weak games. Let v be weak, and let x be in its core. By the definition of "balanced," we have

$$\gamma_1 x(S_1) + \dots + \gamma_p x(S_p) \equiv x(N).$$

From this, (9) follows at once, using (1) and (2) and the positivity of the γ_j.

Theorem 1 characterizes the weak games However, it is not a sharp result, because many of the conditions (9) will be redundant in general.

5. WEIGHTS OF BALANCED SETS

Theorem 1 gives us the motivation for studying sets $\mathcal{S} = \{S_1, \ldots, S_p\}$ that are balanced in the sense of Sec. 1. In that section we proved that a minimal balanced set has a unique weight vector. By an extension of that argument, we can prove the following:

> LEMMA 2. Let $\mathcal{S}$ be a balanced set. The closure of the set of weight vectors for $\mathcal{S}$ is compact and convex, and its extreme points are precisely the weight vectors of the minimal balanced sets included in $\mathcal{S}$.

Proof. Convexity is obvious, as is boundedness, and hence compactness of the closure. Points in the closure are just the weight vectors for balanced sets $\mathcal{T} \subseteq \mathcal{S}$. Let γ be the weight vector for a minimal balanced set $\mathcal{T} \subseteq \mathcal{S}$. If γ were not extreme, then points of the form $\gamma \pm \rho$ would be weight vectors for subsets of $\mathcal{S}$, with $\rho \neq 0$. Since $\rho_j = 0$ for $S_j \notin \mathcal{T}$, this would contradict the uniqueness of γ. Conversely, let γ be a weight vector of a nonminimal balanced set $\mathcal{T} \subseteq \mathcal{S}$. Let $\mathcal{U} \subset \mathcal{T}$ be minimal balanced, with weight vector δ. Then points of the form $\gamma \pm (\delta - \gamma)t$ will also be weight vectors for $\mathcal{T}$, showing that γ is not extreme.

> COROLLARY. A balanced set is the union of the minimal balanced sets that it contains.

It is useful sometimes to regard weight vectors as points in a linear space of $2^n - 1$ dimensions with coordinates γ_S, $0 \subset S \subseteq N$, giving γ_N the value -1 and all other nonpositive coordinates the value 0. Let Γ denote the set of all weight vectors of all balanced sets. Then Γ is compact and convex, and its extreme points are the unique weight vectors of all minimal balanced sets. (Lemma 2, with $\mathscr{S} = \{S | 0 \subset S \subset N\}$.) The set of all nonnegative multiples of points in Γ forms a cone W^* which is <u>dual</u> to the cone W of weak games, since, by Theorem 1, v is in W if and only if

$$(10) \qquad \sum_{S \subseteq N} \gamma_S v(S) \leq 0, \qquad \text{all} \quad \gamma \in W^*.$$

But since W^* is a proper cone (contains no line), it is spanned by its extremal rays, and it suffices to verify the inequality in (10) for the extreme points of Γ, i.e., for the weight vectors for the minimal balanced sets. Thus we have shown the following:

<u>THEOREM 2</u>. The game v has a nonempty core if and only if, for every minimal balanced set $\mathscr{S} = \{S_1, \ldots, S_p\}$, we have

$$\sum_{j=1}^{p} \gamma_j v(S_j) \leq v(N),$$

where γ is the unique weight vector for $\mathscr{S}$.

Clearly, no smaller set of inequalities will serve. Geometrically, each minimal balanced set corresponds to a face of dimension $2^n - 2$ of the polyhedral cone W.

6. PROPER WEAK GAMES

We call a minimal balanced set proper if no two of its elements are disjoint.

THEOREM 3. The proper game v has a nonempty core if and only if, for every proper minimal balanced set $\mathscr{S} = \{S_1, \ldots, S_p\}$, we have

$$(11) \qquad \sum_{j=1}^{p} \gamma_j v(S_j) \le v(N),$$

where γ is the unique weight vector for $\mathscr{S}$.

Proof. If $\mathscr{S}$ is a minimal balanced set, let us write $L(\mathscr{S}, v)$ for $\Sigma_{S \in \mathscr{S}} \gamma_S v(S)$, where $\{\gamma_S | S \in \mathscr{S}\}$ are the unique weights for $\mathscr{S}$. The heart of the proof is the following lemma.

LEMMA 3. If $\mathscr{S}$ is an improper minimal balanced set, then $L(\mathscr{S}, v) \le v(N)$ is implied by $L(\mathscr{T}, v) \le v(N)$ together with the superadditivity conditions (3), where $\mathscr{T}$ is some minimal balanced set having fewer pairs of disjoint elements than $\mathscr{S}$.

Proof of Lemma 3. Let γ be the weight vector for $\mathscr{S}$, and assume that $Q, R \in \mathscr{S}$ with $Q \cap R = 0$ and $\gamma_Q \le \gamma_R$. Let $T = Q \cup R$. Then T is not in $\mathscr{S}$, for if it were we could eliminate it by transferring its weight to Q and R, contradicting the minimality of $\mathscr{S}$. Define $\mathscr{T}$ by

$$\mathscr{T} = \begin{cases} \mathscr{S} \cup \{T\} - \{Q\} & \text{if } \gamma_Q < \gamma_R, \\ \mathscr{S} \cup \{T\} - \{Q, R\} & \text{if } \gamma_Q = \gamma_R. \end{cases}$$

We assert: (a) $\mathscr{T}$ is balanced; (b) $\mathscr{T}$ is minimal balanced; and (c) $\mathscr{T}$ has fewer disjoint pairs than $\mathscr{S}$.

(a): Assign a weight of γ_Q to T, and a weight of $\gamma_R - \gamma_Q$ to R if $\gamma_Q < \gamma_R$. Assign all other sets in $\mathscr{T}$ the same weight as they received in $\mathscr{S}$. Then these weights balance $\mathscr{T}$.

(b): Suppose $\mathscr{U} \subseteq \mathscr{T}$ is balanced. Since $\mathscr{S}$ is minimal, we must have $T \in \mathscr{U}$. Define $\mathscr{V} = (\mathscr{U} - \{T\}) \cup \{Q, R\}$. Then $\mathscr{V}$ is balanced, using the same weights as for $\mathscr{U}$ but transferring the weight of T to Q and R. But $\mathscr{V} \subseteq \mathscr{S}$. Hence $\mathscr{V} = \mathscr{S}$. Hence either $\mathscr{U} = \mathscr{T}$, or we have $\mathscr{U} = \mathscr{S} \cup \{T\} - \{Q, R\}$ and $\gamma_Q < \gamma_R$. But, by the corollary to Lemma 2, we could in this case have chosen $\mathscr{U}$ originally so that $R \in \mathscr{U}$. Hence $\mathscr{U} = \mathscr{T}$.

(c): Each disjoint pair in $\mathscr{T}$ that is not also in $\mathscr{S}$ is of the form $\{S, T\}$. Corresponding to it is the disjoint pair $\{Q, T\}$, which is in $\mathscr{S}$ but not in $\mathscr{T}$. But $\mathscr{S}$ also contains the disjoint pair $\{Q, R\}$, which is not in $\mathscr{T}$.

To complete the proof of the lemma, we need only note that

$$L(\mathcal{J}, v) = L(\mathcal{S}, v) + \gamma_Q v(T) - \gamma_Q v(Q) - \gamma_Q v(R) \geq L(\mathcal{S}, v),$$

using the weights for $\mathcal{J}$ found in (a) above, and superadditivity.

Using the lemma, we can eliminate all the inequalities (11) based on improper minimal balanced sets, without changing the class of <u>proper</u> games defined by the remaining inequalities. This completes the proof of the theorem.

We believe that Theorem 3 is sharp, in the sense that no smaller set of inequalities (11) will serve, but we do not have a proof.

7. THE CASE $n = 4$

A compilation of minimal balanced sets for $n \leqq 6$ will be given in a separate Memorandum. Here we describe the situation for $n = 4$.

There are actually 41 different minimal balanced sets, but they can be grouped into nine equivalence classes, corresponding to the nine sets in the table. Permutations of the players generate the other 32. We have used integer weights in the table, multiplying each set of γ_j by its least common denominator, or "depth." Thus, the partitions are the minimal balanced sets of depth 1.

The table illustrates the simple fact that if the elements of any minimal balanced set are replaced by their complements, the result is minimal balanced. For example, "f" and "h" are related in this way, also "c" and "d," and "e" and "i," while complementation transforms "g" into a permuted form of itself.

Only "d," "h," and "i" in the list are proper. Consulting the multiplicities, we see that eleven inequalities are required to delimit the set of proper 4-person games having nonvacuous cores, in contrast to the single inequality (8) that sufficed for proper 3-person games.

Table 1

BALANCED SETS FOR $n = 4$

		Weights	Depth	Multiplicity
a.	$\{\overline{12}, \overline{34}\}$	1, 1	1	3
b.	$\{\overline{123}, \overline{4}\}$	1, 1	1	4
c.	$\{\overline{12}, \overline{3}, \overline{4}\}$	1, 1, 1	1	6
d.	$\{\overline{123}, \overline{124}, \overline{34}\}$	1, 1, 1	2	6
e.	$\{\overline{1}, \overline{2}, \overline{3}, \overline{4}\}$	1, 1, 1, 1	1	1
f.	$\{\overline{12}, \overline{13}, \overline{23}, \overline{4}\}$	1, 1, 1, 2	2	4
g.	$\{\overline{123}, \overline{14}, \overline{24}, \overline{3}\}$	1, 1, 1, 1	2	12
h.	$\{\overline{123}, \overline{14}, \overline{24}, \overline{34}\}$	2, 1, 1, 1	3	4
i.	$\{\overline{123}, \overline{124}, \overline{134}, \overline{234}\}$	1, 1, 1, 1	3	1
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