A BOMBER-FIGHTER DUEL

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The problem considered here is that of a fighter, capable of firing a single rocket burst, attacking a bomber, which defends itself by firing intermittently. The value of the game and a good strategy for the bomber are presented for certain cases, the main restriction being that the bomber has a relatively small amount of ammunition. Even for this case, a good strategy for the fighter has not yet been discovered. The strategies for the bomber are those discovered by H. K. Weiss at Aberdeen; thus our results are in part a verification of Weiss' work.

The payoff function. Consider a fighter $F$ and a bomber $B$ approaching each other along predetermined paths at predetermined speeds during a time interval $(0, 1)$. $F$ chooses a time $t$ at which to fire his rockets, and leaves the scene of action immediately thereafter. $B$ chooses a subset $S$ of $(0, 1)$ of measure $\delta$ (fixed) on which to concentrate his fire: the points in $S$ specify the times at which the trigger is depressed. Suppose a bullet fired by $B$ at time $x$ reaches $F$ at time $T(x) > x$. If $F$ fires his rockets at $T(x)$, only bullets fired by $B$ before $x$ can be effective. We describe this by saying that $F$ chooses $x$, $0 \leq x \leq 1$, meaning that he decides to fire at time $T(x)$, and denote $F$'s accuracy when he chooses $x$ (i.e., when he fires at $T(x)$) by $a(x)$. If the value of the conflict to $F$ is $\alpha > 0$ if $B$ is destroyed, $-\beta \leq 0$ if $F$ is destroyed, and 0 if both survive (both cannot be destroyed), the
payoff to F is

\[ v(x, S) = \alpha a(x) \Phi(x, S) - \beta \left[ 1 - \Phi(x, S) \right] \]

\[ = \left[ \alpha a(x) + \beta \right] \Phi(x, S) - \beta, \]

where \( \Phi(x, S) \) is the probability that F survives all bullets fired before \( x \), when B uses firing policy S. Conventionally, the accuracy of B is specified by a function \( r(x) \), such that \( r(x)dx \) is the probability of a kill during the time interval \( x, x + dx \) if the trigger is depressed throughout the interval. Then

\[ - \int_0^x s(y) r(y) dy \]

\[ \Phi(x, S) = e \]

where \( s(y) \) is the characteristic function of S. Now if \( p(y) \) is any function, \( 0 \leq p(y) \leq 1 \), \( \int_0^1 p(y) dy = \delta \), there is a set S of measure \( \delta \) such that \( \Phi(x, S) \) approximates

\[ - \int_0^x p(y) r(y) dy \]

\[ \Phi(x, p) = e \]

uniformly in \( x \). We therefore extend B's admissible pure strategies to the class of p's defined above. \( p(y) \) may now be interpreted as the intensity of fire at time \( y \), and the restriction \( p(y) \leq 1 \) means simply that there is a maximum intensity of fire which cannot be exceeded.

Omitting the additive constant \( -\beta \), which does not influence the analysis of the game, we have the payoff
\[ v(x, p) = A(x) \Phi(x, p), \]

\[ 0 \leq x \leq 1, \ 0 \leq p(y) \leq 1, \ \int_{0}^{1} p(y)dy = \delta. \] We shall suppose that \( A(x), r(y) \) are continuous, non-negative, and monotone increasing.

**Solution of the game.** Shapley [RM-118] has proved that \( B \) never needs to use mixed strategies. In fact, if \( G(p) \) is any mixed strategy, he shows that the pure strategy \( p^*(y) = \int p(y)dG(p) \) is at least as effective as \( G(p) \) against every \( x \). Thus

\[ \inf_{G(p)} \sup_{x} v[x, G] = \inf_{p} \sup_{x} v(x, p) = \nu, \]

where \( \nu \) is the value of the game.

We now prove that if one pure strategy \( p \) is better than another against some \( x \), there is another \( x \) against which it is worse. Formally,

**Lemma.** For any two strategies \( p_1(y), p_2(y), v(x, p_1) \geq v(x, p_2) \) for all \( x \) implies \( p_1 = p_2 \) almost everywhere (so that \( v(x, p_1) = v(x, p_2) \) for all \( x \)).

**Proof.** If \( v(x, p_1) \geq v(x, p_2) \) for all \( x \), then

\[ \int_{0}^{x} p_1(y)r(y)dy \leq \int_{0}^{x} p_2(y)r(y)dy \text{ for all } x. \]

Writing \( p_2 - p_1 = f, \int_{0}^{x} f(y)r(y)dy = \Psi(x) \), we have \( f = \frac{\Psi'}{r} \), so that

\[ \int_{0}^{1} \frac{d\Psi(x)}{r(x)} = 0, \ \Psi(x) \geq 0 \text{ for all } x. \] Integrating by parts yields

\[ \lim_{\epsilon \to 0} \left[ \frac{\Psi(1)}{r(1)} - \frac{\Psi(\epsilon)}{r(\epsilon)} + \int_{\epsilon}^{1} \frac{\Psi(x)dr(x)}{r^2(x)} \right] = 0. \]
But \( \Psi(\varepsilon) \leq r(\varepsilon) \int_0^\varepsilon |f(x)| \, dx \) (\( r \) is monotone increasing), so that
\[
\frac{\Psi(\varepsilon)}{r(\varepsilon)} \to 0 \text{ as } \varepsilon \to 0.
\]
Thus \( \int_\varepsilon^1 \frac{\Psi(x)dr(x)}{r^2} = 0 \) for every \( \varepsilon > 0 \),

\[
\frac{\Psi}{r^2} = 0 \text{ a.e., } \Psi = 0 \text{ a.e., } fr = 0 \text{ a.e., } \text{ and } f = 0 \text{ a.e.}
\]

Suppose the functions \( A(x), r(x) \) satisfy the following condition

(1): there is a \( c_0 < 1 \) such that \( m(x) = \frac{A'(x)}{A(x)r(x)} \leq 1 \) for \( c_0 \leq x \leq 1 \).

This condition is satisfied by a large class of functions \( A, r \)
(including some that look reasonable, although actual data will have
to be inspected to check this; cf. the example below). Define

\[
\delta_0 = \int_{c_0}^1 m(x) \, dx.
\]

If \( \delta \leq \delta_0 \), there is a \( c \geq c_0 \) such that

\[
\int_c^1 m(x) \, dx = \delta.
\]

Thus, if (1) holds and if \( \delta \leq \delta_0 \), the function

\[
p_0(x) = \begin{cases} 
0 & \text{for } 0 \leq x \leq c \\
m(x) & \text{for } c < x \leq 1
\end{cases}
\]

is an admissible strategy, where \( c \) is chosen so that \( \int_c^1 m(x) \, dx = \delta \).

**Theorem.** If \( p_0 \) is an admissible strategy, it is a good strategy,
and the value of the game is \( A(c) \).

**Proof.** By substitution, we obtain

\[
v(x, p_0) = \begin{cases} 
A(x) & \text{for } 0 \leq x \leq c \\
A(c) & \text{for } c \leq x \leq 1
\end{cases}.
\]

Thus \( v(x, p_0) \) is constant over the interval \((c, 1)\). It was in fact
this property of \( p_0 \) which led to its suggestion by Weiss. Since
\[ \sup_x v[x, p_0] = A(c), \] the theorem will be proved if we show that, for any \( p(x) \), \( \sup_x v[x, p] \geq A(c) \). It follows from the lemma that, unless \( p = p_0 \) a.e., there is an \( x^* \) with \( v[x^*, p] > v[x, p_0] \). This \( x^* \) must exceed \( c \), since always \( v(x, p) \leq A(x) \), and \( v(x, p_0) = A(x) \) for \( x \leq c \). Then \( v(x^*, p) > v(x^*, p_0) = A(c) \), and the theorem is proved. In fact we have \( \sup_x v(x, p) > A(c) \) unless \( p = p_0 \) a.e., so that \( p_0 \) is the only good strategy.

**Example.** As an example, consider the case \( \beta = 0, \alpha = 1, r(x) = k^2 x \). Then \( \lambda(x) = x, m(x) = (kx)^{-2} \). If \( k > 1 \), (2) is satisfied, so that we suppose \( k > 1 \). \( c_0(k) = k^{-1}, \delta_0 = \int_{k^{-1}}^1 (kx)^{-2} dx = \frac{k^{-1}}{k^2} \). The value \( v(k) \) to \( F \) when \( B \) has the maximal amount of ammunition \( \delta_0 \) is \( \frac{1}{k} \).

The graphs of \( \delta_0(k) \), \( v(k) \) are shown below:

For fixed \( k \), the good strategy for \( B \) is shown below, where \( c \) is chosen so that \( \int_c^1 m(x)dx = \delta \).