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ON A BESSEL FUNCTION INTEGRAL

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PREFACE AND SUMMARY

In the course of RAND's continuing investigations on the propagation of electromagnetic pulses, it became necessary to evaluate an integral involving exponential and Bessel functions. It was found that this integral had been evaluated incorrectly in the literature, and the error has been perpetuated for nearly 30 years. This Memorandum evaluates the integral correctly, and shows that although the incorrect result had been expressed entirely in terms of algebraic and logarithmic functions, the correct result is considerably more complicated, and involves incomplete gamma functions.

ON A BESSEL FUNCTION INTEGRAL

In the course of an investigation on the propagation of electromagnetic pulses, it became necessary to evaluate the integral

$$F_\nu(a,b,p) = \int_b^\infty e^{-pt} \left(\frac{t-b}{t+b}\right)^{\nu/2} K_\nu\left(a\sqrt{t^2-b^2}\right)dt \quad -1 < \mathrm{Re}\,\nu < 1 \tag{1}$$

where K_ν denotes the modified Bessel function of the second kind. While the corresponding integral for Bessel functions of the first kind is well known,(1)

$$\int_b^\infty e^{-pt}\left(\frac{t-b}{t+b}\right)^{\nu/2} I_\nu\left(a\sqrt{t^2-b^2}\right)dt = \frac{e^{-b\sqrt{p^2-a^2}}}{\sqrt{p^2-a^2}}\left[\frac{a}{p+\sqrt{p^2-a^2}}\right]^\nu \tag{2}$$

a correct form for Eq. (1) could not be found. The integral is presented in an early collection of Laplace transforms by McLachlan and Humbert,(2) based on an earlier paper by McLachlan,(3) but the result is incorrect. McLachlan formed a linear combination of the right sides of Eq. (2) and the corresponding expression with ν replaced by $-\nu$, based on the relation

$$K_\nu(z) = \frac{\pi}{2\sin\nu\pi}\left[I_{-\nu}(z) - I_\nu(z)\right] \tag{3}$$

and asserted that it was equivalent to replacing I_ν by K_ν in the left-hand sides. However, the factor multiplying I_ν in the integral depends on ν, and hence the analysis is not correct. While the text of Ref. 1 does not reproduce the erroneous result on the indicated page, it does transcribe the error on a later page (Ref. 1, p. 205, pairs 15-18), the pairs referring to modified Bessel functions of the second kind with argument of phase $\pi/4$(Kelvin functions). This Memorandum will show that the correct result for Eq. (1) involves incomplete gamma functions. The special case $\nu = 0$ has been evaluated correctly previously.[4]

To evaluate the integral, first make the translation $t - b = u$; then employ the integral representation,[5]

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty d\tau\ \tau^{-\nu-1} \exp\left(-\tau - \frac{z^2}{4\tau}\right) \tag{4}$$

The original integral F_ν converges for $|\mathrm{Re}\ \nu| < 1$, $\mathrm{Re}\ p > |\mathrm{Re}\ a|$. Temporarily assume $|\arg a| < \pi/4$, which is necessary for the convergence of Eq. (4). With these substitutions, there results after some cancellation

$$F_\nu = e^{-pb} a^\nu 2^{-\nu-1} \int_0^\infty e^{-pu} u^\nu du \int_0^\infty d\tau\ \tau^{-\nu-1} \exp\left(-\tau - \frac{a^2\ u(u + 2b)}{4\tau}\right) \tag{5}$$

Make the substitution $\tau = a^2u/2x$. The exponential character of the integrand at infinity permits the order of integration to be changed, and the result is

$$F_\nu = \frac{1}{2} e^{-pb} a^{-\nu} \int_0^\infty dx\ x^{\nu-1}\ e^{-bx} \int_0^\infty du \exp\left[-u\left(p + \frac{x}{2} + \frac{a^2}{2x}\right)\right] \tag{6}$$

The integral over u converges if the real part of the coefficient of -u in the exponent is positive. With the previously assumed restriction $|\arg a| < \pi/4$, the requirement is simply Re $p > 0$. Under this condition the u integral may be evaluated, and yields

$$F_\nu = e^{-pb} a^{-\nu} \int_0^\infty \frac{dx\, x^\nu e^{-bx}}{x^2 + 2px + a^2} \tag{7}$$

Define three numbers x_1, x_2, and s by the relations

$$x_1 = p + \sqrt{p^2 - a^2} = p + s \tag{8}$$

$$x_2 = p - \sqrt{p^2 - a^2} = p - s \tag{9}$$

Then the denominator in Eq. (7) may be separated into partial fractions, and yields

$$F_\nu = \frac{e^{-pb}}{2s} \frac{1}{a^\nu} \int_0^\infty dx\, x^\nu e^{-bx} \left[\frac{1}{x+x_2} - \frac{1}{x+x_1}\right] \tag{10}$$

This is now a standard form (Ref. 1, p. 137, pair 7) giving the result

$$F_\nu = \frac{e^{-pb}}{2s} \frac{\Gamma(\nu+1)}{a^\nu} \left[x_2^\nu e^{bx_2} \Gamma(-\nu, bx_2) - x_1^\nu e^{bx_1} \Gamma(-\nu, bx_1)\right] \tag{11}$$

The exponents may be simplified, yielding

$$F_\nu = \frac{\Gamma(\nu+1)}{2s\, a^\nu} \left[x_2^\nu e^{-bs} \Gamma(-\nu, bx_2) - x_1^\nu e^{bs} \Gamma(-\nu, bx_1)\right] \tag{12}$$

Here $\Gamma(-\nu,z)$ denotes the incomplete gamma function, defined for $|\arg z| < \pi$ by the relation[6]:

$$\Gamma(-\nu,z) = \int_z^\infty t^{-\nu-1}\, e^{-t}\, dt \tag{13}$$

The convergence condition on the u integral in Eq. (6) is equivalent to requiring that the real part of the denominator in Eq. (7) must be positive. For $|\arg a| < \pi/4$, which condition was imposed to permit the use of the integral representation of Eq. (4), the real part of the denominator is positive for all x if Re $p > 0$. However, both sides of Eq. (12) are analytic functions of p and a for $|\arg a| < \pi/2$, Re $p >$ Re a, and thus Eq. (12) is valid in the wider region.

It may easily be verified that for $\nu = 0$, Eq. (12) reduces to the result of Ref. 4, which is in terms of exponential integral functions. Other interesting cases are for $\nu = \pm 1/2$, for which the K_ν function becomes an exponential, and the incomplete gamma function becomes an error function. The result becomes

$$\int_b^\infty dt(t + b)^{-\frac{1}{2}} \exp -\left(pt + a\sqrt{t^2-b^2}\right)$$
$$= \sqrt{\pi/2}\,\frac{1}{s}\left[x_2^{\frac{1}{2}}\, e^{-bs}\, \mathrm{erfc}\sqrt{bx_2} - x_1^{\frac{1}{2}}\, e^{bs}\, \mathrm{erfc}\sqrt{bx_1}\right] \tag{14}$$

$$\int_b^\infty dt(t-b)^{-\frac{1}{2}} \exp -\left(pt + a\sqrt{t^2-b^2}\right)$$

$$= \sqrt{\pi/2}\,\frac{1}{s}\left[x_1^{\frac{1}{2}}\,e^{-bs}\,\mathrm{erfc}\,\sqrt{bx_2} - x_2^{\frac{1}{2}}\,e^{bs}\,\mathrm{erfc}\,\sqrt{bx_1}\;\right] \tag{15}$$

Finally, consider the limit as $|\arg a|$ approaches $\pi/2$. If p is positive, x_2 approaches a negative real number. The incomplete gamma function of Eq. (13) satisfies the relation[6]

$$\Gamma(-\nu,z) = \Gamma(-\nu)\,\left[1-z^{-\nu}\,\gamma^*(-\nu,z)\right] \tag{16}$$

where γ^* is a single valued analytic function of ν and z with no finite singularities. For $|\arg a| = \pi/2$, F_ν may be re-expressed in terms of unmodified Bessel and Neumann functions. The Bessel function result is well known.[1] After paying suitable attention to the phase of x_1 and x_2, the Neumann function result is as follows

$$\int_b^\infty dt\; e^{-pt}\left(\frac{t-b}{t+b}\right)^{\frac{\nu}{2}} Y_\nu\left(a\sqrt{t^2-b^2}\right)$$

$$= \frac{1}{(ab)^\nu r \sin\pi\nu}\left[e^{-br}\left\{(bz_2)^\nu \cos\pi\nu - \gamma^*(-\nu,-bz_2)\right\} + e^{br}\left\{(bz_1)^\nu - \gamma^*(-\nu,bz_1)\right\}\right] \tag{17}$$

where the notation has been introduced

$$z_1 = \sqrt{p^2 + a^2} + p = r + p \tag{18}$$

$$z_2 = \sqrt{p^2 + a^2} - p = r - p \tag{19}$$

The function γ^* is graphed on page 261 of Ref. 6.

The result for $\nu = 0$ can be expressed as exponential integral functions, while the result for $\nu = \pm 1/2$ is given in terms of error functions and Dawson's integral. Thus,

$$\int_b^\infty dt\ e^{-pt}\ Y_o(a\sqrt{t^2-b^2}) = \frac{1}{\pi r}\left[e^{-br}\ Ei(bz_2)+e^{br}\ E_1(bz_1)\right] \tag{20}$$

$$\int_b^\infty dt\ e^{-pt}\ \frac{\cos\ a\sqrt{t^2-b^2}}{(t+b)^{\frac{1}{2}}} = \frac{1}{r}\left[\sqrt{\pi/2}\ z_1^{\frac{1}{2}}\ e^{br}\ \mathrm{erfc}\ \sqrt{bz_1}\ -z_2^{\frac{1}{2}}\ e^{-br}\ D(\sqrt{bz_2})\right] \tag{21}$$

$$\int_b^\infty dt\ e^{-pt}\ \frac{\sin\ a\sqrt{t^2-b^2}}{(t-b)^{\frac{1}{2}}} = -\frac{1}{r}\left[\sqrt{\pi/2}\ z_2^{\frac{1}{2}}\ e^{br}\ \mathrm{erfc}\sqrt{bz_1}+z_1^{\frac{1}{2}}\ e^{-br}D(\sqrt{bz_2})\right] \tag{22}$$

Here the notation of Ref. 6 has been employed

$$Ei(z) = -\int_{-z}^\infty \frac{e^{-t}}{t}\ dt \tag{23}$$

$$E_1(z) = \int_z^\infty \frac{e^{-t}}{t}\ dt \tag{24}$$

$$D(z) = \int_0^z e^{t^2}\ dt \tag{25}$$

The integral in Eq. (23) is a principal value. The integral in Eq. (25) times $\exp(-z^2)$ is tabulated on page 319 of Ref. 6.

If desired, all the incomplete gamma functions can be expressed in terms of confluent hypergeometric functions.

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