U. S. AIR FORCE

PROJECT RAND

RESEARCH MEMORANDUM

THE NOISY DUEL, ONE BULLET EACH,
ARBITRARY NON-MONOTONE ACCURACY

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Consider a duel between two opponents, each having one bullet, with accuracies $p_1(t)$, $p_2(t)$ at time t, $a \leq t \leq b$, $0 \leq p_i(t) \leq 1$. We suppose that p_1 and p_2 are continuous functions of t, but not necessarily monotone.

If I fires at time $T < b$ and misses, and II has not yet fired, II's best course is clearly to fire at a time $t \geq T$ for which $p_2(t)$ assumes its maximum value in the interval (T, b). This maximum value of $p_2(t)$ in the interval (T, b) we denote by $m_2(T)$. Similarly $m_1(T) = \max p_1(t)$, $T \leq t \leq b$. With this obvious reduction, the strategy of each player is specified by a single time t, meaning that he will fire at time t if the opponent has not fired previously.

For a pair of strategies (t_1, t_2), the pay-off to I is

$$f(T) = p_1(T) - \left[1 - p_1(T)\right] m_2(T) \text{ if } T = t_1 < t_2$$

$$g(T) = - p_2(T) + \left[1 - p_2(T)\right] m_1(T) \text{ if } T = t_2 < t_1$$

$$h(T) = p_1(T) - p_2(T) \text{ if } T = t_1 = t_2,$$

where $T = \min(t_1, t_2)$.

We have thus reduced the duel to the following game G. We are given three continuous functions $f(t)$, $g(t)$, $h(t)$ defined on the interval $a \leq t \leq b$. Each player chooses a number t in (a, b) without knowing the other's choice. The pay-off to I is $k(t_1, t_2)$, where $k(t_1, t_2)$ is either $f(T)$, $g(T)$, or $h(T)$ according as $T = t_1 < t_2$, $T = t_2 < t_1$, or $T = t_1 = t_2$, where $T = \min(T_1, T_2)$ This game turns out to be the essential element of all duels with perfect information, and of a more general class of conflicts with perfect information to be described in another memorandum. Its solution is given in the following section.

Solution of the game G. Define $F(T) = \max_{t \le T} f(t)$, $G(T) = \min_{t \le T} g(t)$. Then $F(T)$ is monotone non-decreasing and $G(T)$ is monotone non-increasing. There are three cases to be considered.

Case I. $F(T)$ and $G(T)$ intersect. Say the first intersection is at T_o, $F(T_o) = G(T_o) = v$: There is then a $t_1 \le T_o$ with $f(t_1) = v$ and a $t_2 \le T_o$ with $g(t_2) = v$. The value of the game is then v, and approximately good strategies for the two players are to choose random times t near t_1 and t_2 respectively. More precisely, given $\epsilon > 0$, choose intervals $I_1 = (a_1, b_1)$, $I_2 = (a_2, b_2)$ about t_1, t_2 such that

$$f(t) \ge v - \epsilon\ ,\ G(t) \ge v - \epsilon \text{ for } t \in I_1$$

$$g(t) \le v + \epsilon\ ,\ F(t) \le v + \epsilon \text{ for } t \in I_2.$$

Let α, β be the mixed strategies of choosing t in I_1, I_2 with rectangular distributions (or any continuous distributions concentrated on I_1, I_2). Then

(1) $k(\alpha, t_2) \ge v - \epsilon$ for all t_2, and

(2) $k(t_1, \beta) \le v + \epsilon$ for all t_1.

For $k(\alpha, t_2) = \frac{1}{b_1 - a_1} \int_{a_1}^{b_1} k(t, t_2)dt$, and, for $t \in I_1$,

$$k(t, t_2) = g(t_2) \ge G(b_1) \ge v - \epsilon \text{ for } t > t_2$$

$$k(t, t_2) = f(t) \ge v \epsilon \text{ for } t < t_2.$$

Thus the integrand $k(t, t_2)$ over (a_1, b_1) always exceeds $v - \epsilon$, except possibly

at the single point $t = t_2$, so that (1) holds. The proof of (2) is similar.

Notice that, in case I, the function $h(t)$, specifying the result of simultaneous action, has no influence on either the value of the game or the approximately good strategies; these depend only on $f(t)$ and $g(t)$. The only influence of $h(t)$ is that it may force one or both of the players to avoid simultaneous action by using continuous strategies.

Case II. $F(b) < G(b)$. Here the value depends on $h(b)$; it is the median of the three numbers $F(b)$, $G(b)$, $h(b)$.

Case IIa. $h(b) \leq F(b)$. There is a t-interval I in which $f(t) \geq F(b) - \epsilon$; by choosing t in I with a continuous distribution, I can insure getting at least $F(b) - \epsilon$; the argument is identical with that in Case I. By choosing $t_2 = b$, II restricts his loss to $h(b)$ or $f(t_1)$, according as $t_1 = b$ or $t_1 < b$. Since neither $h(b)$ nor $f(t_1)$ exceeds $F(b)$, II can restrict his loss to $F(b)$, which is therefore the value of the game.

Case IIb. $h(b) \geq G(b)$. This is essentially the same as IIa. There is a t-interval J in which $g(t) \leq G(b) + \epsilon$. II restricts his loss to $G(b) + \epsilon$ by using a continuous distribution over J, while I insures getting $G(b)$ by choosing $t_1 = b$.

Case IIc. $F(b) \leq h(b) \leq G(b)$. Here the good strategy for each player is $t_1 = t_2 = b$. If $t_1 = b$, I receives either $h(b)$ or $g(t_2)$, according as $t_2 = b$ or $t_2 < b$. Since $g(t_2) \geq G(b) \geq h(b)$, I insures getting $h(b)$ Similarly II restricts his loss to $h(b)$ by choosing $t_2 = b$.

Case III. $F(a) > G(a)$. This case is similar to II; the value is the median of the three numbers $F(a)$, $G(a)$, $h(a)$. If $h(a) < G(a)$, I uses a continuous strategy over a t interval in which $f(t) > F(a) - \epsilon$, II chooses $t_2 = b$. If $h(a) > F(a)$, II uses a continuous strategy over a t-interval in which $g(t) < G(a) + \epsilon$, I chooses $t_1 = b$. If $G(a) \leq h(a) \leq F(a)$, both players choose $t = a$.

Example. As an illustration, consider the duel with accuracies $p_1(t) = \left| \frac{1}{3} - t \right|$, $p_2(t) = \frac{1}{2} - \left| \frac{1}{2} - t \right|$, $0 \leq t \leq 1$.

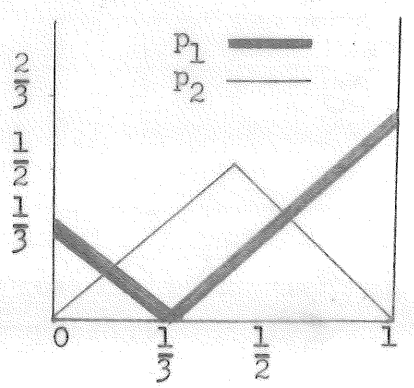


Figure 1.

We have $m_1(t) = \frac{2}{3}$;

$m_2(t) = \frac{1}{2}$ for $t \leq \frac{1}{2}$,

$m_2(t) = 1 - t$ for $t \geq \frac{1}{2}$.

$f(t) = -\frac{3}{2} t$ for $t \leq \frac{1}{3}$,

$f(t) = \frac{3}{2} t - 1$ for $\frac{1}{3} \leq t \leq \frac{1}{2}$,

$f(t) = -t^2 + \frac{10}{3} t - \frac{5}{3}$ for $\frac{1}{2} \leq t \leq 1$.

$g(t) = -\frac{5}{3} t + \frac{2}{3}$ for $t \leq \frac{1}{2}$,

$g(t) = \frac{5}{3} t - 1$ for $t \geq \frac{1}{2}$.

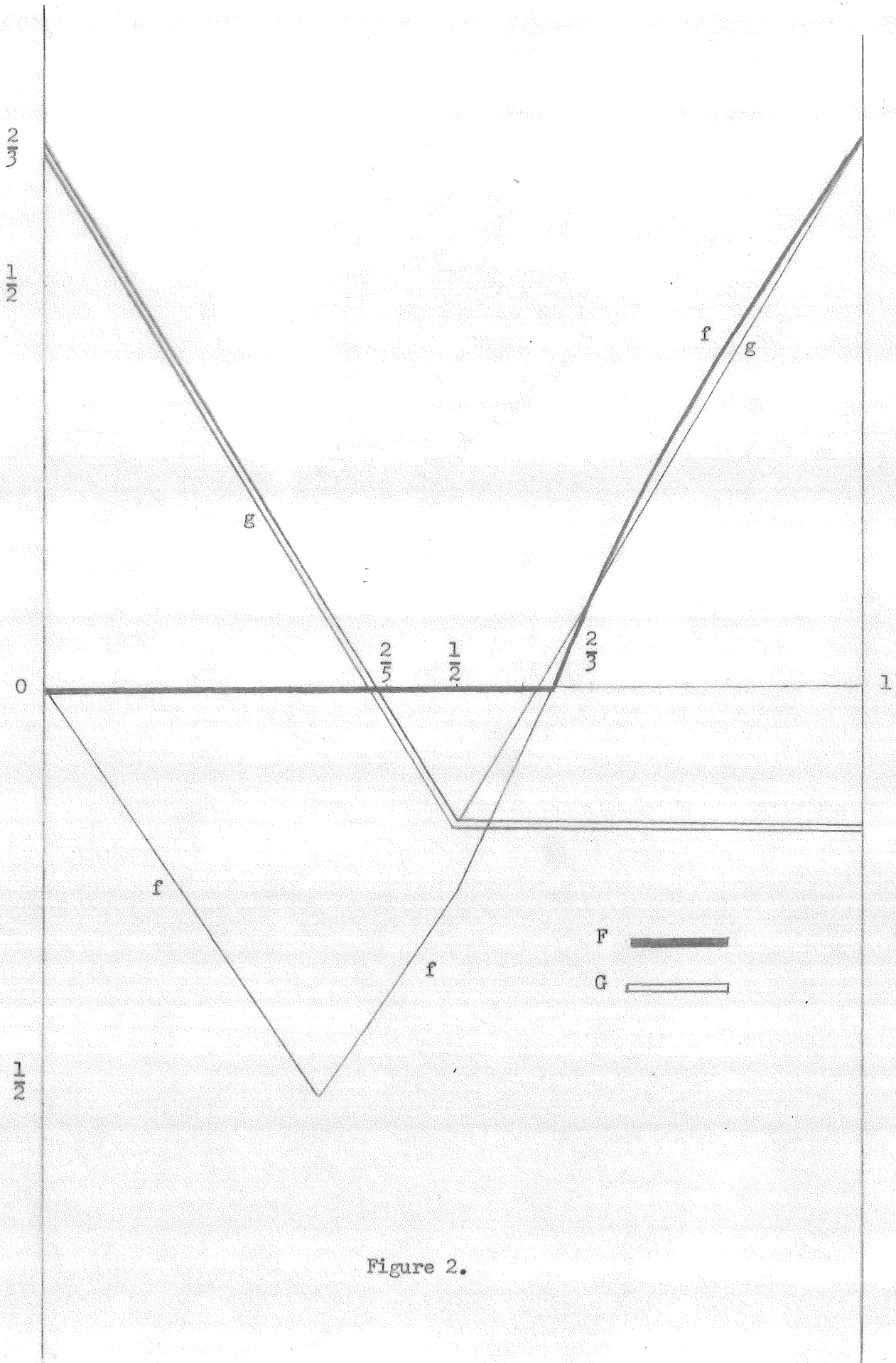


Figure 2.

The graphs of f, g, F, G are shown. The intersection of F and G is at $t = \frac{2}{5}$, and their common value is 0. The value of this duel is therefore 0. The good strategies, as specified by the general solution, are $t_1 = 0$, $t_2 = \frac{2}{5}$. It happens that simultaneous action at $t_1 = 0$ is preferred by I and at $t = \frac{2}{5}$ is preferred by II, so that neither player need resort to mixed strategies to avoid simultaneous action.

A notable fact is that the good strategy as specified by the general solution is not the one that should be used by player II in this duel. If time $t = \frac{2}{5}$ arrives and I has not yet fired (he should have fired at $t = 0$), II should re-evaluate the new situation so as to take advantage of I's mistake. The new situation is a duel over the interval $(\frac{2}{5}, 1)$, with a new F; G is the same as before over this interval. The new F coincides with f throughout $(\frac{2}{5}, 1)$; the intersection of F and G occurs at about $t = .54$ and their common value, which is the value of the new duel is $-\frac{1}{6}$. The appropriate firing times are $t_2 = \frac{1}{2}$, $t_1 = .54$, except that, if $t = .54$ arrives and II has not yet fired, I can increase his expectations above $-\frac{1}{6}$ by waiting. Incidentally, should both players wait until a time $t = a > \frac{2}{3}$ without firing, the duel becomes a Case III game. F(a) already exceeds G(a), and the best strategy for both players is to fire practically instantly (it turns out that $h(a) < G(a)$, so that I will avoid simultaneity).

Thus the strategies specified by the general solutions are good in the sense of insuring the value of the game, but need not be optimal in the sense of taking maximum advantage of the opponent's past mistakes during the game.

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