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AN EXAMPLE OF BLUFFING WITH PURE STRATEGIES

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One form of bluffing is to act, in a state of weakness, in a way usually associated with strength. This can be achieved either by (1) prescribing, for each given weak state, a probability of strong action, or (2) prescribing a subclass of weak states in which strong action will always be taken. We have pointed out [illegible] that in certain circumstances (1) can be approximated by (2): mixed strategies can be approximated by pure strategies. We now describe a game in which (2) replaces (1) exactly, not merely as an approximation.

The game considered is a form of poker in which each player receives a number (his hand) selected at random from the unit interval, with each player's hand unknown to the other. Player I then has a choice of (a) folding, (b) betting a fixed amount $z_1 \geq 1$, or (c) betting a fixed amount $z_2 \geq z_1$. In case (a), I pays II one unit (the ante). In case (b) or (c), II, knowing the amount of I's bet, has the choice of folding or seeing the bet. If II folds, he pays I one unit; if II sees, the hands x, y are compared, and the player with the better hand (the larger number) receives an amount $z + 1$ from the other, where z is the amount of I's bet, either z_1 or z_2.

Formally, a pure strategy for I is a pair (A_1, A_2) of disjoint subsets of the unit interval, specifying the hands x on which he bets z_1, z_2 respectively. A pure strategy for II is a pair (B_1, B_2) of not necessarily disjoint subsets of the unit interval, specifying the hands y on which he will see bets of z_1, z_2 respectively. The payoff

to I is then

$$V(A_1,A_2;\ B_1,B_2) = -1 + \sum_{i=1}^{2} m(A_i)\left[2-m(B_i)\right] + \iint_{\substack{x\in A_i \\ y\in B_i}} (z_i+1)\phi(x,\ y)dxdy,$$

where m(S) denotes the Lebesgue measure of the set S, and $\phi(x,\ y) = 1$ if $x > y$, $\phi = -1$ if $x < y$.

Solution of the game. For any fixed strategy $S = (A_1,\ A_2)$ of I, define

$$f_i(S,\ y) = \frac{z_i + 1}{m(A_i)} \int_{A_i} \phi(x,\ y)dx, \qquad i = 1,\ 2.$$

For any fixed strategy $T = (B_1,\ B_2)$ of II, define

$$g_i(x,\ T) = 1 - m(B_i) + (z_i + 1) \int_{B_i} \phi(x,\ y)dy, \qquad i = 1,\ 2.$$

Then f_i is I's conditional expectation for the given S and y when II sees a bet z_i, and g_i is I's conditional expectation for the given T and x when he bets z_i. f_1, f_2 are monotone non-increasing, g_1, g_2 are monotone non-decreasing. We have

$$V(S,\ T) = -1 + \sum_{i=1}^{2} m(A_i)\left\{1 + \int_{B_i} f_i(S,\ y)dy + \int_{C\,B_i} 1\ dy\right\},$$

$$V(S,\ T) = \int_{C(A_1+A_2)} -1\ dx + \sum_{i=1}^{2} \int_{A_i} g_i(x,\ T)dx,$$

where C(S) denotes the complement in the unit interval of S. We see

from these formulas for V(S, T) that

(α) $\inf_{T'} V(S, T') = V(S, T)$ if and only if

$$\min\left[f_i(S, y), 1\right] = \begin{matrix} f_i \text{ on } B_i \\ 1 \text{ on } C\,B_i \end{matrix}$$

(β) $\sup_{S'} V(S', T) = V(S, T)$ if and only if

$$\max\left[g_1(x, T), g_2(x, T), -1\right] = \begin{matrix} g_1 \text{ on } A_1 \\ g_2 \text{ on } A_2 \\ -1 \text{ on } C(A_1 + A_2). \end{matrix}$$

We now describe a pair of strategies S_0, T_0 which are optimal against each other, i.e., such that

$$\inf_T V(S_0, T) = \sup_S V(S, T_0) = V(S_0, T_0).$$

These strategies were discovered partly by looking for strategies with the properties which these will be shown below to have, and partly by divination. The strategies are most simply described in terms of the numbers $c_i = \frac{2}{z_i + 2}$, $i = 1, 2$. Since $z_2 \geq z_1 \geq 1$, $0 \leq c_2 \leq c_1 \leq 2/3$. We define

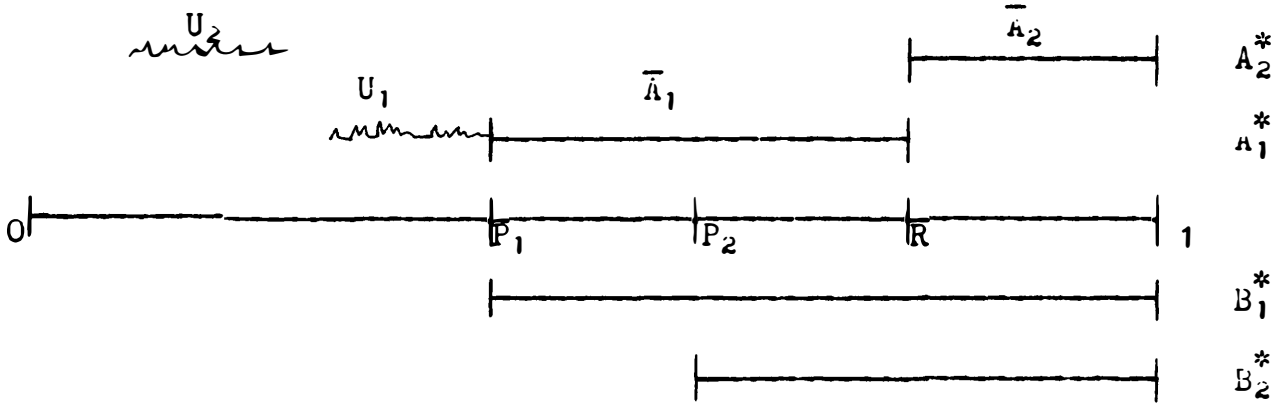
$A_i^* = \bar{A}_i + U_i$, where

$$\bar{A}_1 = \left\{1 - c_1 \leq x \leq 1 - \frac{c_1 c_2}{2}\right\}, \quad \bar{A}_2 = \left\{1 - \frac{c_1 c_2}{2} \leq x \leq 1\right\},$$

and U_1, U_2 are any two disjoint subsets of $0 \leq x \leq 1 - c_1$ of measures $(1 - c_1)m(\bar{A}_1)$, $(1 - c_2)m(\bar{A}_2)$ respectively. It must be shown that $\sum_{i=1}^{2} (1 - c_i)m(\bar{A}_i) \leq 1 - c_1$; this we defer until later. Define

$$B_i^* = 1 - c_i \le y \le 1 \quad .$$

Graphically, the strategies are



where $P_1 = 1 - c_1$, $P_2 = 1 - c_2$, $R = 1 - \frac{c_1 c_2}{2}$. Notice that II's strategy is uniquely specified and does not involve bluffing. I's strategy is uniquely specified only for $x \ge P_1$; the measures of U_1, U_2 are specified, but their precise location is immaterial. The subset U_2 of A_2^* represents bluffing – making a high bet with a mediocre hand, while making a low bet on somewhat stronger hands.

To verify that $S_0 = (A_1^*, A_2^*)$, $T_0 = (B_1^*, B_2^*)$ are good strategies, we compute f_1, f_2, g_1, g_2 and verify (α), (β). Direct computation, using $z_i + 1 = \frac{2 - c_i}{c_i}$ and setting $m(\bar{A}_i) = \alpha_i$, yields

$$f_i(S_0, P_i) = \frac{(2 - c_i)}{c_i(2 - c_i)\alpha_i} \left[- (1 - c_i)\alpha_i + \alpha_i \right] = 1.$$

Since f_1, f_2 are monotone non-increasing, T_0 satisfies α with $S = S_0$. Again by direct computation,

$$g_i(x, T_0) = 1 - c_i + \frac{2 - c_i}{c_i} \left[2x - 2 + c_i \right] \quad \text{for} \quad 1 - c_i \le x \le 1.$$

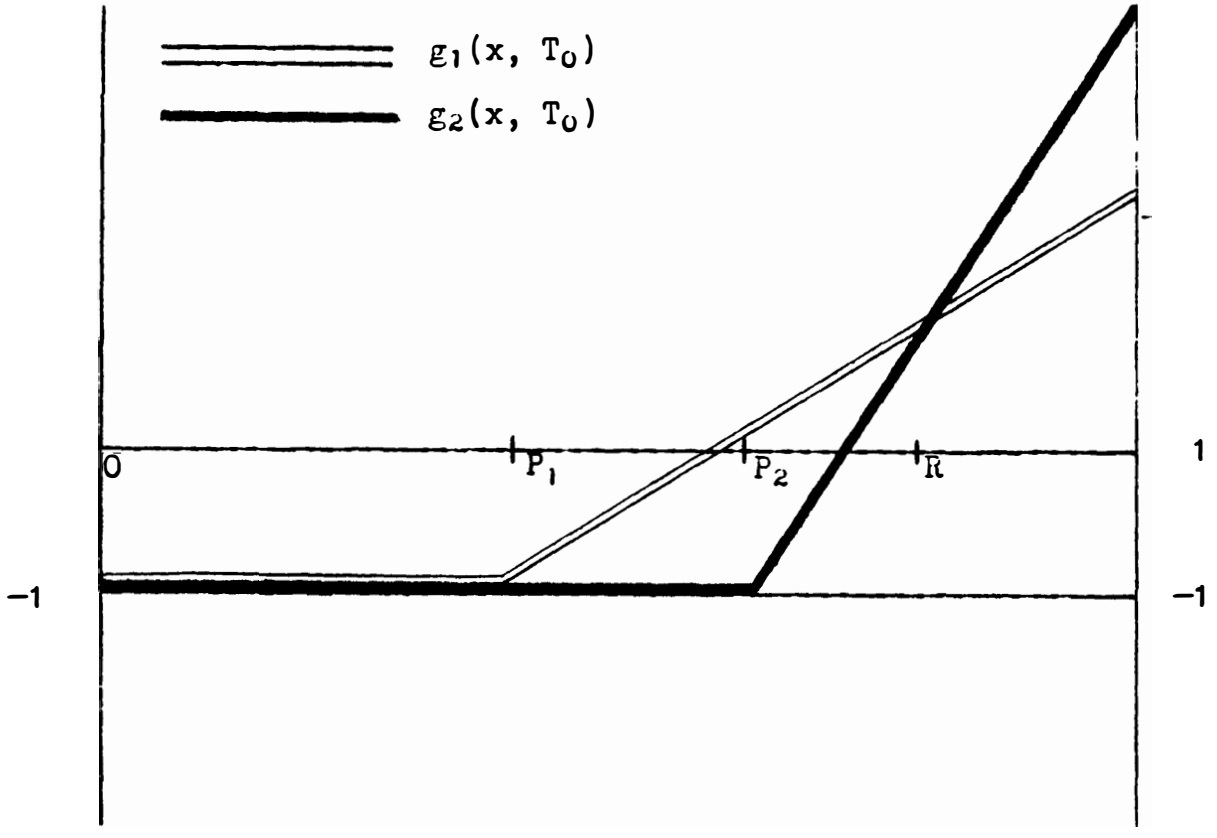
In particular,

$$g_i(P_i, T_0) = 1 - c_i + \frac{2 - c_i}{c_i}\left[2 - 2c_i - 2 + c_i\right] = -1,$$

$$g_i(R, T_0) = 1 - c_i + \frac{2 - c_i}{c_i}\left[2 - c_1c_2 - 2 + c_i\right]$$

$$= 3 - 2c_i - 2c_j + c_ic_j, \quad \text{where} \quad j = 3 - i.$$

Since the last expression is symmetric in c_1, c_2, $g_1(R, T_0) = g_2(R, T_0)$. Finally, since no points of B_i are left of P_i, $g_i(x, T_0) = g_i(P_i, T_0) = -1$ for $x \leq P_i$, and since for $x > P_i$, g_i has slope $2\left(\frac{2 - c_i}{c_i}\right) = 2(z_i + 1)$, we have $g_2 \geq g_1$ for $x \geq R$, $g_2 \leq g_1$ for $x \leq R$. Graphically,



Thus S_0 satisfies (β) with $T = T_0$.

This completes the proof, except for verifying that $m(U_1) + m(U_2) \leq 1 - c_1$, i.e., that

$$(1 - c_1)\left(c_1 - \frac{c_1 c_2}{2}\right) + (1 - c_2)\left(\frac{c_1 c_2}{2}\right) \leq 1 - c_1.$$

Now fix $c_1 = c$ and write $c_2 = x$; we must show that $f(x) \leq 0$ for $0 \leq x \leq c$, $0 \leq c \leq \frac{2}{3}$, where

$$f(x) = c(1 - c)(2 - x) + cx(1 - x) - 2 + 2c$$

$$= -cx^2 + c^2 x - 2c^2 + 4c - 2.$$

$\max_x f(x)$ occurs at $x = \frac{c}{2}$ and equals

$$h(c) = \frac{c^3}{4} - 2c^2 + 4c - 2,$$

so that we must show that $h(c) \leq 0$ for $0 \leq c \leq \frac{2}{3}$. In fact the smallest root of $h(c) = 0$ occurs at approximately $c = .76$, corresponding to $z_1 = \frac{2}{c} - 2 = .62$. Thus our solution is valid for all $z_1 \geq .62$, $z_2 \geq z_1$. For $z_1 < .62$, the solution has not been discovered. It may be verified that, for $z_1 \geq .62$, $z_2 > z_1$, the only solutions are those we have exhibited: II's strategy is unique and pure, I has a set of good pure strategies corresponding to the set of possible choices of U_1, U_2, and I's most general good mixed strategy is an arbitrary mixture of his good pure strategies.

The value of the game is easily computed, and turns out to be exactly $h(c)$. Thus our solution is valid over precisely that range of z_1 for which the value of the game is non-positive for all $z_2 \geq z_1$.