U. S. AIR FORCE

PROJECT RAND

RESEARCH MEMORANDUM

A GENERALIZATION OF THE SILENT DUEL, TWO OPPONENTS, ONE BULLET EACH, ARBITRARY ACCURACY

M. A. Girshick

RM-206

1 August 1949

Assigned to _____

This is a working paper. It may be expanded, modified, or withdrawn at any time. The views, conclusions, and recommendations expressed herein do not necessarily reflect the official views or policies of the United States Air Force.

The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

9 pp

SUMMARY

We consider a silent duel between two opponents I and II with accuracies $P_1(x)$ and $P_2(x)$, $0 \leq x \leq 1$, which has a payoff function

$$V(x,y) = P_1(x) - P_2(y) + \epsilon P_1(x)P_2(y)$$

where $\epsilon = r_1$ if $x < y$, 0 if $x = y$ and $\mp r_2$ if $x > y$ where $r_i > 0$ and $P_i(x)$ are continuous and strictly monotone with $P_i(0) = 0$, $r_iP_i(1) \leq 1$, $(i = 1, 2)$. The good strategies are defined as follows: Player I has density function

$$f(x) = \frac{1}{\Phi(b)} \frac{P_2'(x)}{P_2^2(x)P_1(x)}$$

and player II has a density function

$$g(y) = \frac{1}{\Psi(b)} \frac{P_1'(y)}{P_1^2(y)P_2(y)}$$

where

$$\Phi(b) = \int_a^1 f(x)dx \text{ and } \Psi(b) = \int_a^1 g(y)dy.$$

Choose b_1 and b_2 such that

$$\Phi(b_1) = \frac{r_1}{P_2(b_1)} + \frac{r_2}{P_2(1)}, \quad \Psi(b_2) = \frac{r_2}{P_1(b_2)} + \frac{r_1}{P_1(1)}$$

and define $a = \max(b_1, b_2)$. If $a = b_1$, player I's optimum strategy is to fire according to density $f(x)$ and player II's strategy is a mixture of $g(x)$ and firing at 1 where $a \leq x \leq 1$. However if $a = b_2$ then II uses $g(x)$ and I

mixes f(x) and firing at 1. If $a = b_1 = b_2$, then neither fire at 1 with positive probability. This solution generalizes a duel considered by Blackwell.

. . . . . .

In a previous memorandum, ("The Silent Duel, One Bullet Each, Arbitrary Accuracy", GA-MM-302, 9-30-48) David Blackwell considered a duel between two opponents I and II, accuracies $P_1(x)$ and $P_2(x)$, $0 \leq x \leq 1$, and a payoff function.

$$V(x,y) = P_1(x) - P_2(y) - \epsilon P_1(x)P_2(y)$$

where (a) $P_1(x)$ and $P_2(x)$ are continuous and strictly monotone, (b) $P_i(0) = 0$ $(i = 1, 2)$, (c) $P_i(1) = 1$ and (d) $\epsilon = 1$ if $x < y$, $\epsilon = 0$ if $x = y$ and $\epsilon = -1$ if $x > y$.

In this note we replace condition (d) by (d'): $\epsilon = r_1$ if $x < y$, $\epsilon = 0$ if $x = y$, $\epsilon = -r_2$ if $x > y$ where $r_1 > 0$ and $r_2 > 0$. We also replace condition (c) by condition (c'): $r_1P_1(1) \geq 1$, $r_2P_2(1) \geq 1$.

The following example will illustrate the necessity for modifying conditions (c) and (d).

Consider a silent duel between two opponents I and II with accuracies $R_1(x)$ and $R_2(x)$ $(R_i(0) = 0, R_i(1) = 1)$ which has the following payoff. If I kills II, the value to I is $\alpha(\alpha > 0)$. If II kills I, the value to I is $-\beta(\beta > 0)$. If neither are killed, the value is zero. In addition we assume that (a) A hit by the i^{th} opponent has a probability π_i of being lethal. (b) The gun of the i^{th} opponent has a probability p_i of containing a live bullet and $1 - p_i$ of containing a blank. The π_i's and p_i's are known to each opponent.

The payoff as a function of x and y is given by

$$V(x,y) = \alpha p_1 \pi_1 R_1(x) - \beta p_2 \pi_2 R_2(y) + \beta p_1 p_2 \pi_2 R_1(x) R_2(y)$$

if $x < y$ and

$$V(x,y) = \alpha p_1 \pi_1 R_1(x) - \beta p_2 \pi_2 R_2(y) - \alpha p_1 p_2 \pi_1 R_1(x) R_2(y)$$

if $x > y$.

Setting $P_1(x) = \alpha p_1 \pi_1 R_1(x)$, $P_2(y) = \beta p_2 \pi_2 R_2(y)$, $r_1 = 1/\alpha\pi_1$, $r_2 = 1/\beta\pi_2$, we see that this problem reduces itself to the one under consideration.

The optimum strategies will be obtained by a slight generalization of Blackwell's argument.

We assume that, at least in part, players I and II employ densities $f(x)$ and $g(y)$ respectively, $b \le x, y \le 1$. Then, if player I uses $f(x)$ and player II chooses a y.

$$(2) \qquad V(f,y) = Q_1 - (1+r_2 Q_1) P_2(y) + (r_1+r_2) P_2(y) \int_b^y P_1(x) f(x)\,dx$$

where $$Q_1 = \int_b^1 P_1(x) f(x)\,dx \ .$$

Differentiating equation (2) twice and solving the resulting differential equation for $f(x)$ we get

$$(3) \qquad f(x) = \frac{1}{\Phi(b)} \cdot \frac{P_2'(x)}{P_2^2(x) P_1(x)} \qquad b \le x \le 1$$

where

$$\Phi(b) = \int_b^1 \frac{P_2'(x)\,dx}{P_2^2(x) P_1(x)} \ .$$

Now

$$(4) \qquad \int_b^y P_1(x) f(x)\,dx = \frac{1}{\Phi(b)} \left[\frac{1}{P_2(b)} - \frac{1}{P_2(y)}\right]$$

and hence

$$(5) \qquad Q_1 = \frac{P_2(1) - P_2(b)}{P_2(1) P_2(b) \Phi(b)} \ .$$

Substituting (4) and (5) into (2) we get

$$(6) \qquad V(f,y) = P_2(y)\left[\frac{r_1P_2(1) + r_2P_2(b) - P_2(1)P_2(b)\Phi(b)}{P_2(1)P_2(b)\Phi(b)}\right] + \frac{P_2(1) - P_2(b) - (r_1 + r_2)P_2(1)P_2(b)}{P_2(1)P_2(b)\Phi(b)} .$$

Similarly

$$(7) \qquad V(x,g) = P_1(x)\left[1 + r_1Q_2\right] - Q_2 - (r_1 + r_2)P_1(x)\int_b^x P_2(y)g(y)dy$$

where

$$Q_2 = \int_b^1 P_2(y)g(y)dy .$$

From (7) we obtain

$$(8) \qquad g(y) = \frac{1}{\psi(b)}\,\frac{P_1'(y)}{P_1^2(y)P_2(y)}$$

where

$$\psi(b) = \int_b^1 g(y)dy .$$

From (8) we obtain

$$(9) \qquad \int_b^x P_2(y)g(y)dy = \frac{1}{\psi(b)}\left[\frac{1}{P_1(b)} - \frac{1}{P_1(x)}\right]$$

and hence

$$(10) \qquad Q_2 = \frac{P_1(1) - P_1(b)}{P_1(1)P_1(b)\psi(b)}$$

Substituting (9) and (10) into (7) we get

$$(11) \qquad V(x,g) = P_1(x)\left[\frac{P_1(1)P_1(b)\psi(b) - r_2P_1(1) - r_1P_1(b)}{P_1(1)P_1(b)\psi(b)}\right] + \frac{(r_1+r_2)P_1(1)P_1(b) + P_1(b) - P_1(1)}{P_1(1)P_1(b)\psi(b)}$$

In order for (6) to be independent of y, we must have

$$\Phi(b) = \frac{r_1}{P_2(b)} + \frac{r_2}{P_2(1)} \quad . \tag{12}$$

Similarly, in order for (11) to be independent of x, we must have

$$\Psi(b) = \frac{r_2}{P_1(b)} + \frac{r_1}{P_1(1)} \quad . \tag{13}$$

We now repeat and somewhat generalize Blackwell's lemma.

Lemma. The equations

$$\Phi(x) = \frac{r_1}{P_2(x)} + \frac{r_2}{P_2(1)} \quad , \quad \Psi(x) = \frac{r_2}{P_1(x)} + \frac{r_1}{P_1(1)} \tag{14}$$

have unique solutions b_1, b_2, $0 < b_1, b_2 < 1$. $\Phi(x) - \frac{r_1}{P_2(x)}$ and $\Psi(x) - \frac{r_2}{P_1(x)}$ are strictly monotone decreasing functions of x for $o < x \leq 1$.

Proof: Write $u(x) = \Phi(x) - \frac{r_1}{P_2(x)}$. For any c_1 and c_2 with $0 < c_1 < c_2 < 1$

$$u(c_1) - u(c_2) = \int_{c_1}^{c_2} \frac{P_2'(y)dy}{P_2(y)P_1(y)} - r_1 \left[\frac{1}{P_2(c_1)} - \frac{1}{P_2(c_2)}\right] \tag{15}$$

$$\geq \frac{1}{P_1(c_2)} \int_{c_1}^{c_2} \frac{P_2'(y)dy}{P_2^2(y)} - r_1 \left[\frac{1}{P_2(c_1)} - \frac{1}{P_2(c_2)}\right]$$

$$= \frac{1-r_1P_1(c_2)}{P_1(c_2)} \left[\frac{1}{P_2(c_1)} - \frac{1}{P_2(c_2)}\right]$$

Now $P_1(x)$ and $P_2(x)$ are strictly monotone. And since $1-r_1P_1(c_2) > 1-r_1P_1(1) \geq 0$ by assumption, we see that $u(c_1) - u(c_2)$ is positive, so that $u(x)$ is strictly monotone decreasing. Letting $c_1 \to 0$ we see that $u(x) \to \infty$ as $x \to 0$.

Since $u(x)$ is continuous and $u(1) = -\frac{r_1}{P_1(1)}$, there is a unique b_1 such that $\Phi(b_1) - \frac{r_1}{P_2(b_1)} = \frac{r_2}{P_2(1)}$. A similar argument proves the other assertion of the lemma.

We define $a = \max(b_1, b_2)$ where b_1 and b_2 are the numbers described in the lemma.

We consider three cases:

Case 1. $a = b_1 = b_2$. Then the optimum strategy for player I is to employ the density $f(x)$ given by (3) and for player II to employ the density $g(y)$ given by (8) $a \leq x_1 \leq 1$.

Proof

$$V(f,y) = \frac{P_2(1)-P_2(a)-(r_1+r_2)P_2(1)P_2(a)}{r_1P_2(1)+r_2P_2(a)} = V_1 \tag{16}$$

and

$$V(x,g) = \frac{(r_1+r_2)P_1(1)P_1(a)+P_1(a)-P_1(1)}{r_2P_1(1)+r_1P_1(a)} = V_2 \ . \tag{17}$$

Now since

$$\int V(f,y)g(y)dy = \int_a^1 V_1 g(y)dy = V_1 \tag{18}$$

$$\int V(x,g)f(x)dx = \int_a^1 V_2 f(x)dx = V_2 \tag{19}$$

we must have $V_1 = V_2 = V$.

Also if $y < a$

$$V(f,y) = \frac{1}{\Phi(a)} \int_a^1 P_1(x)f(x)dx - P_2(y)\left[1+r_2\frac{1}{\Phi(a)}\right]\int_a^1 P_1(x)f(x)dx \tag{20}$$

$$\geq \frac{1}{\Phi(a)}\int_a^1 P_1(x)f(x)dx - P_2(a)\left[1+r_2\frac{1}{\Phi(a)}\right]\int_a^1 P_1(x)f(x)dx =$$

$$= \frac{P_2(1)-P_2(a)}{\Phi(a)P_2(1)P_2(a)} \left[1-r_2P_2(a)\right] - P_2(a)$$

$$= \frac{\left[P_2(1)-P_2(a)\right]\left[1-r_2P_2(a)\right] - P_2(a)\left[r_1P_2(1) + r_2P_2(a)\right]}{r_1P_2(1) + r_2P_2(a)}$$

$$= V.$$

Similarly we can show that if $x < a$, $V(x,g) \leq V$. The last two results are, of course, justifiable on simpler grounds.

Case 2. $a = b_1 > b_2$. Then the optimum strategy for player II is to employ the density $f(x)$ $a \leq x \leq 1$ and for player II to employ the density $g(y)$ a fraction γ of the times and fire at 1, a fraction $1-\gamma$ of the times where

$$\gamma = \frac{r_1P_2(1) + 1}{r_1P_2(1) + 1 - A} \tag{21}$$

and

$$A = \frac{P_1(1)P_1(a)\psi(a) - r_1P_1(a) - r_2P_1(1)}{P_1(1)P_1(a)\psi(a)} \tag{22}$$

The value of the game is given by (16).

Proof: From (11) we have $V(x,g) = AP_1(x) + B$ where A is given by (22) and

$$B = \frac{(r_1+r_2)P_1(1)P_1(a) + P_1(a) - P_1(1)}{P_1(1)P_1(a)\psi(a)} \tag{23}$$

Since $a = b_1 > b_2$, the lemma implies that $A < 0$. If II has strategy I_1, always firing at 1,

$$V(x,I_1) = \left[r_1P_2(1) + 1\right] P_1(x) - P_2(1) \tag{24}$$

for $x < 1$. There is then a constant γ, $0 \leq \gamma \leq 1$ such that

$$\gamma A + (1-\gamma)\left[r_1P_2(1) + 1\right] = 0 . \tag{25}$$

The remainder of the argument is similar to that given in case 1.

Case 3. $a = b_2 > b_1$. Then the optimum strategy for player II is to employ the density $g(y)$ and for player 1 to employ $f(x)$ a fraction γ of the times and fire at 1 a fraction $1-\gamma$ of the times where

$$\gamma = \frac{1 + r_2 P_1(1)}{1+C+r_2 P_1(1)} \tag{26}$$

and

$$C = \frac{r_1 P_2(1)+r_2 P_2(a)-P_2(1)P_2(a)\Phi(a)}{P_2(1)P_2(a)\Phi(a)} \tag{27}$$

The value of the game is given by (17).

Proof: From (6) we have $V(f,g) = CP_2(y) + D$ where C is given by (27) and

$$D = \frac{P_2(1)-P_2(b)-(r_1+r_2)P_2(1)P_2(b)}{P_2(1)P_2(b)\Phi(b)} \tag{28}$$

Since $a = b_2 > b_1$, the lemma implies that $C > 0$. If I has a strategy I_1 always firing at 1,

$$V(I_1 y) = -P_2(y)\left[1+r_2 P_1(1)\right] + P_1(1) \tag{29}$$

for $y < 1$. There is then a constant γ, $0 \leq \gamma \leq 1$ such that

$$\gamma C - (1-\gamma)\left[1+r_2 P_1(1)\right] = 0 \quad .$$

The remainder of the argument is similar to that given in case 1.

MAG: je