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THEORY OF FIRING. I

A. N. Kolmogorov, Editor
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Introduction

In many complicated problems in the theory of firing, one must deal with probabilities which are expressed by multiple integrals depending on a number of parameters. As a rule, these integrals cannot be evaluated by elementary means and do not lead to functions for which tables have been constructed. For example, the expressions

\[ S_1 = 1 - \frac{\rho}{\sqrt{\pi} E_U} \int_{-\infty}^{+\infty} e^{-\rho^2 y^2 / E_U^2} \left( 1 - \frac{\rho}{\sqrt{\pi} E} \int_{y-b}^{y+b} e^{-\rho^2 x^2 / E^2} \, dx \right) dy \]  

or

\[ R = 1 - \frac{\rho}{\sqrt{\pi} E_U} \int_{-\infty}^{+\infty} e^{-\rho^2 y^2 / E_U^2} \left( 1 - \frac{\rho}{\sqrt{\pi} E} \int_{y-b}^{y+b} e^{-\rho^2 (x-A)^2 / E^2} \, dx \right) ^n \left( 1 - \frac{\rho}{\sqrt{\pi} E} \int_{y-b}^{y+b} e^{-\rho^2 (x+A)^2 / E^2} \, dx \right) \, dy \]  

for the probability of at least one hit in the case of "two groups of errors," one-dimensional dispersion, and firing at one or more targets, are encountered in the majority of textbooks on the theory of firing. (See also the second article published below by academician A. N. Kolmogorov.)

The integral (1) depends upon three parameters: \( K = E / E_U \), \( c = b / E_U \), and \( n \). For purposes of orientation, one can construct sufficiently detailed tables of the function \( S_1(K, c, n) \) defined by (1). (Professor N. V. Smirnov of the Math. Inst. Acad. Sci. U.S.S.R. has developed methods for the construction of such tables,
Expression (2) depends upon four parameters:

\[ K = \frac{E}{E_u}, \quad c = \frac{b}{E_u}, \quad n, \text{ and } a = \frac{A}{E_u}. \]

Construction of corresponding tables would be yet more cumbersome. In more complex cases, which are common in the theory of firing, the number of parameters may be considerably larger.

Ordinarily, in considering similar questions, the calculations are shortened by numerical integration of a comparatively small number of typical examples. Sometimes this gives an accurate qualitative insight into the problem. For this reason, certain approximation methods have great interest, which involve either simplifying the question or idealizing the problem. For example, in the case of a large number of shots, instead of considering the exact point of impact of each shell, one may consider the density of the points of impact. This method is often encountered in the literature, but the systematic mathematical development of such methods remained in a rudimentary state until the work of the co-workers of the Artillery Scientific Research Naval Institute and of the Mathematical Inst. Acad. Sci. U.S.S.R., some of which are published in the present collection.

Both in the A.S.R.N.I. and in the M.I.A.S. U.S.S.R., a large number of calculations were carried out for comparison of results obtained by the methods proposed for approximation with those obtained by precise numerical integration. Certain of these
materials, and also further theoretical investigations, will be published in the following collection.

The tables for the articles of A. N. Kolmogorov were computed by co-workers, graduate students, and pupils of the Mechanical-mathematical faculty of the M G Ü. The editorial work on the whole collection was done by co-workers of the Math. Inst. U.S.S.R., D. A. Vacil'kov.
THE NUMBER OF HITS IN A GIVEN NUMBER OF SHOTS
AND GENERAL PRINCIPLES OF EVALUATING
THE EFFECTIVENESS OF A SYSTEM OF FIRING

A. N. Kolmogorov

Introduction

§1. Choice of index of effectiveness of fire.

§2. The case of mutual independence of hits for individual shots.

§3. Classification of factors influencing the result of firing, and the meaning of dependence between hits for individual firings.


§5. Probability of destruction of the target in the case \[ P(A|m) = 1 - e^{-\alpha m} \].

Introduction

In the entire sequel we shall consider a group of \( n \) shots, each of which may result in a hit or a non-hit upon a target. Clearly the number of hits, \( \mu \), obtained can assume only the values \( m = 0, 1, 2, \ldots, n \).

The corresponding probability of obtaining exactly \( m \) hits we designate by

\[
(1) \quad P_m = \bar{F}_\mu(\mu = m).
\]
Using the probability $P_m$, we obtain the mathematical expectation of the number of hits $M(\mu)$, which can be written in the form

\[(2) \quad M(\mu) = P_1 + 2P_2 + \cdots + nP_n,\]

and the probability $R_m$ of obtaining at least $m$ hits, which appears as

\[(3) \quad R_m = \overline{P}(\mu \geq m) = \sum_{i=1}^{n} = mP_1 = 1 - \sum_{i=0}^{m-1} P_i.\]

In particular, the probability $R_1$ of obtaining at least one hit is equal to

\[(4) \quad R_1 = 1 - P_0.\]

The mathematical expectation $M(\mu)$ and the probabilities $R_m$ are the principal characteristics used in the extant military literature for evaluating effectiveness of systems of firing. By firing, I mean, here and in the sequel, a group of shots directed at a common target. The firing may consist of one or several volleys or of consecutive single shots. By a system of firing, I mean the method of firing, established ahead of time, which may provide for a preassigned number of shots (as is presupposed further in this work), or cessation of firing upon obtaining an assigned result, distribution of shots between different aimings (when several aimings are made), the arrangement of range-finding, etc. The result of every actual firing is in large measure accidental. Therefore the effectiveness of a system of firing cannot be characterized by the actual result of some
individual firing carried out by this system. The evaluation of effectiveness of a system of firing can depend only upon the distribution of probabilities for possible results of individual firings. Any quantity defined uniquely in terms of these probabilities can be considered as a certain characteristic of the given systems of firing. The numbers \( M(\mu) \), \( P_m \), and \( R_m \) are such characteristics in our case of firings consisting of \( n \) shots, each resulting in a "hit" or a "non-hit" on a target. On the other hand, the number of hits \( \mu \) could be considered a characteristic in the case (not interesting for us) when some \( P_m = 1 \) and all others are zero.\]

\( M(\mu) \) and \( R_m \) are defined in (2) and (3) by means of the probabilities \( P_m \), that is, by the distribution of the probability of the random variable \( \mu \). Hence it is first of all essential to analyze this question: Under what conditions is it sufficient to know the numbers \( P_0, P_1, \ldots, P_n \) in order to evaluate the effectiveness of a system of firing? Generally speaking, this is not always true. For example, in bombarding a target of large area, it is often far from simple to obtain a sufficiently large number of hits on the target, and the problem is to obtain such a distribution of projectiles in the area of the target as to guarantee the destruction of a sufficiently large portion of this area. We shall not treat cases of this kind in the present work.

Limiting the evaluation of effectiveness of systems of firing to methods based on the numbers \( P_0, P_1, \ldots, P_n \), it is natural to raise the question of whether one can calculate a single real-valued
function of these probabilities,
\[ W = f(P_0, P_1, \ldots, P_n), \]
which would be an index of effectiveness of the system of firing. The expectation \( M(\mu) \) or the number \( R_m \) (\( m \) being the number of hits necessary for destruction of the target) are often used as such functions. The widespread discussions in the literature concerning the advantages and drawbacks of "evaluation by expectation" and "evaluation by probabilities" often fail to be clear. This has led me to devote §1 to a survey of this question.

§§2 and 3 are devoted to the purely mathematical questions of exact and approximate calculations of \( P_m \) and \( R_m \). The tables of the Poisson distributions and the corrections thereto, which appear at the end of this work, are related to this question, and it seems to me that these may attain considerable usefulness in solving problems in the theory of firing.

In §4 is given a general statement of the problem of firing with artificial dispersion, on the basis of the considerations of §1 and the formulas developed in §3. In particular, the concept of artificial dispersion is introduced. The ideas of this somewhat abstractly written paragraph will be developed in other works of the writer.

§1. Choice of index of effectiveness of firing.

We begin with a consideration of two typical cases, between which there exist many intermediate ones.
First case. The firing is carried out to solve a completely
definite problem (the sinking of a ship, the shooting down of an
airplane, etc.), which can be only solved or unsolved, and we are
concerned only with the probability $\overline{P}(A)$ that the problem is solved.
(Here $A$ denotes the event consisting in the successful solution of
the given problem.)

We denote by $\overline{P}(A|m)$ the conditional probability that the given
problem is solved, under the condition that exactly $m$ hits are
received. Then, by the theorem on complete probability (we suppose
that $\overline{P}(A|0) = 0$),

$$\overline{P}(A) = \sum_{i=1}^{n} P_i \overline{P}(A|i).$$

If in particular, solution of the problem is certain for
$\mu \geq m$ and impossible for $\mu < m$, that is, if

$$\overline{P}(A|r) = \begin{cases} 1 & \text{for } r \geq m \\ 0 & \text{for } r < m \end{cases},$$

then the general formula (5) assumes the form

$$\overline{P}(A) = R_m.$$ 

For example, if solution of the problem is certain when at least
one hit is obtained, then

$$\overline{P}(A) = R_1.$$ 

For $m > 1$, the assumption (6) is highly artificial: it is
difficult to imagine, for example, a concrete situation in firing
ir which a solution of the problem is guaranteed by obtaining ten 
hits and is impossible when only nine hits are obtained. It is 
more natural to suppose that the probability $P(A|m)$ increases 
gradually with the number of hits obtained. In §5, we shall con-
sider the special case

$$P(A|m) = 1 - e^{\alpha m},$$

where $\alpha$ is a certain constant. This type of dependence of $P(A|m)$ 
on $m$ is selected by us because it permits us to obtain, in many 
cases, fairly simple expressions for the probabilities $P(A)$, which 
are the objects of our major interest. At the same time, the for-
mula (9), under various interpretations of the constant $\alpha$, provides 
us with a means of approximating certain relations which are ac-
tually encountered; and (9) is just as appropriate as (6). It is 
quite natural, furthermore, to suppose that $P(A|m)$ does not de-
crease with increasing $m$, i.e., that

$$D_m = P(A|m) - P(A|(m-1)) \geq 0.$$  

Using (10) and a special case of Abel's transformation, we may 
conveniently rewrite (5) in the form

$$P(A) = D_1 R_1 + D_2 R_2 + \ldots + D_n R_n.$$  

(The form of Abel's transformation used here is this: if $a_0 = 0,$
$\varphi_m = a_m - a_{m-1}$ and $R_m = P_m + \ldots + P_n$, then $\sum_{i=1}^{n} a_i p_i = \sum_{i=1}^{n} v_i R_i.$)

If, for example, we have $n = 10$ and $P(A|1) = P(A|2) = 0$, $P(A|3) = 1/3$,
$P(A|4) = 2/3$, and $P(A|5) = \ldots = P(A|10) = 1$, then by (5) we obtain
\[ F(A) = \frac{1}{2} P_3 + \frac{2}{3} P_4 + P_5 + P_6 + \ldots + P_{10} . \]

Formula (11) leads us to the simpler expression

\[ F(A) = \frac{1}{2} R_3 + \frac{1}{3} R_4 + \frac{1}{3} R_5 . \]

These remarks suffice to clarify the meaning of the probabilities \( R_m \) for calculations of success probabilities in the case of problems of firing consisting of \( n \) shots.

**Second case.** The firing consists of only one out of a number of similar and mutually independent firings, and we are interested only in the average damage inflicted on the enemy by these firings. In this case, it suffices to know for each individual firing the mathematical expectation \( M(\xi) \) of the damage \( \xi \) inflicted by it.

We denote by \( M(\xi|m) \) the conditional mathematical expectation of the damage \( \xi \) under the assumption that exactly \( m \) hits are obtained. By a known formula concerning complete mathematical expectation (we suppose \( M(\xi|0) = 0 \)), we obtain

\[ M(\xi) = P_1 M(\xi|1) + P_2 M(\xi|2) + \ldots + P_n M(\xi|n) . \]

If we suppose that

\[ M(\xi|m) = K_m , \]

i.e., that the expectation of damage is proportional to the number of hits obtained, then

\[ M(\xi) = KM(\mu) . \]
There are many special cases in which (13) and (14) may be considered sufficiently accurate, and in these cases the effectiveness of firing from the point of view of expectation of damage is defined by the expectation of number of hits.

However, there are a great many special cases in which one observes a considerable deviation from (13). In such cases it would be wrong to replace (12) by (14).

It is quite natural to suppose that \( M(\xi|m) \) is a non-decreasing function of \( m \); thus

\[
(15) \quad c_m = M(\xi|m) - M(\xi|\text{m-1}) \geq 0
\]

Thus we find, using Abel’s transformation, that

\[
(16) \quad M(\xi) = \sum_{i=1}^{n} c_i R_i
\]

Our remarks suffice to indicate in which cases we can evaluate \( M(\xi) \) by using \( M(\mu) \) and in which we must use (16).

In both of the typical cases considered above, one may characterize the effectiveness of firing with considerable completeness by a single quantity, which can be called the index of effectiveness of the system of firing. In the first case, this was the probability \( P(A) \), and in the second case it was \( M(\xi) \). In both cases the index of effectiveness was an expression of the form

\[
(17) \quad W = c_1 R_1 + c_2 R_2 + \ldots + c_n R_n
\]

where the \( c_n \) are certain non-negative coefficients. One may believe
that (17) is sufficiently flexible to embrace the intermediate cases which we have not considered.

Once the index of effectiveness for a given concrete situation of firing has been chosen, it is natural to search for that system of firing, among all those with the given number of shots \( n \), which yields the largest value for the index of effectiveness and to call this the most advantageous. We shall consider such problems with a given "expenditure of military supplies" in §5.

In concluding this paragraph, we remark that

\[
M(\mu) = R_1 + R_2 + \ldots + R_n
\]

and that \( R_m \) and \( M(\mu) \) are related by the celebrated inequality of Chebycheff: ("Chebycheff")

\[
R_m \leq \frac{M(\mu)}{m}.
\]

§2. The Case of Mutual Independence of Hits in Individual Shots

We suppose that the numbers

\[
i = 1, 2, 3, \ldots, n
\]

are assigned to the \( n \) shots under consideration. We denoted by \( E_i \) the chance event consisting in the obtaining of a hit with the \( i^{th} \) shot and by \( E_i^c \) the opposite event, consisting in a non-hit with the \( i^{th} \) shot. The corresponding probabilities are denoted by

\[
\begin{align*}
p_i &= P(E_i) \\
q_i &= P(E_i^c) = 1 - p_i
\end{align*}
\]
As is known, the mathematical expectation can be written as

\[ M(\mu) = p_1 + p_2 + \ldots + p_n \]

The use of (21) makes all calculations concerning the mathematical expectation \( M(\mu) \) especially simple.

In contrast to the mathematical expectation, which can be written as in (21), the probabilities \( P_m \) and \( R_m \) cannot be defined, generally speaking, uniquely by the probabilities \( p_i \). In order to define \( P_m \) and \( R_m \) it is necessary, generally speaking, to know the probabilities \( p_i \) and also the type of dependence that exists among these probabilities. In this paragraph, we consider the case of mutual independence among the events \( B_1, \ldots, B_n \).

In this case the probability that hits are obtained with the shots numbered \( i_1, \ldots, i_n \) \((i_1 < i_2 < \ldots < i_n)\) and non-hits with all other shots is equal to

\[ p_{i_1} \cdot p_{i_2} \cdot \ldots \cdot p_{i_m} \cdot \prod_{j \neq i_s} q_j , \]

or, equivalently

\[
\begin{pmatrix}
\frac{p_{i_1}}{q_{i_1}} & \frac{p_{i_2}}{q_{i_2}} & \ldots & \frac{p_{i_m}}{q_{i_m}}
\end{pmatrix}
\frac{n}{j=1} q_j .
\]

[Translator's comment: this formula is of course invalid if any \( q_j \) is zero.] Adding up all such products, we obtain the formula

\[ P_m = \left( \sum_{i_1 < \ldots < i_m} \frac{p_{i_1}}{q_{i_1}} \ldots \frac{p_{i_m}}{q_{i_m}} \right) \frac{n}{j=1} q_j . \]
In particular,

$$P_0 = \prod_{j=1}^{n} q_j,$$

$$P_1 = \left( \sum_{i=1}^{n} \frac{p_i}{q_i} \right) \prod_{j=1}^{n} q_j,$$

$$P_2 = \left( \sum_{i_1 < i_2} \frac{p_{i_1} p_{i_2}}{q_{i_1} q_{i_2}} \right) \prod_{j=1}^{n} q_j.$$

The number of terms in parentheses in (22) is

$$\binom{m}{n} = \frac{n!}{m! (n-m)!}.$$

Hence if (23) $p_1 = \ldots = p_n = p$, we obtain from (22) the well-known binomial formula

$$P_m = \binom{m}{n} p^m q^{n-m},$$

where

$$q = 1 - p.$$

The general formula (22) is so complicated that one could not recommend its use except in case of urgent necessity. In case all of the numbers $p_i$ are small enough, (22) can be replaced by much more convenient approximate formulas. These approximations become better as the maximum

$$\lambda = \max (p_1, \ldots, p_n)$$

decreases. The simplest of these approximations is the formula of Poisson:
\( P_m = \frac{a^m}{m!} e^{-a} + O(\lambda) \),

where

\( a = p_1 + p_2 + \ldots + p_n = M(\mu) \).

(See infra for a proof of (27).) (27) represents \( P_m \) up to an additive error of order \( \lambda \). The practical application of (27) is simplified by the use of tables of the function

\( \psi(m,a) = \frac{a^m}{m!} e^{-a} \).

(See K. Pearson, "Tables for Statisticians and Biometricians". In abbreviated form, reprinted in "Technique of Statistical Calculation" by A. K. Mitropolski (Sel'hozgiz, 1931).)

One may also suggest the more complicated but more exact formula

\( P_m = \psi(m,a) - \frac{b}{2} \nabla^2 \psi(m,a) + O(\lambda^2) \),

where

\( b = \sum_{i=1}^{n} p_i^2 \)

and

\( \nabla^2 \psi(m,a) = \psi(m,a) - 2\psi(m-1,a) + \psi(m-2,a). \) \( (\psi(m,a) = 0 \) for \( m < 0 \).

In the case (23) where all \( p_i \)'s are equal, we have \( a = np, b = np^2 = \frac{a^2}{n} \), \( \lambda = p = a/n \), and (27) and (30) reduce to

\( P_m = \psi(m,a) + O\left(\frac{1}{n}\right) \),

\( P_m = \psi(m,a) - \frac{a^2}{2n} \nabla^2 \psi(m,a) + O\left(\frac{1}{n^2}\right) \).
(These remainder terms \(O\left(\frac{1}{n}\right)\) and \(O\left(\frac{1}{n^2}\right)\) can be used in other cases if a merely remains bounded for all values of \(n\).)

The advantage of using (30) and (34) as against (27) and (33) is shown by the following example: \(n=5; p_1 = \ldots = p_5 = 0.3; a = 1.5; b = .45\). Let \(P_n' = \psi(m,a)\) and \(P''(m,a) = \psi(m,a) - \frac{b}{2} \nabla^2 \psi(m,a)\). Values of \(P, P',\) and \(P''\) are given in Table 1.

<table>
<thead>
<tr>
<th>(m)</th>
<th>(P_m)</th>
<th>(P'_m)</th>
<th>(P_m - P'_m)</th>
<th>(P''_m)</th>
<th>(P_m - P''_m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.16807</td>
<td>0.22313</td>
<td>-0.05506</td>
<td>0.17293</td>
<td>-0.00486</td>
</tr>
<tr>
<td>1</td>
<td>0.36015</td>
<td>0.02546</td>
<td>0.02546</td>
<td>0.35960</td>
<td>0.00035</td>
</tr>
<tr>
<td>2</td>
<td>0.30870</td>
<td>0.25102</td>
<td>0.05766</td>
<td>0.29495</td>
<td>0.01375</td>
</tr>
<tr>
<td>3</td>
<td>0.13230</td>
<td>0.12551</td>
<td>0.00679</td>
<td>0.13495</td>
<td>-0.00265</td>
</tr>
<tr>
<td>4</td>
<td>0.02835</td>
<td>0.04707</td>
<td>-0.01872</td>
<td>0.03648</td>
<td>-0.00813</td>
</tr>
<tr>
<td>5</td>
<td>0.00243</td>
<td>0.01412</td>
<td>-0.00829</td>
<td>0.00388</td>
<td>-0.00145</td>
</tr>
</tbody>
</table>

Since the probabilities \(p_1\) in this example are fairly large, (27) and (33) yield only very crude approximations to \(P_m\) but (30) and (34) give approximations \(P''_m\) which differ from \(P_m\) by less than 0.015.

We consider another example: \(n = 50; p_1 = p = .03; a = 1.5; b = 0.045\). The results for \(m = 0, 1, 2, 3, 4, 5\) are set forth in Table 2.

<table>
<thead>
<tr>
<th>(m)</th>
<th>(P_m)</th>
<th>(P'_m)</th>
<th>(P_m - P'_m)</th>
<th>(P''_m)</th>
<th>(P_m - P''_m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.21805</td>
<td>0.22312</td>
<td>-0.00508</td>
<td>0.21811</td>
<td>-0.00006</td>
</tr>
<tr>
<td>1</td>
<td>0.33726</td>
<td>0.33469</td>
<td>0.00257</td>
<td>0.33720</td>
<td>0.00006</td>
</tr>
<tr>
<td>2</td>
<td>0.25550</td>
<td>0.25102</td>
<td>0.00448</td>
<td>0.25541</td>
<td>0.00009</td>
</tr>
<tr>
<td>3</td>
<td>0.12643</td>
<td>0.12551</td>
<td>0.00092</td>
<td>0.12645</td>
<td>-0.00002</td>
</tr>
<tr>
<td>4</td>
<td>0.04603</td>
<td>0.04707</td>
<td>-0.00104</td>
<td>0.04601</td>
<td>0.00002</td>
</tr>
<tr>
<td>5</td>
<td>0.01308</td>
<td>0.01412</td>
<td>-0.00109</td>
<td>0.01310</td>
<td>-0.00002</td>
</tr>
</tbody>
</table>
Here the ordinary Poisson formula gives acceptable results
\(|P_m - P'_m| < .005\) and the more precise formula (30) and (34) give
\(P_m\) with an error < 0.0001.

For the probabilities \(P_m\) of obtaining at least \(m\) hits, we
obtain from (27), (30), (33), and (34)

\[ R_m = H(m,a) + O(\lambda) \; ; \]
\[ R_m = H(m,a) - \frac{b}{2} \nabla^2 H(m,a) + O(\lambda^2) \; ; \]
\[ R_m = H(m,a) + O\left(\frac{1}{n}\right) \; ; \]
\[ R_m = H(m,a) - \frac{\sigma^2}{2n} \nabla^2 H(m,a) + O\left(\frac{1}{n^2}\right) \; , \]

where

\[ H(m,a) = 1 - \psi(0,a) - \psi(1,a) - \ldots - \psi(m-1,a) \; . \]

(For \(m < 1\), let \(H(m,a) = 1\)). Tables of the functions

\[ H(1,a) = 1 - e^{-a} \]
\[ H(2,a) = 1 - e^{-a}(1+a) \]

are to be found in a number of works on the theory of firing. (See,
for example, P. A. Gel'vih, "Firing", vol. 1).

Values of \(H(m,a)\) and

\[ \nabla^2 H(m,a) = -\nabla \psi(m-1,a) \]

for \(m \leq 11\) are found in tables I and II in the appendix. (See also
chart I.) With the aid of these tables, determination of \(R_m\) by the
use of (36) and (38) is carried out with a very small expenditure of
effort.
For example, let \( n = 24 \),

\[
\begin{align*}
p_1 &= \ldots = p_6 = 0.2 \\
p_7 &= \ldots = p_{12} = 0.1 \\
p_{13} &= \ldots = p_{18} = 0.15 \\
p_{19} &= \ldots = p_{24} = 0.05,
\end{align*}
\]

and let it be required to find \( R_3 \). Then \( a = 3.0 \), \( b = 0.45 \), and by the tables, we find, for \( m = 3 \) and \( a = 3 \), \( H = 0.577 \), \( \nabla^2 H = -0.075 \). Then \( R_3 \approx 0.577 + \frac{0.45}{2} \times 0.075 = 0.594 \), in view of (36).

Carrying out elementary calculations as required by (22), we find \( R_3 = 0.59503 \).

Returning to the verification of formulas (27) and (30), we find that they are obtained by expanding the probabilities \( P_m \) in a Charlier series:

\[
(42) \quad P_m = \sum_{k=0}^{\infty} A_k \nabla^k \psi(m, t),
\]

where

\[
(43) \quad A_k = e^t \sum_m \nabla^k \psi(m, t) P_m,
\]

\[
(44) \quad \nabla^0 \psi(m, t) = \psi(m, t),
\]

\[
\nabla^{k+1} \psi(m, t) = \nabla^k \psi(m, t) - \nabla^k (m-1, t).
\]

Translator's comment: '(44) was evidently incorrect in the Russian text and has been altered.

The coefficients \( A_k \) can be written in the form
(45) \[ A_k = \sum_{i=0}^{k} (-1)^i \frac{t^i}{i!} \frac{F_{k-i}}{(k-i)!} \]

where the numbers \( F_s \) are the "factorial moments" of the random variable, i.e., \( F_0 = \sum_{m=0}^{n} P_m = 1 \), \( F_1 = \sum_{m=0}^{n} mP_m = a \), and in general

(46) \[ F_s = \sum_{m=0}^{n} m(m-1)\ldots(m-s+1)P_m \]

Formulas (42)–(46) are applicable not only to our present case, in which \( P_m \) have the form (22), but also to the probabilities \( F(\mu=m) \) corresponding to an arbitrary random variable \( \mu \) which can assume only a finite number of non-negative integral values. The parameter \( t \) is arbitrary in the above expansions. Ordinarily we set \( t = a \).

Then the formulas for the first coefficients are somewhat simplified. Specifically, for \( t = a \), we have

(48) \[
\begin{align*}
A_0 &= 1 , \\
A_1 &= 0 , \\
A_2 &= \frac{F_2}{2} - aF_1 + \frac{a^2}{2} , \\
A_3 &= \frac{F_3}{6} - a\frac{F_2}{2} + \frac{a^2}{2}F_1 - \frac{a^3}{6} .
\end{align*}
\]

(For a discussion of the Charlier series, see V. I. Romanovskii, "Matematicheskaya statistika", GONTI, 1938, §60.)

(If \( \mu \) assumes an infinite set of values \( m = 0, 1, 2, 3, \ldots \), then (42)–(46) can be applied only if certain restrictions are placed on the probabilities \( P_m \). For example, it suffices to demand convergence of the series)
\[ \sum_{n=0}^{\infty} \frac{p_m}{(m+1) \sqrt{\psi(m,t)}}. \]

For our special case, we set
\[ a = \sum_{i=1}^{n} p_1 \]
\[ b = \sum_{i=1}^{n} p_1^2 \]
\[ c = \sum_{i=1}^{n} p_1^3 \]
\[ d = \sum_{i=1}^{n} p_1^4 \]

and obtain, for \( t = a \),
\[ A_0 = 1, A_1 = 0, A_2 = -\frac{b}{2}, \]
\[ A_3 = -\frac{c}{3}, A_4 = -\frac{d}{4} + \frac{b^2}{8}. \]

Since \( b, c, d \) are quantities of order not exceeding \( \lambda, \lambda^2, \lambda^3 \), respectively, we obtain
\[ A_2 = O(\lambda), A_3 = O(\lambda^2), A_4 = O(\lambda^2). \]

One can prove that, for any \( k \geq 1 \),
\[ A_{2k-1} = O(\lambda^k), A_{2k} = O(\lambda^k). \]

Closer analysis reveals not only that the coefficients of (42) have orders as in (51a) for the case we are considering (i.e., \( p_m \) given by (22) and \( t = a \)) but that the same bounds apply to the sums of all succeeding terms. In this fashion, in our case, breaking off the
series at the term with subscript $2k-2$, we are left with an error whose order is not greater than $\lambda^k$. From this and (50) we obtain (27) and (30).

In questions of an applied character, the order of an error in relation to some parameter (in our case $\lambda$) chosen as principal infinitesimal gives only an extremely preliminary indication concerning the usefulness of this or that approximating formula. In order to give one's self a definite insight into the practical applicability of an approximating formula for small but finite values of the parameter, it is necessary either to calculate a sufficient number of typical examples or to obtain bounds on the error in the form of inequalities, valid for finite values of the parameter. Up to the present time, the author has obtained reasonably simple evaluations of this kind only for the case $m = 1$. We now give without proof these inequalities for (35) and (38). (See von Mises, Wahrscheinlichkeitsrechnung, 1931, page (49).)

For the case $m = 1$, (35) and (36) yield

\begin{align}
(52) \quad R_1 &= 1 - e^{-a} + O(\lambda) \\
(53) \quad R_1 &= 1 - e^{-a}(1 - \frac{b}{2}) + O(\lambda^2) .
\end{align}

(We have $\nabla^2 H(a,1) = H(a,1) - 1$ since $H(a,m) = 1$ for $m < 1$.) The corresponding evaluations of $R_1$ in the form of inequalities are:

\begin{align}
(54) \quad 1 - e^{-a} &\leq R_1 \leq (1 - e^{-a})(1 + \frac{\lambda}{2(1 - \lambda)}) , \\
(55) \quad 1 - e^{-a}(1 - \frac{b}{2}) &\leq R_1 \leq \left[ 1 - e^{-a}(1 - \frac{b}{2}) \right] \left( 1 + \frac{\lambda^2}{3(1 - \lambda)} \right) .
\end{align}
TABLE 3

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( \frac{\lambda}{2(1-\lambda)} )</th>
<th>( \frac{\lambda^2}{3(1-\lambda)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>0.010</td>
<td>0.00014</td>
</tr>
<tr>
<td>0.05</td>
<td>0.026</td>
<td>0.00037</td>
</tr>
<tr>
<td>0.10</td>
<td>0.056</td>
<td>0.0038</td>
</tr>
<tr>
<td>0.20</td>
<td>0.125</td>
<td>0.017</td>
</tr>
<tr>
<td>0.30</td>
<td>0.214</td>
<td>0.043</td>
</tr>
</tbody>
</table>

From table 3, it is seen, for example, that with \( \lambda = 0.2 \) we risk an error less than or equal to 12\(\frac{1}{2}\)% in calculating \( R_1 \) by (52) and an error less than or equal to 1.7% in using (53).

It would be extremely desirable to obtain such compact evaluations in the form of inequalities for the probabilities \( R_m \) with \( m > 1 \).

§3. Classification of factors influencing the result of firing, and the meaning of dependence between hits for individual shots.

It is convenient to divide the factors which determine the result of firing into the following four groups:

1. Factors which, in the problem under consideration, are not known and which are fixed beforehand.

2. Factors which are under our control. Among these factors are, as a rule: the number of shots and their distribution in time (within the limits imposed by the number of guns, their rate of fire, and the available supply of shells); azimuth, elevation, fuze (in the case of explosive shells), for each shot. For definiteness, we consider the case of non-time-fuzed shells, and firing consisting of
n shots, each occurring at a fixed time. Thus for each shot two parameters are under our control: azimuth and elevation, which we denote, for the \( i \)th shot, by \( \alpha_i \) and \( \beta_i \) respectively. Then the mathematical problem of selecting a system of firing which is rational under the given conditions consists in the most advantageous choice of the 2n parameters

\[
\alpha_1, \alpha_2, \ldots, \alpha_n, \\
\beta_1, \beta_2, \ldots, \beta_n.
\]

3. Random factors which simultaneously influence the results of all or some of the shots. Such factors include, for example, errors in the location of the target, if the positioning of the gun depends upon this for various shots (and, in general, the so-called "repeating errors"), the type of maneuvering employed by a moving target, etc. In the mathematical study of problems of firing it is supposed that all factors of this third group are defined by the values assigned to a certain number of parameters

\[
\varphi_1, \varphi_2, \ldots, \varphi_s,
\]

which obey a certain law of probability distribution. This law is usually given in the form of a description of the corresponding density of probabilities:

\[
f(\varphi_1, \varphi_2, \ldots, \varphi_s).
\]

The set of all parameters \( \varphi_r (r = 1, \ldots, s) \) we denote by the single letter \( \varphi \).

4. Random factors which are independent of each other and of the factors of the 3rd group, which affect only one shot. These
factors, which produce so-called "technical dispersion", include aiming errors in the case of independent aimings on each shot.

The calculation of $P_m$, $R_m$, $p_1$, $M(\mu)$, and $\bar{W}$, discussed in §1 and §2, is carried out under the assumption that all factors of the first two groups are determined: it does not make sense, for example, to speak of obtaining three hits in a firing under completely unknown conditions or under determined external conditions but with unknown positioning of the gun. Thus the probabilities $P_m$, $R_m$, $p_1$, and the quantities $M(\mu)$ and $\bar{W}$ are considered as functions of the parameters $\sigma_1$ and $\beta_1$ (their dependence on the factors of the first group need not be specified explicitly, since these factors are to be considered as constant throughout the entire operation.)

Along with the unconditional probabilities $P_m$, $R_m$, and $p_1$ and the mathematical expectation $M(\mu)$, we will consider the conditional probabilities.

\begin{align*}
(56) \quad P_m(\infty) &= \bar{F}(\mu = m | \varrho_1, \ldots, \varrho_s), \\
(57) \quad R_m(\infty) &= \bar{F}(\mu > m | \varrho_1, \ldots, \varrho_s), \\
(58) \quad p_1(\infty) &= \bar{F}(B_1 | \varrho_1, \ldots, \varrho_s),
\end{align*}

and the conditional mathematical expectation

\begin{equation}
(59) \quad M(\mu | \Phi) = M(\mu | \varrho_1, \ldots, \varrho_s),
\end{equation}

for fixed values of the parameters $\varrho_1, \varrho_2, \ldots, \varrho_s$. They are related to the unconditional probabilities and expectation by the known formulas

\begin{equation}
(60) \quad P_m = \int \int \cdots \int P_m(\infty) f(\infty) d\varrho_1 \ldots d\varrho_s,
\end{equation}
(61) \[ R_m = \int \int \cdots \int R_m(\theta)f(\theta)d\theta_1 \cdots d\theta_s, \]

(62) \[ P_1 = \int \int \cdots \int P_1(\theta)f(\theta)d\theta_1 \cdots d\theta_s, \]

(63) \[ M(\mu) = \int \int \cdots \int M(\mu|\theta)f(\theta)d\theta_1 \cdots d\theta_s. \]

For the quantities

(64) \[ W(\theta) = \sum_{i=1}^{n} c_i R_i(\theta), \]

and

(65) \[ \mathcal{W} = \sum_{i=1}^{n} c_i R_i, \]

we have the analogous formula

(66) \[ W = \int \int \cdots \int W(\theta)f(\theta)d\theta_1 \cdots d\theta_s. \]

From the point of view of the present §, one may say that when we considered in §2 the events \( B_1 \) as being independent, we neglected the existence of factors of the third group. Speaking generally, it follows that we must consider the random events \( B_1 \) as only conditionally independent for fixed values of the parameters \( \theta_1, \ldots, \theta_s \).

In view of this, all the formulas of §2 must be applied not to the unconditional probabilities \( P_m, R_m, \) and \( P_1 \), but to the corresponding conditional probabilities. For example, we combine (22) and (60) to obtain

(67) \[ P_m = \int \int \cdots \int f(\theta) \left( \sum_{1 \leq i_1 < i_2 < \cdots < i_m} \frac{P_{i_1}(\theta)}{\lambda_{i_1}(\theta)} \cdots \frac{P_{i_m}(\theta)}{\lambda_{i_m}(\theta)} \right) \times \]

\[ \prod_{k=1}^{n} q_k(\theta)d(\theta)_1 \cdots d(\theta)_s, \]
which is valid without any special assumption concerning the unconditional independence of the events $B_1$. (We set $q_1(\theta) = 1 - p_1(\theta).$)

Naturally, all of the conditional probabilities $P_m(\theta), R_m(\theta), p_1(\theta),$ and the mathematical expectation $M(\mu | \theta)$ depend upon $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ as well as upon $\theta_1, \ldots, \theta_s$.


We turn now to the problem which was mentioned at the end of §1: to determine the system of firing which will lead to the largest index of effectiveness $\mathcal{W}$. Mathematically, this problem consists in finding the maximum value, $\max \mathcal{W}$, of the function

(68) \hspace{1em} \mathcal{W}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n),

and such combinations $\alpha_1^*, \ldots, \alpha_n^*, \beta_1^*, \ldots, \beta_n^*$ that

(69) \hspace{1em} \mathcal{W}(\alpha_1^*, \ldots, \alpha_n^*; \beta_1^*, \ldots, \beta_n^*) = \max \mathcal{W}.

Let us assume that the probability $p_1$ depends only upon $\alpha_1$ and $\beta_1$ and not on any $\alpha_j, \beta_j$ for $j \neq 1$. (This can be faulty, for example, if the maneuvers of the target are influenced by the firing.) Let us further assume that, as happens very often, the function $p_1(\alpha_1, \beta_1)$ assumes its maximum

(70) \hspace{1em} \max p_1 = p_1(\overline{\alpha}_1, \overline{\beta}_1)

for a unique pair $(\overline{\alpha}_1, \overline{\beta}_1)$ of values. In this case, the question naturally arises as to whether or not

(71) \hspace{1em} \max \mathcal{W} = \mathcal{W}(\overline{\alpha}_1, \overline{\alpha}_2, \ldots, \overline{\alpha}_n; \overline{\beta}_1, \ldots, \overline{\beta}_n).
In two special cases, the answer to this question is affirmative:

I. The equality (71) is valid for the case \( W = M(\mu) \).

II. The equality (71) is valid when \( W \) is given by \( W = c_1R_1 + \ldots + c_nR_n \), the \( c_i \)'s being \( \geq 0 \), if the events \( B_1 \) are mutually independent, i.e., if it is possible to ignore factors which were placed in the third group.

The assertion I is an immediate consequence of (21). To verify II, it suffices to observe that if the events \( B_1 \) are independent, the probabilities \( R_m \) are single-valued functions

\[
R_m = F_m(p_1, \ldots, p_n)
\]

of the probabilities \( p_1 \), and these functions are non-decreasing in each argument (as one may easily prove).

In the following discussion, however, we shall see that the cases in which

\[
W = W(\overline{a}_1, \ldots, \overline{a}_n; \overline{\beta}_1, \ldots, \overline{\beta}_n)
\]

is less than max \( W \) are just as important. These are systems of firing in which obtaining the maximum probability of obtaining a hit on each shot does not give the best system of firing. For cases of this kind one must deliberately choose \( a_1 \) and \( \beta_1 \) for various shots which are different from the values \( \overline{a}_1 \) and \( \overline{\beta}_1 \) giving maximum probability of hitting. Such a system of firing is called firing with artificial dispersion. From I and II, we now obtain:

1. Artificial dispersion is of no value if the index of effectiveness is \( M(\mu) \).
2. Artificial dispersion is useless if hits obtained by individual shots are independent random events.

Usually in these cases artificial dispersion not only is useless but also decreases the effectiveness of fire.

Typical situations in which artificial dispersion can be useful are those in which:

1°. the maximum effectiveness is obtained with a number of hits considerably smaller than \( n \);

2°. the most important random factors influencing the result of firing are those affecting the whole firing simultaneously.

The first of these conditions occurs in a particularly striking way when one hit alone will accomplish the desired result, i.e., when it is natural to put

\[
(74) \quad W = R_1.
\]

[Translator's comment: this seems to be at variance with II above, since in the present case \( c_1 = 1 \) and \( c_i = 0 \) for \( i > 1 \).]

§5. Probability of destruction of the target in the case \( P(A|m) = 1 - e^{-\alpha m} \).

Let us assume that

\[
(75) \quad \lim_{m \to \infty} P(A|m) = 1.
\]

In this case, with unbounded increase in the number of hits, the target must surely, sooner or later, be destroyed (with probability 1).
If in this case the hits occur one after another and not several at once, then the mathematical expectation of the number of hits after which the target will be destroyed is equal to

\[
\omega = \sum_{r=1}^{\infty} r D_r ,
\]

where \( D_r = P(A|r) - P(A|r-1) \) is the probability of destroying the target with exactly \( r \) hits. For the sequel, it is useful to note that \( \omega \) can also be written in the form:

\[
\omega = \sum_{r=0}^{\infty} 1 - P(A|r) .
\]

We call the quantity \( \omega \) the mean necessary number of hits. Obviously

\[
\omega \geq 1 .
\]

The equality

\[
\omega = 1
\]

occurs only in the case

\[
P(A|r) = \begin{cases} 
0 & \text{for } r = 0 \\
1 & \text{for } r \geq 1
\end{cases} .
\]

We obtain \( \omega = m \) in particular where

\[
P(A|r) = \begin{cases} 
0 & \text{for } r < m \\
1 & \text{for } r \geq m
\end{cases} .
\]

For \( \omega > 1 \) the case (80) is exceptional; more often, \( P(A|r) \) gradually increases with \( r \). We consider in this § the special case

\[
P(A|r) = 1 - e^{-\alpha r} ,
\]
where \( a \) is a certain positive constant. \((81)\) is no less (but no more!) arbitrary than \((80)\), but it leads to \textbf{decisively simpler results}.

\((81)\) and \((77)\) lead to

\[
\omega = \frac{1}{1-e^{-a}} ,
\]

\[(83)\]
\[a = -\log(1 - \frac{1}{\omega}) .
\]

We turn to the case of mutual independence of hits between the individual shots, and obtain

\[(84)\]
\[
\mathbb{P}(\overline{A}) = 1 - \mathbb{P}(A) = \sum_{i=1}^{n} p_i \left[ 1 - \mathbb{P}(A|1) \right] .
\]

Using \((22)\) and \((81)\), we obtain, after certain transformations,

\[(85)\]
\[
\mathbb{P}(\overline{A}) = \prod_{i=1}^{n} \left( p_i e^{-a} + q_i \right) .
\]

Having set

\[(86)\]
\[
p_i' = \frac{p_i}{\omega} = p_i (1 - e^{-a})
\]
\[
q_i' = 1 - p_i'
\]

we obtain from \((85)\)

\[(87)\]
\[
\mathbb{P}(\overline{A}) = \prod_{i=1}^{n} q_i' = \prod_{i=1}^{n} \left( 1 - \frac{p_i}{\omega} \right) ,
\]

\[(88)\]
\[
\mathbb{P}(A) = 1 - \prod_{i=1}^{n} \left( 1 - \frac{p_i}{\omega} \right) .
\]

\((88)\) shows that, under the given assumptions, the \underline{probability of destroying the target is equal to the probability which would appear}
If destruction were certain with one hit but $p_1$ were replaced by $p'_1 = \frac{p_1}{\omega}$.

The same conclusion is valid if dependence exists among the events $B_i$, when

\begin{equation}
(80) \quad p'_1(\omega) = \frac{p_1(\omega)}{\omega}.
\end{equation}
ARTIFICIAL DISPERSION IN THE CASE OF DESTRUCTION WITH
ONE HIT AND OF DISPERSION IN ONE DIMENSION

A. N. Kolmogorov

§1. General formulas.

§2. The principal questions which underlie the studies made in the present article.

§3. The example of P. A. Gel'vih.

§4. A second example.

§5. Conditions under which it is expedient to introduce artificial dispersion.

§6. An upper bound for R.

§7. The case of small b/E.

§8. Approximate calculation of reasonable amount of artificial dispersion in the case of a small target.

§1. General formulas.

In this work we shall assume that the positioning of the weapon for the $i^{th}$ shot is determined by a single parameter, which we call $x_i$. This permits us to investigate the principal questions in the choice of a rational amount of artificial dispersion, and determination of conditions under which artificial dispersion may be useful, without having recourse to enormously complicated mathematical computations. Formulas which relate to such a "one-dimensional" case have immediate practical applications. We mention a few of them:

1. In bombarding a long thin strip, placed perpendicular to the plane of firing, errors in azimuth are unimportant, and the probability of a hit depends only upon the choice of elevation.
Indeed, this case was selected for illustrating questions connected with artificial dispersion in the second volume of "Firing" by P. A. Gel'vih. In our further exposition we shall first of all consider this case and shall designate by the parameter $x_i$ the elevation used for the $i^{th}$ shot.

2. In the firing of naval torpedoes the choice of azimuth is the only essential. It is possible, indeed, that the considerations of the present work will find their most important immediate applications in just this field.

Questions concerning the dropping of bombs upon a long narrow strip have been subjected to detailed mathematical study. (See V. L. Gončarov, "Theory of Probabilities," §§52–53, 2nd edition, 1939, and V. L. Gončarov, "On the question of serial bombardment," Works of VVA, no. 24 (1938).)

We limit ourselves in the present chapter to the consideration of the dependence of the probability of obtaining at least one hit

\[(1) \quad R = R(x_1, \ldots, x_n)\]

upon the elevations $x_1, x_2, \ldots, x_n$. In accordance with the general formulas of §3 of the preceding work, we have

\[(2) \quad R = \int \int \cdots \int f(\Theta)R(\Theta)d\Theta_1 \cdots d\Theta_n\]

\[(3) \quad R(\Theta) = 1 - \prod_{i=1}^{n} [1 - p_i(\Theta)] .\]
We shall assume that $p_1(\theta)$ depends upon only the parameters $\theta_1, \cdots, \theta_8$ and the elevation $x_1$:

$$p_1 = p_1(x_1, \theta).$$

We now consider more closely the case of bombarding a long thin strip placed perpendicular to the plane of firing. We introduce the following notation:

$\eta = $ distance from the gun to the center of the strip,

$\beta = $ half the width of the strip,

$\xi = $ distance from the gun to point of impact of the shell,

$\xi_1 = $ distance from the gun to mean point of impact for elevation $x_1$,

$B_\theta = $ probable deviation in distance.

We shall consider $\beta$ and $B_\theta$ as known and $\eta$ and $\xi_1$ as random quantities. The conditional probability of obtaining a hit on the strip with the $i$th shot and given $\eta$ and $\xi_1$ is

$$(5) \quad F(B_1 | \eta, \xi_1) = \frac{\rho}{\sqrt{\pi} B_\theta} \int_{\eta - \beta}^{\eta + \beta} e^{- \rho^2 \left( \frac{\xi - \xi_1}{B_\theta} \right)^2} \, d\xi.$$

($\rho = 0.476936 \cdots$, as usual).

We now designate by

$\Delta$, the "value" of one division in elevation, i.e., the change in mean distance $\xi_1$ when the elevation is changed by one division.
(We consider $\Delta$ as constant and known. This is allowable if all the distances under consideration differ only slightly from the approximate value $\eta'$ of the distance $\eta$, which is assumed to be known.)

$y$, the elevation to which corresponds the mean distance $\eta$.

$x$, the elevation to which corresponds the mean distance $\xi$.

$b = \frac{\beta}{\Delta}, \quad E = \frac{\beta}{\Delta}$.

With this notation, (5) can be rewritten in the form

$$P(B_1 | y) = \frac{\rho}{\sqrt{\eta} E} \int_{y-b}^{y+b} e^{-\rho^{2} \left(\frac{x-x_1}{E}\right)^{2}} \, dx.$$  

(6)

With the hypotheses that have been made, the system of parameters $\Theta$ leads to the single parameter $y$. For defining $R$ on the basis of (2), (3), and (6) we obtain finally

$$R = 1 - \int_{-\infty}^{+\infty} f(y) \prod_{i=1}^{n} \left[1 - p_i(y)\right] dy,$$

(7)

$$p_i(y) = \frac{\rho}{\sqrt{\eta} E} \int_{y-b}^{y+b} e^{-\rho^{2} \left(\frac{x-x_1}{E}\right)^{2}} \, dx.$$  

(8)

The quantities $b$ and $E$ are nothing but the half-width of the target and the probable deviation in distance, only measured in "value" of the elevation, $\Delta$, taken as the unit of distance.
In conformity with this we shall for brevity call the quantity \( y \) the distance of the target.

We assume, finally, that the "distance" \( y \) itself is governed by the Gaussian law of probability distribution:

\[
(9) \quad f(y) = \frac{\rho^2}{\sqrt \pi \, E_u} \left( \frac{y - y_o}{E_u} \right)^2 \cdot e^{-\rho^2 \left( \frac{y - y_o}{E_u} \right)^2},
\]

when \( y_o \) is the probable distance of the target and \( E_u \) the probable deviation of the target.

Completely analogous formulas are obtained, under known hypotheses, for the cases of naval torpedo firing and aerial bombardment of a narrow strip. The reader will easily divine what concrete interpretations to place on the quantities \( b, E, E_u, x, \) and \( y \) in these cases.

§2. The principal questions which underlie the studies made in the present article. Setting, as usual,

\[
(10) \quad \varphi(z) = \frac{\rho}{\sqrt \pi} e^{-\rho^2 z^2},
\]

\[
(11) \quad F(z) = \frac{2\rho}{\sqrt \pi} \int_0^z e^{-\rho^2 u^2} \, du,
\]

and introducing the notations

\[
(12) \quad z = \frac{y - y_o}{E_u}, \quad a_1 = \frac{x_1 - y_o}{E_u}, \quad c = \frac{b}{E_u}, \quad K = \frac{E}{E_u},
\]

we write the relations (7), (8), (9) in a more compact form:
\[ R = 1 - \int_{-\infty}^{\infty} \phi(z) \prod_{i=1}^{n} \left[ 1 - p(z-a_i) \right] \, dz \]

\[ p(u) = \frac{1}{2} \left[ \Phi \left( \frac{u+c}{K} \right) - \Phi \left( \frac{u-c}{K} \right) \right] . \]

In the limiting case \( K = 0 \), formula (14) becomes

\[ p(u) = \begin{cases} 
1 \text{ for } |u| \leq c, \\
0 \text{ for } |u| > c.
\end{cases} \]

Formula (13) corresponds to firing with \( n \) elevations \( x_1, \ldots, x_n \) (which are in general all different.) If these \( n \) shots are divided into \( s \) groups consisting of \( n_1, \ldots, n_s \) \( (\sum_{r=1}^{s} n_r = n) \) and all \( n_r \) shots of the \( r \)th group are fired with a single elevation \( x^{(r)} \), then setting

\[ a^{(r)} = \frac{x^{(r)} - y_0}{E_u}, \]

we write (13) in the form

\[ R = 1 - \int_{-\infty}^{\infty} \phi(z) \prod_{r=1}^{s} \left[ 1 - p(z-a^{(r)}) \right] \, dz . \]

In particular, for firing with a single elevation \( x \) we obtain

\[ R = 1 - \int_{-\infty}^{+\infty} \phi(z) \left[ 1 - p(z-a) \right] \, dz, \quad a = \frac{x}{E_u} . \]
It is easy to show that the expression (16) assumes its largest value for \( a = 0 \). This maximum, evidently, is equal to

\[(17) \quad S_1 = S_1(K, c, n) = 1 - \int_{-\infty}^{\infty} \varrho(z) \left[ 1 - p(z) \right]^n dz,\]

where \( p(z) \) is defined by (14).

The elevation (18) \( \bar{x} = y_0 \), corresponding to \( a = 0 \), is easily shown to be that elevation which yields the largest possible value for the probabilities

\[(19) \quad p_1 = \int_{-\infty}^{\infty} \varrho(z) p(z - s_1) dz = \frac{1}{2} \left[ I \left( \frac{a_1 + c}{\sqrt{1 + K^2}} \right) - I \left( \frac{a_1 - c}{\sqrt{1 + K^2}} \right) \right] \]

of a hit for each individual shot. The general maximal values of the probabilities \( p_1 \), corresponding to \( a_1 = 0 \), \( x_1 = \bar{x} = y_0 \), are clearly equal to

\[(20) \quad \max p_1 = \bar{F} \left( \frac{c}{\sqrt{1+K^2}} \right) = \bar{F} \left( \frac{b}{E_n} \right),\]

where

\[(21) \quad E_n = \sqrt{E_u^2 + E_p^2} = E_u \sqrt{1 + K^2}\]

is the full probable deviation for each shot. Hence a firing with a single elevation \( \bar{x} = y_0 \) will be a firing without artificial dispersion in the sense of §4 of our preceding article.
We shall denote by

\[(22) \quad \max R = \max R(K, c, n)\]

the maximal value of \( R \) for given \( K, c, n \) and variable \( s_1 \) (that is, variable elevations \( x_1 \)). It is easy to prove that this maximal value actually exists.

More detailed analysis shows that \( \max R \) for given \( K, c, n \) exists for only one system of elevations \( x_1, x_2, \ldots, x_n \) (up to permutations of indices). Clearly

\[(23) \quad S_1 \leq \max R,\]

i.e., firing without artificial dispersion (with one elevation \( \overline{x} = y_0 \)) cannot be more advantageous than firing with the best choice of elevations. Cases are possible in which

\[(24) \quad S_1 = \max R,\]

i.e., in which firing with a single aiming is the best possible. In these cases, the use of artificial dispersion could be only damaging. This, for example, is the situation for \( n = 2, K = 1, c = \frac{1}{4} \), that is, with two shots, dispersion of the target position equal to the individual "technical" dispersion of each shot \( (K = 1, \text{i.e., } E_u = E) \) and width of target equal to one half of
the probable deviation of the target \((c = 1/4, \text{i.e., } 2b = \frac{1}{2} E_u)\). In this case any deviation from the choice of elevations \(s_1 = x_2 = y_0\) leads to a decrease of the probability \(R\).

Cases are also possible, along with \((24)\), in which

\[(25) \quad S_1 < \max R,\]

i.e., the introduction of rationally determined artificial dispersion leads to an increase of the probability \(R\) in comparison with the maximum value obtained by \(R\) with a single elevation. Such, for example, is the situation with \(n = 24, k = 1, c = 1/4,\) i.e., when the situation is just as in the preceding example, but 24 shots are allowed.

Since the exact calculation of \(\max R\) is an extremely laborious process for large values of \(n\) (in contrast to \(S_1\), which is obtained by \((17)\) and numerical integration), it is natural to seek criteria to determine whether we are dealing with \((24)\) or \((25)\). We consider this matter in \(\S 5\). Here it is shown that for fixed \(n > 1\) and \(c\) there always exists a

\[(26) \quad K_0 = K_0(c, n)\]

such that for

\[(27) \quad K \geq K_0,\]
artificial dispersion is harmful, and for

$$K < K_0 ,$$

it is beneficial. In case (28), max R is obtained always for elevations $x_1, x_2, \ldots, x_n$ symmetrically distributed about $y_0$.

The exact solution of the problem of determining max R and the corresponding values $x_1, \ldots, x_n$ of the elevation presents large difficulties even in the case $n = 4$. Furthermore, actual firing in which every shot is the result of a single aiming is inexpedient in view of the enormous complications in directing fire. Hence it is highly essential that one be able to obtain close approximations to max R with only two or three different elevations in the entire firing. In the case $n = 2m$, firing $m$ shots on each of the elevations

$$x^{(1)} = y_0 + \frac{a}{2} E_u ,$$

$$x^{(2)} = y_0 - \frac{a}{2} E_u ,$$

we obtain

$$R = 1 - \int_{-\infty}^{\infty} \phi(z) \left[ 1 - p(z + \frac{a}{2}) \right]^m \left[ 1 - p(z - \frac{a}{2}) \right]^m dz .$$
In the case $n = 3m$, firing $m$ shots with each of the three elevations

\begin{align*}
  x^{(1)} &= y_o, \\
  x^{(2)} &= y_o + aE_u, \\
  x^{(3)} &= y_o - aE_u,
\end{align*}

we obtain

\begin{equation}
  R = 1 - \int_{-\infty}^{\infty} \phi(z) \left[1 - p(z)\right]^m \left[1 - p(z + \varepsilon)\right]^m \left[1 - p(z - \varepsilon)\right]^m dz.
\end{equation}

There are no insuperable difficulties in defining the "steps" $\alpha_1$ and $\alpha_2$ of the elevation $a$, for which (29) and (30) attain their maxima $S_2$ and $S_3$, or in determining the maximal values themselves. It is clear that

\begin{equation}
  S_1 \leq S_2 \leq \max R,
\end{equation}

and that

\begin{equation}
  S_1 \leq S_3 \leq \max R.
\end{equation}

The relations $S_2 \leq \max R$ and $S_3 \leq \Max R$ often give good lower bounds on $\max R$. Upper bounds on $\max R$, which can in many cases be extremely useful, are given in §6.
53. The example of P. A. Gell'vih.

As an example, we consider the case \( \frac{E}{E_u} = \frac{1}{2}, \frac{b}{E_u} = \frac{5}{14}. \)

The probability of a hit for one shot with the best elevation \( \overline{x} = y_0 \) is equal in this case, by (20), to

\[
p = \frac{1}{\pi} \left( \frac{\frac{5}{14}}{\sqrt{1 + \left(\frac{1}{2}\right)^2}} \right) = \Phi (0.320) = 0.171.
\]

For the probability \( S_1 \) of at least one hit in \( n \) shots with one elevation \( \overline{x} = y_0 \), we find by numerical integration of formula (17) results given in Table 1.

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>6</th>
<th>12</th>
<th>18</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_1 )</td>
<td>0.171</td>
<td>0.294</td>
<td>0.385</td>
<td>0.453</td>
<td>0.544</td>
<td>0.669</td>
<td>0.724</td>
<td>0.756</td>
</tr>
<tr>
<td>( 1-(1-p)^n )</td>
<td>0.171</td>
<td>0.312</td>
<td>0.409</td>
<td>0.504</td>
<td>0.674</td>
<td>0.894</td>
<td>0.966</td>
<td>0.989</td>
</tr>
</tbody>
</table>

In the last line are given the probabilities for the indicated \( p \) and \( n \), of obtaining at least one hit if the probabilities of obtaining hits with individual shots are mutually independent. Comparison of \( S_1 \) with these figures shows that the dependence between hit probabilities which exists in our case is extremely disadvantageous for large \( n \).
This is completely understandable: since \( E_u = 2E \), i.e., "the dispersion of position of the target" is twice as large as the dispersion of shots about the mean point of impact, it is natural that firing many shots with one elevation is disadvantageous. In place of this, it is reasonable to turn to artificial dispersion, i.e., firing with several elevations, in the case of large \( n \). Numerical integration of formula (29) for \( n \) shots divided equally between two elevations

\[
x^{(1)} = y_0 + \frac{a}{2} E_u, \\
x^{(2)} = y_0 - \frac{a}{2} E_u,
\]

leads to the following values of the probability \( R \) of obtaining at least one hit (table 2):

<table>
<thead>
<tr>
<th>( a )</th>
<th>( n = 2 )</th>
<th>( n = 4 )</th>
<th>( n = 6 )</th>
<th>( n = 12 )</th>
<th>( n = 18 )</th>
<th>( n = 24 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.294</td>
<td>0.453</td>
<td>0.544</td>
<td>0.669</td>
<td>0.724</td>
<td>0.756</td>
</tr>
<tr>
<td>0.8</td>
<td>0.295</td>
<td>0.466</td>
<td>0.568</td>
<td>0.707</td>
<td>0.763</td>
<td>0.792</td>
</tr>
<tr>
<td>1.6</td>
<td>0.288</td>
<td>0.474</td>
<td>0.596</td>
<td>0.771</td>
<td>0.834</td>
<td>0.863</td>
</tr>
<tr>
<td>2.4</td>
<td>0.259</td>
<td>0.442</td>
<td>0.572</td>
<td>0.781</td>
<td>0.866</td>
<td>0.904</td>
</tr>
<tr>
<td>3.2</td>
<td>0.215</td>
<td>0.373</td>
<td>0.489</td>
<td>0.699</td>
<td>0.803</td>
<td>0.861</td>
</tr>
<tr>
<td>4.0</td>
<td>0.166</td>
<td>0.291</td>
<td>0.385</td>
<td>0.562</td>
<td>0.668</td>
<td>0.722</td>
</tr>
<tr>
<td>4.8</td>
<td>0.121</td>
<td>0.213</td>
<td>0.284</td>
<td>0.421</td>
<td>0.501</td>
<td>0.554</td>
</tr>
<tr>
<td>5.6</td>
<td>0.034</td>
<td>0.148</td>
<td>0.198</td>
<td>0.298</td>
<td>0.358</td>
<td>0.401</td>
</tr>
</tbody>
</table>
Analogously, formula (30) for \( n \) shots, divided equally among three elevations:

\[
x^{(1)} = y_0 \\
x^{(2)} = y_0 + \alpha E_u \\
x^{(3)} = y_0 - \alpha E_u,
\]

gives us

<table>
<thead>
<tr>
<th>( n )</th>
<th>3</th>
<th>6</th>
<th>12</th>
<th>18</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.385</td>
<td>0.544</td>
<td>0.669</td>
<td>0.724</td>
<td>0.756</td>
</tr>
<tr>
<td>0.4</td>
<td>0.389</td>
<td>0.560</td>
<td>0.695</td>
<td>0.750</td>
<td>0.781</td>
</tr>
<tr>
<td>0.8</td>
<td>0.394</td>
<td>0.588</td>
<td>0.749</td>
<td>0.809</td>
<td>0.840</td>
</tr>
<tr>
<td>1.2</td>
<td>0.383</td>
<td>0.597</td>
<td>0.790</td>
<td>0.862</td>
<td>0.893</td>
</tr>
<tr>
<td>1.6</td>
<td>0.357</td>
<td>0.578</td>
<td>0.799</td>
<td>0.882</td>
<td>0.926</td>
</tr>
<tr>
<td>2.0</td>
<td>0.323</td>
<td>0.537</td>
<td>0.775</td>
<td>0.883</td>
<td>0.933</td>
</tr>
<tr>
<td>2.4</td>
<td>0.287</td>
<td>0.486</td>
<td>0.726</td>
<td>0.848</td>
<td>0.912</td>
</tr>
<tr>
<td>2.8</td>
<td>0.252</td>
<td>0.433</td>
<td>0.663</td>
<td>0.791</td>
<td>0.865</td>
</tr>
</tbody>
</table>

By the foregoing considerations it is possible to determine the optimum values of \( s = \alpha_2 \) and \( s = \alpha_3 \) in the case of two and three different elevations respectively, with sufficient accuracy for
practical purposes, as well as the corresponding values of $S_2$ and $S_3$. In the following table we give for comparison the values of $S_1$ calculated above. (For the last column, see the following discussion.)

**TABLE 4**

<table>
<thead>
<tr>
<th>$n$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
<th>$P(N^*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.6</td>
<td>—</td>
<td>0.292</td>
<td>0.296</td>
<td>—</td>
<td>0.292</td>
</tr>
<tr>
<td>3</td>
<td>—</td>
<td>0.7</td>
<td>0.385</td>
<td>—</td>
<td>0.394</td>
<td>0.397</td>
</tr>
<tr>
<td>4</td>
<td>1.4</td>
<td>—</td>
<td>0.453</td>
<td>0.476</td>
<td>—</td>
<td>0.476</td>
</tr>
<tr>
<td>6</td>
<td>1.6</td>
<td>1.2</td>
<td>0.544</td>
<td>0.596</td>
<td>0.597</td>
<td>0.599</td>
</tr>
<tr>
<td>12</td>
<td>2.0</td>
<td>1.5</td>
<td>0.669</td>
<td>0.784</td>
<td>0.801</td>
<td>0.803</td>
</tr>
<tr>
<td>14</td>
<td>2.2</td>
<td>1.7</td>
<td>0.724</td>
<td>0.868</td>
<td>0.890</td>
<td>0.695</td>
</tr>
<tr>
<td>24</td>
<td>2.4</td>
<td>1.9</td>
<td>0.756</td>
<td>0.904</td>
<td>0.934</td>
<td>0.940</td>
</tr>
</tbody>
</table>

From this table, it is seen that for $n = 2, 3,$ and 4 the transition from two to three elevations is of very little use. For $n = 6$, the transition from one to two elevations is more marked: $S_2 - S_1 = 0.052$, but giving from two to three elevations gives only a very small increase in the probability of obtaining at least one hit: $S_3 - S_2 = 0.001$. For larger values of $n$, there are greater advantages gained in going from one to two and from two to three elevations.
This question naturally arises: can one not produce significant increases in the probability of obtaining at least one hit by going from firing with three elevations to firing with four or more elevations or by distributing the shots with each aiming not equally, but in some other way (e.g., for \( n = 24, 6 + 12 + 6 \))? Direct calculations in the case of more than three elevations are unwieldy in the extreme and are practically difficult. However, for the values of \( K, c, \) and \( n \) which we have selected, these calculations are almost unnecessary, since by other methods one may prove that the absolute maximum of \( R \), which cannot be exceeded for any choice of elevations \( x_i \), is in all cases which we consider only very little in excess of \( S_3 \) (for \( n = 3, 6, 12, 18, 24 \)) or \( S_2 \) (\( n = 2, 4 \)). For this purpose it is necessary to make use of the bound

\[
\max R \leq P(N^*),
\]

where

\[
N^* = \frac{2p}{\sqrt{\pi}} \tau(\frac{c}{K}) cn = 0.5382 \tau(\frac{c}{K}) cn,
\]

which is established in §6. The determination of \( P(N^*) \) for these formulas presents no difficulties, since tables of \( P(N) \) and \( \tau(u) \) are given in the appendix to the present collection (Tables III and IV). The values of \( P(N^*) \) corresponding to the values which we have selected for \( K, c, \) and \( n \), are presented in the last column of Table 4. Since we always have
\[
S_2 \leq \begin{cases} 
\max R \leq P(N^*) \\
S_3 \leq \end{cases}
\]

the numbers given in Table 4 permit us in all cases which we consider to determine \( \max R \) to within 1%.

We now turn to the second and third columns of Table 4. They show that, although \( K = E/E_u \) and \( c = b/E_u \) are constant, it is still necessary, for maximum effectiveness, to choose the differences between elevations differently for different values of \( n \); specifically, these differences increase.

This observation evidently has validity more generally: it seems likely that for fixed \( K, c \), and number of different elevations, among which the shots are equally distributed, the most advantageous differences between these elevations will increase as \( n \) increases.

In order to give this conjectural assertion a precise meaning, it is necessary to introduce some kind of characteristic of the "magnitude" of artificial dispersion. As such a characteristic it is natural to choose the mean square deviation \( \sigma_n \) of the elevations \( \bar{x}_i \) from \( \bar{x} = y_0 \), defined by the formula

(33) \[ \sigma_n^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2 = \frac{1}{n} \sum_{r=1}^{n} n_r (x(r) - \bar{x})^2 \].

Translator’s note: formula (33) is a mass of misprints in the Russian text: this is my best guess at the correct expression.

It is evident that, putting
\[ c_n^2 = \frac{1}{n} \sum_{i=1}^{n} a_i^2 = \frac{1}{n} \sum_{r=1}^{n_r} n_r (a(r))^2, \]

we will have

\[ \sigma_n = d_n E_u. \]

For the cases examined by us in detail, of firing with two or three elevations, distributed symmetrically about \( \bar{x} = y_o \), with an equal number of shots for each elevation, we obtain from (34)

\[ d_n = a \text{ in the case of two elevations,} \]

\[ d_n = \sqrt{\frac{2}{3}} a = 0.816a \text{ in the case of three elevations.} \]

The notations introduced here will be used in §§5 and 8.

§4. A second example.

As a different example we consider the case

\[ K = E/E_u = 1, \quad c = b/E_u = 1/4. \]

In this case, the probability of a hit with one shot at an elevation \( \bar{x} = y_o \) is \( p = 0.095 \). For \( n \) shots distributed equally between two elevations \( x^{(1)} = y_o + \frac{a}{2} E_u \) and \( x^{(2)} = y_o - \frac{a}{2} E_u \), we obtain values of \( R \) as set forth in Table 5.
TABLE 5

<table>
<thead>
<tr>
<th>a</th>
<th>6</th>
<th>12</th>
<th>24</th>
<th>48</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.436</td>
<td>0.660</td>
<td>0.846</td>
<td>0.940</td>
</tr>
<tr>
<td>0.8</td>
<td>0.431</td>
<td>0.657</td>
<td>0.848</td>
<td>0.941</td>
</tr>
<tr>
<td>1.6</td>
<td>0.418</td>
<td>0.648</td>
<td>0.851</td>
<td>0.953</td>
</tr>
<tr>
<td>2.4</td>
<td>0.395</td>
<td>0.627</td>
<td>0.847</td>
<td>0.962</td>
</tr>
<tr>
<td>3.2</td>
<td>0.360</td>
<td>0.588</td>
<td>0.825</td>
<td>0.963</td>
</tr>
<tr>
<td>4.0</td>
<td>0.316</td>
<td>0.531</td>
<td>0.777</td>
<td>0.947</td>
</tr>
</tbody>
</table>

For $a = 0$, $x(1)$ and $x(2)$ coincide. Hence the first row of Table 5 gives the probabilities $S_1$ of at least one hit for one elevation.

One may determine $a_2$ and $S_2$ approximately by using Table 5. The results are shown in Table 6.

TABLE 6

<table>
<thead>
<tr>
<th>n</th>
<th>$a_2$</th>
<th>$S_2$</th>
<th>$S_2$</th>
<th>P(N*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>0.0</td>
<td>0.435</td>
<td>0.436</td>
<td>0.462</td>
</tr>
<tr>
<td>12</td>
<td>0.0</td>
<td>0.660</td>
<td>0.660</td>
<td>0.671</td>
</tr>
<tr>
<td>24</td>
<td>1.6</td>
<td>0.846</td>
<td>0.851</td>
<td>0.859</td>
</tr>
<tr>
<td>48</td>
<td>2.8</td>
<td>0.940</td>
<td>0.963</td>
<td>0.968</td>
</tr>
</tbody>
</table>
For \( n = 6 \) and \( n = 12 \), we have \( a_2 = 0 \) and \( S_2 = S_1 \). This means that it is impossible to increase the effectiveness of firing by going from one elevation to two elevations. One could show that for the given values of \( K \) and \( c \) and \( n = 6 \) or \( n = 12 \), in general \( \max R = S_1 \); i.e., artificial dispersion is of no value. In these cases, the bound \( P(N^*) \) is very large. For \( n = 24 \), in contrast to \( n = 6 \) and \( n = 12 \) firing with two elevations yields some increase in the effectiveness, although it is not significant. From the value of \( P(N^*) \) one can see that increase in the number of elevations could result only in a highly insignificant bettering of the effectiveness.

For \( n = 48 \), the advantage in introducing two elevations is very real: the probability of getting no hits in the case of one elevation is

\[
1 - S_1 = 0.060 ,
\]

and with two aimings it falls to

\[
1 - S_2 = 0.037 ,
\]

i.e., more than 35\%.

The example which we have selected is typical. Indeed, it is possible to prove that for any \( K < \infty \) and \( c < \infty \), there exists a positive integer \( n_o(K,c) \) such that artificial dispersion is useless for

\[
(38) \quad n \leq n_o(K,c)
\]
but becomes useful for

\[ n > n_0(K, c) \] (39)

In some cases, \( n_0(K, c) = 1 \), e.g., in the example of §3. Sometimes \( n_0(K, c) \) is so large that its existence has no practical interest—but theoretically, it always exists. In view of the calculations made in the present §5, \( n_0(1, 1/4) \) lies between 12 and 24. A precise examination of this example by the method expounded in §5 shows that \( n_0(1, 1/4) = 19 \).

§5. Conditions under which it is expedient to introduce artificial dispersion.

We shall take \( c \) and \( n > 1 \) as constant, and \( K \) variable. In the extreme case \( K = 0 \); i.e., in the case where there is only dispersion in the position of the target \( (E_u > 0) \) and total absence of technical dispersion \( (E = 0) \), it is easy to discover that the most advantageous firing is with \( n \) different elevations

\[ x_1 = y_o + (-n + 1)b \]
\[ x_2 = y_o + (-n + 3)b \]
\[ x_3 = y_o + (-n + 5)b \]
\[ \cdots \cdots \cdots \]
\[ x_n - 1 = y_o + (n - 3)b \]
\[ x_n = y_o + (n - 1)b \] (40)
distributed with an interval $2b$ equal to the width of the target and symmetrically distributed about $y_0$. The corresponding value of $\max R$ is

$$\max R = \frac{\Phi}{\Phi} \left( \frac{\delta}{\delta u} \right) = \Phi(nc),$$

since the target will be destroyed for this distribution of elevations if and only if the center $y$ of the target lies in the interval $y_0 - nb \leq y \leq y_0 + nb$, and the probability that this will occur is expressed by (41).

It appears extremely likely that for increase in $K$, the most advantageous distribution of the points $x_i$ changes in a continuous manner, that their distribution is always symmetrical about $y_0$, and that all points $x_i$ can be the same only when they simultaneously take on the value $y_0$.

If these conjectures, which we have not succeeded in proving rigorously up to the present time, are correct, then confluence of all $x_i$ will surely take place for some finite value $K = K_0$. For $K \geq K_0$, the introduction of artificial dispersion will be harmful; i.e., the best distribution of the points $x_i$ for $K \geq K_0$ is simply

$$x_1 = x_2 = \ldots = x_n = y_0.$$ 

In figure 1, we present schematically a possible movement of the best possible positions of $x_1, \ldots, x_n$ for $n = 4$. 
In this chart, the differences between the points $x_i$ decrease continuously as $K$ increases from zero to $K_0$. One must not think that this is always so: for some values of $c$, the changes in $x_i$ with changes in $K$ are quite different. Indeed, cases are known in which the points $x_i$, with increasing $K$, at first go farther apart from each other and only later begin to approach each other.

To determine $K_0 = K_0(c,n)$, we consider $R(a_1, \ldots, a_n)$ for small $a_i$. For $a_1 = a_2 = \ldots = a_n = 0$
\[
\begin{align*}
\frac{\partial R}{\partial a_1} &= 0, \\
\frac{\partial^2 R}{\partial a_1 \partial a_j} &= T' = -\int_{-\infty}^{+\infty} \varphi(z) \left[p'(z)\right]^2 \left[1 - p(z)\right]^{n-2} dz, \quad i \neq j, \\
\frac{\partial^2 R}{\partial a_1^2} &= T'' = -\int_{-\infty}^{+\infty} \varphi(z) p''(z) \left[1 - p(z)\right]^{n-1} dz.
\end{align*}
\]

Hence, for small \(a_1\), we have, up to terms of the 3rd order,

\[
R = S_1 + \frac{1}{2} T' \sum_{i \neq j} a_i a_j + \frac{1}{2} T'' \sum_1 a_1^2
\]

\[
= S_1 + \frac{1}{2} T' \sum_{i \neq j} a_i a_j + \frac{1}{2} T'' \sum_1 a_1^2,
\]

where

\[
T = T(K, c, n) = T'' - T'
\]

\[
= \int_{-\infty}^{+\infty} \varphi(z) \left\{ \frac{\left[p'(z)\right]^2}{1 - p(z)} + p''(z) \right\} \left[1 - p(z)\right]^{n-1} dz.
\]

In another fashion, one may represent
\[ R = S_1 + \frac{n^2}{2} T' \bar{a}^2 + \frac{n}{2} T (a_n^2 + \bar{a}^2), \]

or

\[ R = S_1 + \frac{1}{2} \left[ (n^2 - n) T' + nT'' \right] \bar{a}^2 + \frac{n}{2} T a_n^2. \]

For \( T < 0 \), as anyone can plainly see, the coefficient of \( \bar{a}^2 \) in (44) is necessarily negative. Consequently, in this event any small artificial dispersion is deleterious, since it leads to a diminution of \( R \) in comparison with \( S_1 \).

In the case \( T > 0 \), by introducing small artificial dispersion with the added restriction that \( \bar{a} = 0 \), we obtain an increase in \( R \) in comparison with \( S_1 \). Translator's remark: \( \bar{a} \) is the arithmetic mean of \( a_1, \ldots, a_n \).

This result is correct even without any improved hypotheses.

Analyzing (43), one can establish that the expression \( T(K, c, n) \) vanishes only for one value \( K_o' = K_o'(c, n) \), is positive for \( K < K_o' \) and negative for \( K > K_o' \).

Thus, for \( K < K_o' \), small artificial dispersions with \( \bar{a} = 0 \) are useful, and for \( K > K_o' \), they are deleterious.

Under the hypotheses set forth above, we have \( K_o' = K_o \), that is, \( K_o(c, n) \) is defined by means of the equality

\[ T(K_o, c, n) = 0. \]
§6. An upper bound for \( R \).

In this § we shall show that always

\[
R \leq P(N^*),
\]

where

\[
N^* = \tau(C)N = \tau(b)N,
\]

\[
N = \frac{2p}{\sqrt{\pi}} \frac{nc}{n} = 0.5382 \frac{nc}{n},
\]

\[
\tau(u) = -\frac{1}{2u} \int_{\infty}^{\infty} \log \left\{ 1 - \frac{1}{2} \left[ \Phi(z+u) - \Phi(z-u) \right] \right\} dz,
\]

and the function \( P(N) \) is defined by the equalities

\[
P(N) = \Phi(S_o) - 2\Phi(S_o),
\]

\[
S_o = \frac{1}{p} \left( \frac{4\pi}{4} \right)^{\frac{1}{3}} \frac{1}{3} (N^*)^{\frac{1}{3}} = 2.305 (N^*)^{\frac{1}{3}}.
\]

Tables of the functions \( \tau(u) \) and \( P(N) \) are given in the appendix. Determination of \( P(N^*) \) for given \( K, c, \) and \( n \) with the help of these tables and (47) and (48) is extremely simple. Furthermore, as we saw in §§3 and 4, the inequality (46) often gives a sufficiently precise bound on \( R \).
For proof of (46) we set

\begin{equation}
D(z) = -\frac{1}{2c} \sum_{i=1}^{n} \log [1-p(z-a_i)] .
\end{equation}

With (52), (13) can be rewritten as

\begin{equation}
R = 1 - \int_{-\infty}^{+\infty} \phi(z)e^{-2cD(z)} \, dz .
\end{equation}

Clearly

\begin{equation}
D(z) \geq 0 ,
\end{equation}

and

\begin{equation}
\int_{-\infty}^{\infty} D(z) \, dz = n T^{(c)} = \frac{\sqrt{\pi} N^*}{2pc} = 1.856 \frac{N^*}{c} .
\end{equation}

Putting to one side the definition of \( D(z) \) by (52), we consider the variational problem of finding the maximum of (53) where \( D(z) \) is subjected to the restrictions (54) and (55). One can show (see also the articles of I. A. Gubler and A. V. Svesnikov, following this article in the present collection) that this maximum is exactly equal to the quantity \( P(N^*) \) defined by (50) and is attained for

\begin{equation}
D(z) = \begin{cases} 
\frac{2}{2c} (S_o^2 - z^2) & \text{for } |z| < S_o , \\
0 & \text{for } |z| \geq S_o ,
\end{cases}
\end{equation}
where

\[ S_0 = \frac{1}{\rho} \left( \frac{34/\pi}{4} \right)^{\frac{1}{3}} (N^*)^{\frac{1}{3}}. \]

As a matter of fact, the function \( D(z) \) cannot be defined exactly by the use of (56), but from the foregoing, in any event, it follows that for any choice of \( a_1, \ldots, a_n \) we have

(57) \[ R \leq P(N^*), \]

as it was required to prove.

67. The case of small \( b/E \).

If \( c/K = b/E \) is small, then we have approximately

(58) \[ p(z-a_1) = \frac{c}{K} \varphi \left( \frac{z-a_1}{K} \right), \log \left[1-p(z-a_1)\right] = -\frac{c}{K} \varphi \left( \frac{z-a_1}{K} \right), \]

and (53) can be replaced by the approximate formula

(60) \[ R = 1 - \int_{-\infty}^{\infty} \varphi(z)e^{-2cD(z)}dz, \]

where

(61) \[ D(z) = \frac{1}{K} \sum_{i=1}^{n} \varphi \left( \frac{z-a_i}{K} \right). \]
It is natural to call the function $D(z)$ the induced density of fire, since $D(z)$ is the mathematical expectation of the number of hits in the strip

$$\xi_0 + zE_u \Delta < \xi < \xi_0 + (z + dz)E_u \Delta,$$

where $\xi_0$ is the mean distance for elevation $y_0$.

From (51) it follows that

(62) \hspace{1cm} D(z) \geq 0,

(63) \hspace{1cm} \int_{-\infty}^{\infty} D(z)dz = n = \frac{\sqrt{\pi}}{2\rho^2} N.

As in the preceding $\xi$, the maximum of (60) for all functions $D(z)$ satisfying (62) and (63) is equal to $P(N)$ and is attained for

(64) \hspace{1cm} D(z) = \begin{cases} \frac{L^2}{2c} (S_0^2 - z^2) & \text{if } |z| < S_0 \\ 0 & \text{if } |z| \geq S_0 \end{cases},

where

(65) \hspace{1cm} S_0 = \frac{1}{\rho} \left( \frac{3\sqrt{\pi}}{4} \right)^{\frac{1}{3}} N^{\frac{1}{3}} = 2.305 N^{\frac{1}{3}}.

The inequality
(66) \[ R \leq P(N) \]

obtains special interest in view of the fact that for small \( c/K \) and small \( K \), i.e., \( E_u \) large by comparison with \( E \), we have approximately

(67) \[ \max R = P(N) . \]

(See also the cited articles by I. A. Gubler and A. A. Svesnikov.)

We limit ourselves here to indicating the values of \( P(N) \) for \( K = 1/2 \), \( c = 5/14 \), which were considered in §3. These values are given in Table 7.

<table>
<thead>
<tr>
<th>( n )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>6</th>
<th>12</th>
<th>18</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K )</td>
<td>0.38</td>
<td>0.58</td>
<td>0.77</td>
<td>1.15</td>
<td>2.31</td>
<td>3.46</td>
<td>4.61</td>
</tr>
<tr>
<td>( P(K) )</td>
<td>0.26</td>
<td>0.36</td>
<td>0.44</td>
<td>0.55</td>
<td>0.76</td>
<td>0.86</td>
<td>0.92</td>
</tr>
</tbody>
</table>

Comparison with the results of §3 shows that the evaluation of \( \max R \) by (66) gives in this case crude (since \( c/K = 5/7 \) and \( K = 1/2 \) cannot be called very small!) but useful results for a first orientation. We have seen that evaluating \( \max R \) by \( P(N^*) \) gives much more exact results.
§8. Approximate calculation of reasonable amount of artificial dispersion in the case of a small target.

We shall proceed from formula (60), which is applicable, as we have indicated, to the case of small \( c/K = b/E \), that is, in the case of target size small by comparison with the technical dispersion. We define the complete dispersion of firing by the help of the mean square deviation \( \tilde{\sigma}_n \) of the actual point of impact \( x \) of the shells from the point \( \bar{x} = y_0 \). As is easily shown, \( \tilde{\sigma}_n \) is defined by the formula

\[
(68) \quad \tilde{\sigma}_n^2 = \sigma^2 + \sigma_r^2 ,
\]

where

\[
(69) \quad \sigma = \frac{1}{\sqrt{2\pi}} E = 1.483 E ,
\]

and \( \sigma_n \) is the mean quadratic deviation of the elevations \( x_i \) from \( \bar{x} \):

\[
(70) \quad \sigma_n^2 = \frac{1}{n} \sum_{1=1}^{n} (\bar{x} - x_i)^2 .
\]

In accordance with (35), we have

\[
(71) \quad \sigma_n = d_n E_u ,
\]

and by (61)
(72) \[ d_n^2 = \frac{1}{n} \int_{-\infty}^{\infty} z^2 D(z) dz , \]

whence from (68)

(73) \[ \tilde{d}_n^2 = d^2 + d_n^2 , \]

where

(74) \[ d = \frac{1}{\sqrt{2\rho}} K = 1.483 K. \]

In the foregoing \( \text{§} \) it was stated that (60) attains its maximum for \( D(z) \) of the form (64). We define the number \( \bar{d}_n \) corresponding to this maximally advantageous \( D(z) \) (which is, however, unattainable with the help of selecting the quantities \( a_i \)) by the formula

(75) \[ \bar{d}_n^2 = \frac{1}{n} \int_{-\infty}^{\infty} z^2 D(z) dz . \]

It is easy to show that \[ \bar{d}_n^2 = \frac{1}{5} S_0^2 = d^2(N) . \]

The function \( d^2(N) \) and its values are given in Table IV of the appendix.

It is natural to assume that for the most rational artificial dispersion, it is approximately true that

(77) \[ d_n^2 = \bar{d}_n^2 , \]
when this equality is possible, i.e., when

\[ d^2 \leq \frac{1}{\sigma_n^2}. \]

If

\[ d^2 > \frac{1}{\sigma_n^2}, \]

then in the first approximation, it is natural to assume that introduction of artificial dispersion has no value.

In this way, we arrive at the following method of calculating artificial dispersion: if

\[ \overline{d}_n^2 = d^2(\eta) \]

is less than or equal to

\[ d^2 = \frac{1}{2\rho^2} K^2 = 2.19 K^2, \]

then artificial dispersion is useless; if \( \overline{d}_n^2 > d^2 \), then rational amounts of artificial dispersion are defined by the equality

\[ d_n^2 = \overline{d}_n^2 - \overline{d}_n^2. \]
In the case \( K = 1/2, \ c = 5/14 \), even with \( n = 2 \), we have

\[
\overline{d^2}_n = d^2(N) = d^2(0.384) = 0.560,
\]

which is a little larger than \( d^2 = 2.19K^2 = 0.548 \); i.e., the criterion just stated, in conformity with the precise results of \( \S 3 \), shows that for the \( c \) and \( K \) under consideration, artificial dispersion is useful beginning with \( n = 2 \).

In the case \( K = 1, \ c = 1/4 \), we have \( d^2 = 2.19 \); the values of \( N \) and \( \overline{d^2}_n \), corresponding to various values of \( n \), are given in Table 5.

<table>
<thead>
<tr>
<th>( n )</th>
<th>6</th>
<th>12</th>
<th>24</th>
<th>48</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N )</td>
<td>0.61</td>
<td>1.61</td>
<td>3.23</td>
<td>6.46</td>
</tr>
<tr>
<td>( \overline{d^2}_n )</td>
<td>0.92</td>
<td>1.45</td>
<td>2.32</td>
<td>3.50</td>
</tr>
</tbody>
</table>

For \( n = 6 \) and \( n = 12 \), we have \( \overline{d^2}_n < d^2 \), which shows that artificial dispersion is useless in these cases. For \( n = 24 \) and \( 48 \), on the other hand, we have \( \overline{d^2}_n > d^2 \), which shows that the introduction of artificial dispersion is reasonable. This corresponds entirely with the results of the exact calculations of \( \S 4 \).

In this way, the approximate methods used in the last two \( \S \S \) are capable of giving a first rough orientation to the question of the usefulness of artificial dispersion.
DETERMINATION OF THE BEST MEANS OF INTRODUCING
ARTIFICIAL DISPERSION IN FIRING
(FOR VARIOUS SPECIAL CASES)

A. A. Svesnikov

Introduction

1. Reduction of the problem of artificial dispersion to a problem of the calculus of variations.
2. Solution of the variational problem.
3. Analysis of the results obtained and deduction of an approximate formula for the density distribution $N(\xi)$.
4. Solution of the problem of artificial dispersion in the two- and three-dimensional cases.
5. Extension of certain of the results obtained to targets of arbitrary size.

Introduction

The concept of the necessity of introducing artificial dispersion in anti-aircraft firing is not new, although there do not exist generally accepted and sufficiently proven concepts concerning the method of introducing such dispersion. One of the most popular ideas about artificial dispersion in anti-aircraft fire is that the best results in anti-aircraft fire will be obtained if one artificially increases the dispersion of shell-bursts (abandoning as
formerly dispersion according to the Gaussian law) in such a way that the size of the mean error of dispersion of shell-bursts is equal to the mean error of target position (in the case of firing time-fuzed shells the mean ellipsoid is used). This rule was held to be applicable in all cases, independently of the concrete conditions of firing: number of possible volleys, dimensions of the target, magnitude of the mathematical expectation for one shot and for all shots.

Ignoring the fact that the attempt to compare sizes of the semi-axes of the ellipsoid of dispersion with the size of the ellipsoid of target dispersion seems to us unjustified, it is clear that any solution of the problem of the best dispersion of shell-bursts cannot enjoy universal applicability, but must depend significantly upon the concrete conditions of firing.

The aim of the present work consists in investigating the question of the most advantageous artificial dispersion for shell-bursts and its dependence upon concrete conditions of the firing.

§1. Reduction of the problem of artificial dispersion to a problem of the calculus of variations.

For a better clarification of the principal aspects of the question and for simplification of the problem, we consider in the present article only two categories of errors:

1. random non-repeating errors of shooting (dispersion of shots).
2. random repeating errors of shooting (distribution of the target).
It is assumed that all errors are governed by the Gaussian law. We shall designate the semi-axes of the unit ellipsoid of dispersion by \( r_x, r_y, r_z \), and the semi-axes of the unit ellipsoid of distribution of the target by \( E_x, E_y, E_z \).

In solving the problem of artificial dispersion of shots where we are dealing with time-fuzed shells, we shall take the most advantageous method to be that which gives the largest probability of obtaining at least one hit within a certain space surrounding the target, which has the form of a parallelepiped. The linear dimensions of the parallelepiped \( (2l_x, 2l_y, 2l_z) \) must be chosen in correspondence with the probability of destroying the target with a single shell. The probability of obtaining a hit with a single shell in the given space must equal the probability of destroying the target with a single shot. In this way, the examination of the destruction of the target is in actuality converted into an investigation of the probability of obtaining at least one hit (shell-burst) in the given space. The legitimacy of this conversion will be proved in the sequel.

In the case where the firing under consideration consists of shells without time fuzing (this case we shall frequently describe in the sequel as the two-dimensional case), the semi-axes of the ellipse of dispersion and of distribution of the target will be denoted by \( r_x, r_y, E_x, E_y \), respectively, and the dimensions of the target by \( 2l_x \) and \( 2l_y \).

Likewise in the linear case (errors in only one direction), \( B_0 \) is designated by the letter \( r_x \), the mean error in target location by
the letter $E_x$, and the size of the target by $2l_x$.

We begin the solution of our problem with the linear case. The two- and three-dimensional cases are amenable to treatment by the same method that is used in the simplest case of linear errors, but require only longer calculations, which will be carried out in the appropriate place.

The probability of a miss in a firing consisting of $n$ shots is given by the known formula

\begin{equation}
Q = \frac{\rho}{\sqrt{\pi E_x}} \int_{-\infty}^{+\infty} e^{-\frac{\rho^2 x^2}{E_x^2}} \left[ \prod_{i=1}^{n} \left( 1 - p(x + \xi_i) \right) \right] dx,
\end{equation}

where

\begin{equation}
p(x + \xi_i) = \frac{1}{2} \left[ \Phi \left( \frac{x + \xi_i + l_x}{r_x} \right) - \Phi \left( \frac{x + \xi_i - l_x}{r_x} \right) \right];
\end{equation}

\begin{equation}
\Phi(x) = \frac{2\rho}{\sqrt{\pi}} \int_{0}^{x} e^{-\rho^2 u^2} du \quad \text{and} \quad \rho = 0, 477, \ldots.
\end{equation}

(2) gives the probability that the target will be destroyed by the $i^{th}$ shot under the condition that the magnitude of the random repeating error of the shot is equal to $x$, and the $i^{th}$ shot is directed to the point which is located at a distance $\xi_i$ from the center of distribution of the target. It is necessary to determine such values of the numbers $\xi_1, \ldots, \xi_n$ that the expression (1) becomes a minimum.

For more convenience in solving the proposed problem, we transform the product found under the integral sign in (1). To this
end, we set

\( \prod_{i=1}^{n} [1 - p(x + \xi_1)] \), and take logs;

\( \log \prod_{i=1}^{n} [1 - p(x + \xi_1)] \).

To simplify this sum, we introduce the distribution function \( n(\xi) \) of the shots, which for every value of \( \xi \) gives the number of shots aimed at points to the left of the point \( \xi \). The function \( n(\xi) \) is a step-function, since it changes value at discrete points \( \xi_1 \) only, and increases at \( \xi_1 \) by the number of shots aimed at \( \xi_1 \).

![Figure 1](attachment:figure1.png)
For illustration, we sketch the growth of the function $n(\xi)$ for a firing with three aimings $\xi_1, \xi_2, \xi_3$, when $(1/4)n$ shots are aimed at $\xi_1$, $(1/2)n$ are aimed at $\xi_2$, and $(1/4)n$ are aimed at $\xi_3$.

To the left of $\xi_1$ there are no shots, and hence $n(\xi) = 0$ for $\xi < \xi_1$. For $\xi_1 \leq \xi < \xi_2$, $n(\xi) = (1/4)n$, since $(1/4)n$ shots are aimed at $\xi_1$. At $\xi_2$, the function $n(\xi)$ takes another jump, this time by an amount $(1/2)n$. The last jump occurs at $\xi = \xi_3$, after which $n(\xi)$ remains constant and equal to $n$.

In the general case, where firing is carried out with a large number of aimings and with non-uniform distribution of shots, the curve will have a more complicated appearance.

By using $n(\xi)$, we can write $\log \pi$ as an integral. Indeed, since the function $n(\xi)$ is constant except for the points $\xi_1$, it follows that the differential $dn(\xi)$ is zero except at the points $\xi_1$, when $dn(\xi)$ is equal to the number of shots aimed at this point. Hence one can write

\[(5) \quad \log \pi = \int_{-\infty}^{\infty} \log \left[1 - p(x+\xi)\right] \, dn(\xi),\]

when the integral is a Stieltjes integral, and differs from the usual integral in that $n(\xi)$ is discontinuous, and hence one cannot write $dn(\xi) = \frac{dn}{d\xi}$ and then perform an ordinary integration.

However, if we substitute a differentiable function $n_1(\xi)$ for the function $n(\xi)$, this maneuver becomes possible. It is easy to see that $n_1(\xi)$ can be so chosen that the integrals (5) using $n(\xi)$ and...
n_{1}(\xi) will differ from each other by as little as we please (see Figure 1).

Setting \( \frac{dn_{1}(\xi)}{d\xi} = N(\xi) \), we obtain the final expression for \( \log \frac{\Omega}{P} \):

\begin{equation}
(6) \quad \log \frac{\Omega}{P} = \int_{-\infty}^{+\infty} \log [1+p(x+\xi)] N(\xi) d\xi.
\end{equation}

Here \( N(\xi) \) is the density of distribution of shots in the vicinity of the point \( \xi \). Hence the integral of \( N(\xi) \), over the whole domain of variation of the variable \( \xi \), must be equal to the total number of shots \( n \), and the function \( N(\xi) \) must be non-negative:

\begin{equation}
(7) \quad N(\xi) > 0,
\end{equation}

\begin{equation}
(8) \quad \int_{-\infty}^{+\infty} N(\xi) \, d\xi = n.
\end{equation}

For the number \( \frac{\Omega}{P} \), we have

\begin{equation}
(9) \quad \frac{\Omega}{P} = e^{-\int_{-\infty}^{+\infty} \log [1-p(x+\xi)] N(\xi) \, d\xi}.
\end{equation}

and the probability \( Q \) of a miss is expressed by...
\[ Q = \frac{p}{\sqrt{\pi \sigma^2}} \int_{-\infty}^{+\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} e^{-\int_{-\infty}^{+\infty} \log[1 - p(x+\xi)]N(\xi) d\xi} dx. \]

Thus the problem is reduced to the problem of finding a function \( N(\xi) \) which makes (10) a minimum, subject to conditions (7) and (8).

\section{2. Solution of the variational problem.} To simplify (10), we remark first that the probability of a hit on the target with one shot

\[ p(x) = \frac{1}{2} \left[ \Phi\left(\frac{x+\ell_x}{r_x}\right) - \Phi\left(\frac{x-\ell_x}{r_x}\right) \right], \]

except for a constant factor \( 2\ell_x \), coincides with the formula giving the additive law of distribution by addition of the law of equal probability to the Gaussian law. (See, for example, P. A. Gel'vih, "Theory of Probability," p. 206.)

It is known that when the parameter \( \ell_x \) of the law of equal probability has the same order of magnitude as the mean error of the Gaussian law, then the additive law may be expressed with great exactitude in the form of a Gaussian distribution, the mean square error of which is equal to the square root of the squares of the mean quadratic errors of the distributions being combined. In other words, one may suppose that
\[ p(x) = \frac{2\rho l_x}{\sqrt{\pi} r_x^*} e^{-\frac{\rho^2 x^2}{r_x^*^2}}, \]

where we have set

\[ r_x^* = \sqrt{r_x^2 + 0.153 l_x^2}. \]

(The coefficient 0.153 is introduced because of the adduction of the parameters of the law of equal probabilities \( l_x \) to the mean error."
See P. A. Gel'vih, "Theory of Probability.")

To simplify the writing of formulas we shall in the future write \( r_x \) instead of \( r_x^* \), setting

\[ p(x) = \frac{2\rho l_x}{\sqrt{\pi} r_x^2} e^{-\frac{\rho^2 x^2}{r_x^2}}. \]

In this way, \( r_x \) must always be replaced by \( \sqrt{r_x^2 + 0.153 l_x^2} \) in the final numerical calculations involving the formulas introduced above.

Before applying ourselves to the actual solution of the problem of finding the form of the function \( N(\xi) \) which makes the integral (10) a minimum, it is convenient to pass from the unknown function \( N(\xi) \) to a new function \( \lambda(x) \):

\[ \lambda(x) = -\int_{-\infty}^{+\infty} \log[1 - p(x+\xi)] N(\xi) d\xi. \]
The function \( \lambda(x) \) cannot be chosen arbitrarily, since conditions (7) and (8) place restrictions on \( N(\xi) \) and hence on \( \lambda(x) \). To elucidate the meaning of these restrictions, we first of all simplify the expression (9), making use of the smallness of the quantity \( p(x+\xi) \), and replacing in the exponent the quantity \( \log [1 - p(x+\xi)] \) by the quantity \(-p(x+\xi)\). The error which this substitution involves will be smaller, the smaller \( p(x+\xi) \) becomes.

To illustrate the possible errors, we introduce the following small table:

<table>
<thead>
<tr>
<th></th>
<th>(-0.1000)</th>
<th>(-0.0500)</th>
<th>(-0.0100)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-p(x))</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\log [1 - p(x)])</td>
<td>(-0.1054)</td>
<td>(-0.0513)</td>
<td>(-0.0101)</td>
</tr>
<tr>
<td>error</td>
<td>0.0054</td>
<td>0.0013</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

As the table shows, the errors involved in replacing \( \log [1 - p(x)] \) by \(-p(x)\) are not large. However, we are not interested merely in these errors, but in the error which results in calculating the number \( Q \) with this substitution. This error will increase as the shots become more thickly distributed (since \( p(x+\xi) \) decreases with increasing \( x+\xi \)). We consider the least favorable case, in which a large number of shots are aimed at a single point. Then the exact formula gives

\[
(16) \quad Q = \frac{\rho}{\sqrt{\pi} E_x} \int_{-\infty}^{+\infty} e^{-\frac{p^2 x^2}{E_x^2}} [1 - p(x)]^p dx ,
\]
and the approximate formula gives

\begin{equation}
Q_1 = \frac{\rho}{\sqrt{\pi} E_x} \int_{-\infty}^{+\infty} e^{-\frac{\rho^2 x^2}{E^2_n}} - np(x) \, dx.
\end{equation}

To evaluate the error committed in using (17) to compute \(Q\), we write \(Q_1 - Q = \Delta Q\), and find that

\begin{equation}
\Delta Q = \frac{\rho}{\sqrt{\pi} E_x} \int_{-\infty}^{+\infty} e^{-\frac{\rho^2 x^2}{E^2_x}} \left\{ e^{-np(x)} - \left[1 - p(x)\right]^n \right\} \, dx.
\end{equation}

To appraise \(\Delta Q\), we use the following inequalities, which are valid for \(0 < p < 1:\)

\begin{equation}
(1 - \frac{np^2}{2}) e^{-np} < (1-p)^n < e^{-np}.
\end{equation}

From these inequalities, it is plain that the quantity in curly brackets in (18) is always positive. [Translator's comment: unless \(p(x) = 0\), when the expression is zero.] For this reason, we see that by (19), we merely increase the error by substituting

\(\frac{n}{2} [p(x)]^2 e^{-np(x)}\)

for the expression \(\{\ldots\}\) in (18). Hence we find

\begin{equation}
0 < \Delta Q < \frac{\rho n}{2\sqrt{\pi} E_x} \int_{-\infty}^{+\infty} [p(x)]^2 e^{-\frac{\rho^2 x^2}{E^2_x}} - np(x) \, dx.
\end{equation}
Finally, removing the negative terms \(-np(x)\) from the exponent, we obtain the still stronger inequality

\[
(21) \quad 0 < \Delta Q < \frac{pn}{2\sqrt{\pi} E_x^2} \int_{-\infty}^{\infty} [p(x)]^2 e^{-\frac{p^2 x^2}{E_x^2}} dx.
\]

To determine the order of magnitude of this expression, we put

\[
p(x) = \frac{2pl_x}{\sqrt{\pi} r_x} e^{-\frac{p^2 x^2}{r_x^2}},
\]

(this under the assumption that \(p(x)\) is small, which makes the maneuver permissible). In this case, the integral (21) is easily evaluated, and we obtain

\[
(22) \quad 0 < \Delta Q < \left(\frac{2pl_x}{\pi r_x}\right)^2 \frac{r_x}{\sqrt{2E_x^2 + r_x^2}} = \frac{2pl_x n}{\sqrt{E_x^2 + \frac{1}{2}r_x^2}} \cdot \frac{pl_x}{2 \pi r_x}.
\]

The first factor has the order of magnitude of the mathematical expectation for the whole firing, and the second factor is by assumption small. In this way it is made clear that the error involved in using \(-p(x+\xi)\) instead of \(\log[1 - p(x+\xi)]\) to calculate \(Q\) is so small that it may be ignored.

We may do this with all the more right, because strictly speaking, the important consideration in the sequel will be the
change in Q resulting from changes in the method of firing (artificial dispersion). It is plain that the amount of change in Q can be subject to errors only of the second order in comparison to the error ΔQ.

The formulas just given for ΔQ refer to the linear case. In the theory evolved further along we must apply them to the two- and three-dimensional cases, where the errors produced by similar substitutions are yet smaller. We make this sufficiently evident assertion without proof, in order not to complicate this article with formulas which do not have first-rate importance.

Thus, instead of (15), we take

\[ (23) \quad \lambda(x) = \frac{2p_k}{\sqrt{\pi}} \frac{x}{r_x} \int_{-\infty}^{\infty} e^{-\frac{p^2(x+\xi)^2}{r_x^2}} N(\xi) d\xi. \]

Integrating (23) with respect to x, we obtain

\[ \int_{-\infty}^{\infty} \lambda(x) dx = 2p_k \int_{-\infty}^{\infty} N(\xi) d\xi. \]

Thus, condition (8) leads to

\[ (24) \quad \int_{-\infty}^{\infty} \lambda(x) dx = 2p_k n. \]
Dealing with condition (7) is more complicated. From this, \( \lambda(x) \) must be expressible in the form (23) with non-negative \( N(\xi) \); hence \( \lambda(x) \) must be such that

\[
\lambda(x) \geq 0.
\]

Replacing (7) by (25), we can obtain smaller values for the minimum of the probability \( Q \) of a miss, obtainable by rational choice of the function \( N(\xi) \). However, we shall see later that in the case where \( r_x \) is small by comparison with \( E_x \), the error in determining the minimum attainable for \( Q \) is not large.

Thus, after a series of transformations, the problem of finding the most advantageous distribution of shots has been reduced to the following variational problem: to find the function \( \lambda(x) \) such that the integral

\[
Q = \frac{1}{\sqrt{\pi E_x}} \int_{-\infty}^{\infty} e^{-\frac{\rho^2 x^2}{E_x^2}} - \lambda(x) \, dx
\]

is a minimum, where \( \lambda(x) \) satisfies the "equation of connection"

\[
\int_{-\infty}^{+\infty} \lambda(x) \, dx = 2l_x n,
\]

as well as the complementary restriction

\[
\lambda(x) \geq 0.
\]
To solve this problem, we must use a method the legitimacy of which is proved in courses dealing with the calculus of variations and which consists in the following:

Instead of the function

\[ F(x) = e^{-\frac{\rho^2 x^2}{E^2_x}} - \lambda(x) \]  

we must form the function

\[ F^*(x) = F(x) + \mu \lambda(x) , \]

where \( \mu \) is a constant, find the partial derivative of \( F^* \) with respect to \( \lambda \) and set this expression equal to zero. This equation, together with (24), permits us to define the function \( \lambda \) and the constant \( \mu \).

Doing all this, we obtain

\[ \frac{\partial F^*(x)}{\partial \lambda} = -e^{-\frac{\rho^2 x^2}{E^2_x}} - \lambda(x) \]

\[ + \mu = 0 ; \]

hence

\[ -e^{-\frac{\rho^2 x^2}{E^2_x}} - \lambda(x) \]

\[ e^{-\frac{\rho^2 x^2}{E^2_x}} = \mu , \]

and taking logarithms of both sides of (30), we find

\[ \lambda(x) = -\log \mu - \frac{\rho^2 x^2}{E^2_x} . \]
If $0 < \mu < 1$, then $\lambda(x)$ is non-negative only for values of $x$ such that

$$ |x| \leq \frac{E_x}{\rho} \sqrt{-\log \mu} .$$

However, $\lambda(x)$ is non-negative in accordance with (25). Hence we define $\lambda(x)$ to be given by (31) where (31) is positive and zero elsewhere:

$$
\lambda(x) = \begin{cases} 
\lambda_0 - \frac{r^2 x^2}{E_x^2} & \text{for } |x| \leq \frac{E_x}{\rho} \sqrt{\lambda_0} , \\
0 & \text{for } |x| > \frac{E_x}{\rho} \sqrt{\lambda_0} , 
\end{cases}
$$

when $\lambda_0 = -\log \mu$. (See E. Goursat, "Cours d'Analyse," Vol. 3, part 2, page 284.)

The solution exhibited here was obtained with the aid of methods of the calculus of variations, which are not applied in the theory of firing and are little known among the majority of artillerists. Hence it is reasonable to prove directly that the solution found does in fact produce the minimum value for $Q$.

For a proof, we assume that we have a solution $\lambda_1(x)$ different from $\lambda(x)$:

$$\lambda_1(x) = \lambda(x) + \eta(x) .$$

It is plain that $\eta(x) \geq 0$ for $|x| > \frac{E_x}{\rho} \sqrt{\lambda_0}$; for other values of $x$, $\eta(x)$
may have arbitrary sign. Let \( \delta Q \) be the change in \( Q \) upon replacing \( \lambda \) by \( \lambda_1 \):

\[
\delta Q = \frac{\rho}{\sqrt{\pi \, E_x}} \int_{-\infty}^{+\infty} e^{-\frac{\rho^2 x^2}{E_x^2}} - \lambda(x) \left\{ e^{-\eta(x)} - 1 \right\} \, dx.
\]

Since \( e^{-z} - (1-z) \geq 0 \) for all values of \( z \), we have,

\[
\int_{-\infty}^{+\infty} e^{-\frac{\rho^2 x^2}{E_x^2}} - \lambda(x) \left\{ e^{-\eta(x)} - [1 - \eta(x)] \right\} \, dx \geq 0.
\]

Thus we have

\[
\int_{-\infty}^{+\infty} e^{-\frac{\rho^2 x^2}{E_x^2}} - \lambda(x) \left\{ e^{-\eta(x)} - 1 \right\} \, dx + \int_{-\infty}^{+\infty} e^{-\frac{\rho^2 x^2}{E_x^2}} - \lambda(x) \eta(x) \, dx \geq 0.
\]

The first integral is simply \( \delta Q \), so that if we prove that the second is always non-positive, we shall have proved that any change in \( \lambda \) produces an increase in \( Q \), i.e., \( \delta Q \geq 0 \).

This is easy to prove if we recall that \( \lambda \) and \( \lambda_1 \)

\[
\int_{-\infty}^{+\infty} \lambda(x) \, dx = -\int_{-\infty}^{+\infty} \lambda_1(x) \, dx = 2\ell \, n.
\]
Thus
\[ -\int_{-\infty}^{\infty} (\lambda_1(x) - \lambda(x))dx = \int_{-\infty}^{\infty} \eta(x)dx = 0. \]

Using this condition on \( \eta(x) \), formula (34), and the fact that
\[ e^{-\frac{\rho x^2}{E_x^2}} < e^{-\lambda_0} \text{ for } |x| > \frac{E_x}{\rho} \sqrt{\lambda_0}, \]
we obtain
\[ \int_{-\infty}^{+\infty} e^{-\frac{\rho x^2}{E_x^2}} - \lambda(x) \eta(x)dx = e^{-\lambda_0} \int_{-\frac{E_x \sqrt{\lambda_0}}{\rho}}^{+\infty} \eta(x)dx + \int_{-\infty}^{-E_x \sqrt{\lambda_0} \rho} \eta(x)dx + \int_{-\infty}^{+\infty} e^{-\frac{\rho x^2}{E_x^2}} \eta(x)dx = 0, \]

which completes the present proof.

The constant \( \lambda_0 \) is easily defined by (24), which gives
\[ \lambda_0 = (\frac{3}{2} \rho \frac{l_{xn}}{E_x})^{2/3}. \]

§3. Analysis of the results obtained and deduction of an approximate formula for the density distribution \( N(\xi) \).
In accord with formulas (34) and (35), the function $\lambda(x)$ is positive for

$$-R_x < x < R_x,$$

where

$$R_x = E_x \sqrt{\frac{3l_n}{2p^2E_x}}.$$  \hspace{1cm} (37)

Outside of the interval (36), we have $\lambda(x) = 0$, and inside (36) the function $\lambda(x)$ is described by the quadratic law

$$\lambda(x) = \lambda_0 - \frac{p_{x}^2}{E_x^2}.$$  \hspace{1cm} (36)

If $R_x$ is large in comparison with $r_x$, then it is easy to see that equation (23) can be approximately satisfied by putting

$$N(c) = \frac{\lambda(c)}{2l_x}.$$  \hspace{1cm} (38)

Since we are principally interested in the case where $r_x$ is small by comparison to $E_x$, and the number $n$ is sufficiently large so that the expression

$$\frac{2pl_x r}{E_x}$$
(This expression is known to be greater than the mathematical expectation of the number of hits, it follows that we may suppose that \( R_x \) is large in comparison to \( r_x \), and we may use (38). From (34), (35), and (37), we obtain finally

\[
N(\xi) = \begin{cases} 
\frac{1}{2r_x E_x^2} (R_x^2 - \xi^2) & \text{for } |\xi| \leq R_x \\
0 & \text{for } |\xi| > R_x.
\end{cases}
\]

Since the mathematical expectation of the number of hits for the entire firing (without artificial dispersion) is equal to

\[
M_0 = n \frac{\rho}{\sqrt{\pi}} \frac{1}{\sqrt{E_x^2 + r_x^2}} \int_{-l_x}^{l_x} e^{-\left(\frac{r_x^2 x^2}{E_x^2 + r_x^2}\right)} dx,
\]

and in the case we consider, \( l_x \) and \( r_x \) are small in comparison to \( E_x \), we have approximately

\[
M_0 = \frac{2\rho l_x n}{\sqrt{\pi} E_x},
\]

and the quantity \( S_1 = R_x/E_x \) depends solely upon \( M_0 \).

Computing the numerical coefficient, we have
(42) \[ S_1 = 2.30 \sqrt[3]{\frac{3}{M_0}}, \]

that is, the region of artificial dispersion increases proportionally to the cube root of the mathematical expectation of hits in firing without artificial dispersion.

§ 4. Solution of the problem of artificial dispersion in the two- and three-dimensional cases.

The results so far obtained relate to the linear case. For the case of the two- and three-dimensional problem a solution is obtained in a precisely analogous way, but the final result has a somewhat different form. We consider this question more closely.

For the two-dimensional case, the probability of a miss is

(43) \[ Q = \frac{\rho^2}{\pi E_x E_y} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\rho^2 \left( \frac{x^2}{E_x^2} + \frac{y^2}{E_y^2} \right)} \prod_{i=1}^{n} \left[ 1 - \rho(x + \xi_i, y + \eta_i) \right] dx dy \]

where \( \xi_i \) and \( \eta_i \) are the displacements of the \( i \)th shot from the center of distribution of the target.

Reasoning just as in the linear case, we transform \( Q \) into the form

(44) \[ Q = \frac{\rho^2}{\pi E_x E_y} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\rho^2 \left( \frac{x^2}{E_x^2} + \frac{y^2}{E_y^2} \right) - \lambda(x,y)} dx dy, \]
where

\[ \lambda(x,y) = - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \log[1 - p(x+\xi, y+\eta)] N(\xi, \eta) \, d\xi \, d\eta , \]

and the "density distribution" \( N(\xi, \eta) \) satisfies the equation of connection

\[ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} N(\xi, \eta) \, d\xi \, d\eta = n . \]

Using the smallness of the probability of a hit with one shot, \( p(x,y) \), we may set

\[ - \log[1 - p(x+\xi, y+\eta)] = p(x+\xi, y+\eta) . \]

We shall suppose that the linear dimensions \( l_x \) and \( l_y \) are not too large in comparison with the semi-axes \( r_x \) and \( r_y \) of the unit ellipse of dispersion, and for simplicity we shall assume that the target is a rectangle whose sides have the same directions as the directions of the axes of the ellipse of dispersion. Then we may set

\[ p(x+\xi, y+\eta) = \left( \frac{2p}{\sqrt{\pi}} \right)^2 \frac{\xi \, \eta}{r_x \, r_y} e^{-p^2 \left[ \frac{(x+\xi)^2}{r_x^2} + \frac{(y+\eta)^2}{r_y^2} \right]} , \]

where \( r_x \) and \( r_y \) are not the semi-axes of the ellipse of dispersion, but are
\[
\begin{align*}
\begin{cases}
  r_x^* = \sqrt{r_x^2 + 0.153 \ell_x^2}, \\
  r_y^* = \sqrt{r_y^2 + 0.153 \ell_y^2}.
\end{cases}
\end{align*}
\]

(See the corresponding remarks in §2 for the linear case.) Replacing \(- \log[1-p(x+\xi, y+\eta)] \) by \(p(x+\xi, y+\eta)\) and using (48), we obtain the integral equation

\[
\frac{(2\rho)^2}{\sqrt{\pi}} \frac{\ell_x \ell_y}{r_x r_y} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\rho^2 \left[ \frac{(x+\xi)^2}{E_x^2} + \frac{(y+\eta)^2}{E_y^2} \right]} N(\xi, \eta)d\xi d\eta = \lambda(x, y).
\]

as in §2, we obtain from (46) the "equation of connection"

\[
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \lambda(x, y) \, dx \, dy = 4 \ell_x \ell_y n.
\]

Hence, the problem leads to the construction of a function \(\lambda(x, y)\) of two variables which gives a minimum for the integral (44) under condition (51) and non-negativity of \(\lambda(x, y)\). The application of the calculus of variations (and also direct verification) gives the following expression for \(\lambda(x, y)\):

\[
\lambda(x, y) = \begin{cases}
  \lambda_0 - \rho^2 (\frac{x^2}{E_x^2} + \frac{y^2}{E_y^2}) & \text{for } \frac{x^2}{E_x^2} + \frac{y^2}{E_y^2} \leq \frac{\lambda_0}{\rho^2} \\
  0 & \text{for } \frac{x^2}{E_x^2} + \frac{y^2}{E_y^2} > \frac{\lambda_0}{\rho^2}.
\end{cases}
\]
Limiting ourselves to the case where \( l_x, l_y, \) and \( r_x, r_y \) are small in comparison to \( E_x \) and \( E_y \), (this being the only case of practical value), one can as in §3 set

\[
N(\xi, \eta) = \frac{\lambda(\xi, \eta)}{4\pi l_x l_y}.
\]

Then, from (52), we obtain

\[
N(\xi, \eta) = \begin{cases} 
\frac{\rho^2}{4\pi l_x l_y} \left[ \frac{2}{\rho} \sqrt{\frac{2l_x l_y}{\pi E_x E_y}} \right] n - \left( \frac{\rho^2}{E_x} + \frac{n^2}{E_y} \right) \log \left( \frac{\rho^2}{E_x} + \frac{n^2}{E_y} \right) < \frac{2}{\rho} \sqrt{\frac{2l_x l_y}{\pi E_x E_y}} n.
\end{cases}
\]

Hence, the greatest artificial deviations of shell-bursts in the \( x- \) and \( y- \)directions are, respectively

\[
\begin{align*}
R_x &= 2.49 \left( E_x \sqrt{\frac{M_0}{\rho}} \right), \\
R_y &= 2.49 \left( E_y \sqrt{\frac{M_0}{\rho}} \right),
\end{align*}
\]

where

\[
M_0 = \frac{4\rho^2 l_x l_y n}{\pi^2 E_x E_y}.
\]
For simplicity, we suppose that the axes of the ellipse of dispersion of the target coincide in direction with the axes of the ellipse of dispersion. This assumption is inaccurate. In coordinate systems whose axes coincide with the axes of the ellipse of dispersion of the target, (54) is correct without this assumption.

The three-dimensional case (firing with time-fuzed shells) does not involve any additional mathematical difficulties, if instead of the probability of destroying an airplane one considers the probability that the shell will burst within a certain region in the neighborhood of the airplane. Indeed, the probability of a miss is

\[
Q = \left( \frac{2\pi}{\sqrt{\pi}} \right)^3 \frac{E_x E_y E_z}{E_x E_y E_z} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-p^2 \left[ \frac{x^2}{E_x^2} + \frac{y^2}{E_y^2} + \frac{z^2}{E_z^2} \right]} \cdot
\]

\[
\prod_{i=1}^{n} \left[ 1 - p(x + \xi_i, y + \eta_i, z + \zeta_i) \right] dx \, dy \, dz,
\]

i.e., it is an integral of the same type as those encountered in the one- and two-dimensional cases. Thus the methods used above are quite applicable to this case. Not lingering over the intermediate calculations, we present the final result for density distribution of shots corresponding to the most advantageous artificial dispersion (for the case when the spread of the artificial dispersion is large in comparison to the unit ellipsoid of dispersion of shell-bursts). (It is easy to show, by the same method as
in the two-dimensional case, that the solution does not change its form when the axes of the ellipsoid of distribution of the target do not coincide with the axes of the ellipsoid of dispersion.)

Our final result is:

\[
N(x, y, z) = \begin{cases} 
\frac{\rho^2}{8E_xE_yE_z} \left[ \frac{1}{\rho^2} \frac{15\rho^3E_xE_yE_z}{\pi n} \right]^{2/5} \left\{ \frac{x^2}{E_x^2} + \frac{y^2}{E_y^2} + \frac{z^2}{E_z^2} \right\}, \\
0, \text{ for } \frac{x^2}{E_x^2} + \frac{y^2}{E_y^2} + \frac{z^2}{E_z^2} > \frac{1}{\rho^2} \left\{ \frac{15\rho^3E_xE_yE_z}{\pi n} \right\}^{2/5}
\end{cases}
\]

and the largest amounts of artificial deviation of shell-bursts in the x-, y-, and z-directions are

\[
\begin{align*}
R_x &= \frac{E_x}{\rho} \left( \frac{15\rho^3E_xE_yE_z}{\pi n} \right)^{1/5} = 2.69 \frac{E_x}{\sqrt{M_0}}, \\
R_y &= \frac{E_y}{\rho} \left( \frac{15\rho^3E_xE_yE_z}{\pi n} \right)^{1/5} = 2.69 \frac{E_y}{\sqrt{M_0}}, \\
R_z &= \frac{E_z}{\rho} \left( \frac{15\rho^3E_xE_yE_z}{\pi n} \right)^{1/5} = 2.69 \frac{E_z}{\sqrt{M_0}},
\end{align*}
\]
where

\[ M_0 = \frac{8\rho^3 l_x l_y l_z n}{\pi^{3/2} E_x E_y E_z}. \]

These formulas, as in the two foregoing cases, permit us to draw extremely important conclusions: first, the artificial dispersion is completely determined by the size \( M_0 \) of the mathematical expectation of the number of hits for the whole firing in the absence of artificial dispersion, and, second, the dimensions of the region of artificial dispersion change slowly with changes in \( M_0 \). Hence, although significant errors are possible in calculating \( M_0 \) (for example, one does not know with sufficient accuracy the number of hits with shell fragments necessary for destruction of an airplane), still the solution obtained proves that these errors can have only a small effect upon the choice of artificial dispersion. (The dimensions of the region shot at increase as the 1/5 power of \( M_0 \); and mistakes in \( M_0 \) can have only a small effect upon the dimensions of the region shot at.)

The formulas introduced above for the density distribution of shots permit us without difficulty to calculate the probability of at least one hit for the best system of firing. This calculation is extremely simple. Indeed, in conformity with the notations introduced, the probability of a miss (in the linear case) is

\[ Q = \frac{\rho}{\sqrt{\pi} E_x} \int_{-\infty}^{+\infty} e^{-\frac{2 x^2}{E_x^2} - \lambda(x)} \, dx, \]

\[ \lambda(x) = \frac{2 x^2}{E_x^2} + \frac{E_x^2}{E_x^2} \]
where as has been proved,

\[
\lambda(x) = \begin{cases} 
\frac{E_x^2}{\pi R_x^2} (R_x^2 - x^2) & \text{for } |x| \leq R_x, \\
0 & \text{for } |x| > R_x.
\end{cases}
\]

The integral (26) can be easily computed, since for \(-R_x \leq x \leq R_x\), the exponent of \(e\) is the constant \(-\rho^2 S_1^2\), where \(S_1 = \frac{R_x}{E_x}\); and for \(|x| > R_x\), \(\lambda(x) = 0\) and the exponent of \(e\) is \(-\rho^2 x^2 E_x^2\). Hence we find

\[
Q = \frac{\rho}{\sqrt{\pi} E_x} \int_{-R_x}^{R_x} e^{-\rho^2 S_1^2} \, dx + \frac{\rho}{\sqrt{\pi} E_x} \int_{-\infty}^{-R_x} e^{-\frac{\rho^2 x^2 E_x^2}{2}} \, dx + \frac{\rho}{\sqrt{\pi} E_x} \int_{R_x}^{+\infty} e^{-\frac{\rho^2 x^2 E_x^2}{2}} \, dx =
\]

\[
\frac{2\rho R_x}{\sqrt{\pi} E_x} e^{-\rho^2 S_1^2} + \left[1 - \Phi(\frac{R_x}{E_x})\right] = \frac{2\rho S_1}{\sqrt{\pi}} e^{-\rho^2 S_1^2} + \left[1 - \Phi(S_1)\right],
\]

and the probability (62) \(P \geq 1 = 1 - Q = \Phi(S_1) - \frac{2\rho S_1}{\sqrt{\pi}} e^{-\rho^2 S_1^2}\).

Analogous calculations can be made in the two- and three-dimensional cases.

In the two-dimensional case, we set

\[
S_2 = \frac{R_x}{E_x} = \frac{R_y}{E_y} = \left\{\frac{\delta x^2 y^2 n}{\pi \rho^2 E_x E_y}\right\}^{1/4},
\]

and after a few elementary
calculations, we find

\[ P \geq 1 = 1 - (1 + \rho^2 \frac{\sigma^2}{\bar{S}_3}) e^{-\rho^2 \frac{\sigma^2}{\bar{S}_3}}. \]

In the three-dimensional case, we set

\[ \bar{S}_3 = \frac{R_x}{E_x} = \frac{R_y}{E_y} = \frac{R_z}{E_z} = \left\{ \frac{15 l_x l_y l_z n}{\pi \rho^2 E_x E_y E_z} \right\}^{1/5}, \]

and again, after elementary calculations, we find

\[ Q = \frac{\rho^3}{\pi^{3/2} E_x E_y E_z} \int \int \int_{-\infty}^{+\infty} e^{-\rho^2 \left( \frac{x^2}{E_x^2} + \frac{y^2}{E_y^2} + \frac{z^2}{E_z^2} \right)} - \lambda(x, y, z) \, dx \, dy \, dz = \]

\[ 1 - \Phi(\bar{S}_3) - \bar{S}_3 (1 + \frac{2}{3} \rho^2 \frac{S_3^2}{\bar{S}_3^3}) \frac{2 \rho}{\sqrt{\pi}} e^{-\rho^2 \frac{S_3^2}{\bar{S}_3^2}}, \]

and hence

\[ P \geq 1 = \Phi(\bar{S}_3) - \bar{S}_3 (1 + \frac{2}{3} \rho^2 \frac{S_3^2}{\bar{S}_3^3}) \frac{2 \rho}{\sqrt{\pi}} e^{-\rho^2 \frac{S_3^2}{\bar{S}_3^2}}. \]

Translator's comment: there is a misprint in the Russian text either in (64) or in the preceding line: caveat lector.
§5. Extension of certain of the results obtained to targets of arbitrary size.

In the preceding §§ it has been shown that to obtain the highest probability of obtaining at least one hit, for the case of small probability of hitting the target with one shot (and in the absence of systematic errors) it is necessary to introduce artificial dispersion of the shots in such a way that

\[
\lambda(x) = \int_{-\infty}^{+\infty} \log\left[1 - p(x+\xi)\right] N(\xi) d\xi = \begin{cases} 
\lambda_0 - \frac{P^2 x^2}{E_x^2} \frac{1}{\rho} \lambda_0 & \text{for } |x| \leq \frac{E_x \sqrt{\lambda_0}}{\rho}, \\
0 & \text{for } |x| > \frac{E_x \sqrt{\lambda_0}}{\rho}.
\end{cases}
\]

(For simplicity, we consider only the linear case, but all the following argument is completely applicable to the two- and three-dimensional cases.) It is easy to see that if the restrictions previously imposed upon \(\lambda(x)\) are removed (i.e., \(N(\xi) \geq 0\) and discrete distribution of shots), then the solution obtained for \(\lambda(x)\) has a universal character, i.e., it is correct for arbitrary values of probability of hitting with one shot, and furthermore this result can be obtained without recourse to the artificial device of introducing the density distribution of shots \(N(\xi)\).

As a matter of fact, if we write

\[
\lambda(x) = -\sum_{i=1}^{n} \log\left[1 - p(x+\xi_i)\right],
\]
then the expression for the probability of a miss will have just the same form as previously;

\[ Q = \frac{\rho}{\sqrt{\pi E_x}} \int_{-\infty}^{\infty} e^{-\frac{\rho^2 x^2}{E_x}} - \lambda(x) \, dx \]

Also the equation of connection for \( \lambda(x) \) remains unchanged. This is easy to show if we integrate both sides of (66) over the interval \((-\infty, +\infty)\):

\[ \int_{-\infty}^{\infty} \lambda(x) \, dx = - \sum_{i=1}^{n} \int_{-\infty}^{+\infty} \log[1 - p(x + \frac{\epsilon}{2})] \, dx \]

By an elementary substitution, we obtain

\[ \int_{-\infty}^{+\infty} \lambda(x) \, dx = - \sum_{i=1}^{n} \int_{-\infty}^{+\infty} \log[1 - p(t)] \, dt = -n \int_{-\infty}^{\infty} \log[1 - p(t)] \, dt \]

Thus we arrive at the following variational problem: to find the form of the function \( \lambda(s) \) such that the integral

\[ Q = \frac{\rho}{\sqrt{\pi E_x}} \int_{-\infty}^{\infty} e^{-\frac{\rho^2 x^2}{E_x}} - \lambda(x) \, dx \]
has a minimum value, under the condition that
\[
\int_{-\infty}^{+\infty} \lambda(x) dx = \left( - \int_{-\infty}^{+\infty} \log[1 - p(t)] dt \right) n.
\]

This problem is entirely analogous to the one which we have solved earlier. The sole difference is that the size of the target \(2l_x\) is replaced in the present case by

\[
(69) \quad S^* = - \int_{-\infty}^{\infty} \log[1 - p(t)] dt.
\]

Hence the solution obtained earlier is completely applicable to the present case — we need merely to substitute \(S^*\) for \(2l_x\) in the final formulas for \(\lambda(x)\) and for the probability of obtaining at least one hit. Similarly, in the two- and three-dimensional cases, we must replace \(4l_xl_y\) and \(8l_xl_yl_z\) by

\[
(70) \quad S^* = - \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \log[1 - p(t, u)] dt\; du
\]

and

\[
(71) \quad S^* = - \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \log[1 - p(t, u, v)] dt\; du\; dv,
\]
respectively. The geometric description of the solution obtained is that artificial dispersion must be introduced in such a way that in the Gaussian curve \( y = \frac{\rho}{\sqrt{\pi F_x}} e^{-\frac{\rho^2 x^2}{F_x^2}} \), we lop off the highest part.

(From the solution given for the variational problem, it is plain that in the case where the distribution of the target does not follow Gauss's law but is described by some other law \( y = \varphi(x) \), then \( \lambda(x) \) is defined by the equation \( \varphi(x)e^{-\lambda(x)} = \text{const.} \), that is, the curve \( y = \varphi(x) \) must be "lopped off" by the factor \( \prod_{i=1}^{n} [1 - p(x + \xi_i)] \).

![Figure 2](image)

Knowing the function \( \lambda(x) \) in the case of small dimensions of the target, we had the means of immediately evaluating the density distribution of shots, \( N(\xi) \), since in this case the integral equation connecting \( N(\xi) \) and \( \lambda(x) \),
\[ \int_{-\infty}^{+ \infty} \log \left[ 1 - p(x + \xi) \right] N(\xi) \, d\xi = -\lambda(x) , \]

could be replaced by the relation \( N(\xi) = \frac{\lambda(\xi)}{2 I_x} \).

In the case of large target dimensions, the integral equation's solution is beset by great difficulties, although certain approximating methods concerning the distribution of shots can be used with great ease.

For example, aiming \( n \) shots at the center of distribution of the target (i.e., no artificial dispersion) we find

\[ (72) \quad \lambda(x) = -n \log \left[ 1 - p(x) \right] , \]

whence, in particular,

\[ (73) \quad \lambda(0) = -n \log \left[ 1 - p(0) \right] . \]

The introduction of artificial dispersion could result only in lessening \( \lambda(0) \) by comparison with (73). Along with this, we have by (65) for the most advantageous choice of \( \lambda(x) \) that \( \lambda(0) = \lambda_0 \), where
\[
\left\{ \frac{3\rho n}{4\varepsilon_x} \int_{-\infty}^{+\infty} \log[1 - p(t)] \, dt \right\}^{2/3} \text{ in the linear case,}
\]

\[
\lambda_0 = \left\{ \frac{-2\rho^2 n}{\pi E_x E_y} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \log[1 - p(t,u)] \, dt \, du \right\}^{1/2} \text{ in the two-dimensional case,}
\]

\[
\left\{ \frac{-15\rho^3 n}{8\pi E_x E_y E_z} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \log[1 - p(t,u,u)] \, dt \, du \, dv \right\}^{2/5} \text{ in the three-dimensional case.}
\]

From this, one may figure approximately that the introduction of artificial dispersion is useless for

\[
(75) \quad n \log[1 - p(0)] < \lambda_0
\]

and is advantageous for

\[
(76) \quad n \log[1 - p(0)] > \lambda_0.
\]
Conclusions

We list our principal conclusions:

1. When the probability of a hit with one shot (in the absence of systematic errors) is small, then in the case of two categories of errors, the problem of the best distribution of shots is easily solved by the calculus of variations.

2. In the case where the dimensions of the region of artificial dispersion exceed four times the axes of the unit ellipsoid of dispersion (time-fuzed shells), or four times the axes of the unit ellipse of dispersion (non-time-fuzed shells), or four times the mean error of dispersion (linear case), the final expression for the density distribution of shots can be calculated by the following formulas:

\[
N(\xi) = \begin{cases} 
\frac{p^2}{2\ell_x} \left( \frac{S_2^2}{1 - \frac{\xi^2}{E_x^2}} \right) & \text{for } \frac{\xi^2}{E_x^2} \leq S_1^2, \\
0 & \text{for } \frac{\xi^2}{E_x^2} > S_1^2,
\end{cases}
\]

(39) where

\[
S_1 = 2.30 \sqrt[3]{M_o},
\]

(42)
(41) \[ M_0 = \frac{2pl_x^n}{\sqrt{\pi} E_x} \; ; \]

in the two-dimensional case,

\[
N(\xi, \eta) = \begin{cases} 
\frac{\rho^2}{4l_x l_y} \left[ \frac{S^2}{2} - \left( \frac{\xi^2}{E_x} + \frac{n^2}{E_y} \right) \right] & \text{for } \frac{\xi^2}{E_x} + \frac{n^2}{E_y} < S^2 \\
0 & \text{for } \frac{\xi^2}{E_x} + \frac{n^2}{E_y} > S^2 
\end{cases} 
\]

when \( S_2 = 2.49 \sqrt{M_0} \), and

(56) \[ M_0 = \frac{4p^2 l_x l_y l_z}{\pi E_x E_y} \; ; \]

in the three-dimensional case,

\[
N(\xi, \eta, \zeta) = \begin{cases} 
\frac{\rho^2}{8l_x l_y l_z} \left[ \frac{S^2}{3} - \left( \frac{\xi^2}{E_x} + \frac{n^2}{E_y} + \frac{\zeta^2}{E_z} \right) \right] & \text{for } \frac{\xi^2}{E_x} + \frac{n^2}{E_y} + \frac{\zeta^2}{E_z} < S^2 \\
0 & \text{for } \frac{\xi^2}{E_x} + \frac{n^2}{E_y} + \frac{\zeta^2}{E_z} > S^2 
\end{cases} 
\]
where \( S_3 = 2.69 \sqrt[5]{M_0} \), and

\[
M_0 = \frac{8 \rho^3 l_x l_y l_z n}{\pi^{3/2} E_x E_y E_z}.
\]

3. Under these assumptions, the probability of at least one hit is approximately given, in our linear, two-dimensional, and three-dimensional cases, by formulas (62), (63), and (64); values for \( S_1 \), \( S_2 \), and \( S_3 \) in these formulas may be taken from the remarks just preceding.
SOLUTION OF THE PROBLEM OF FIRING WITH ARTIFICIAL DISPERSION FOR VARIOUS CASES

I. A. Gubler

Introduction

§1. Principal formulas.

§2. Conditional distribution of the target after $k$ shots which do not give any hits.

§3. Choice of the points $(x_1, y_1, z_1)$.

§4. Density distribution of the points $(x_1, y_1, z_1)$.

§5. Relations between the probability of at least one hit and the mathematical expectation.

§6. Broadening the region fired at with increase in the number of shots fired.

§7. Conditions under which artificial dispersion is advantageous.

§8. Mathematical expectation for firing with the most advantageous artificial dispersion.

§9. Solution for the case of uniform bombardment of a region.

§10. Remark on the applicability of the method.

Introduction

The theoretically exact solution of the problem of firing with artificial dispersion for the case of destruction of the target with one hit must be obtained from studying an integral
which expresses the probability of obtaining at least one hit. In this integral, one finds as parameters the lengths of the semi-axes of the ellipsoids of dispersion and their directions, the dimensions and orientation of the target, and also all the coordinates of the points at which shots are aimed. With such a large number of parameters, the problem of finding the maximum of the integral becomes hopelessly difficult. Hence, we have decided to abandon the mathematically exact solution and to devote our efforts to the production of such approximate solutions as may be satisfactory from a practical point of view. As a matter of fact, it is perfectly evident that a mathematically exact solution would have to be rounded off in one way or another in designing a firing procedure. Hence we are not greatly concerned with the detailed structure of the solution, its properties in small regions of space. On the contrary, we are interested only in the solution as a whole. In other words, we are interested only in the appearance of the region in which we introduce artificial dispersions, and in covering it with shots placed at various points.

In order to make a uniform problem for one, two, and three dimensions, we shall assume that in the three-dimensional case we are looking for the maximal probability of at least one hit in the region where a shell-burst is dangerous.

We solve the problem under the following hypotheses:

1. The dimensions of the target are not large and do not exceed the lengths of the corresponding semi-axes of the ellipsoid of non-repeating errors.
2. We consider the case of two groups of errors (repeating and non-repeating).

3. We consider a statistical problem, that is, the ellipsoid of errors does not change during the time of firing.

4. The semi-axes of the ellipsoids of the repeating errors have the same corresponding directions.

The last assumption is not essential and is adopted merely for the sake of simplicity.

The first assumption is always fulfilled for the case of anti-aircraft fire with non-time-fuzed shells, and also in the case of time-fuzed shells for which the region in which a shellburst is dangerous is not large.

With these limitations, we give formulas which simultaneously permit us to determine when artificial dispersion is not advantageous and give a means of determining the most advantageous dispersion in the case where it is of use.

The method which we apply to the present problem is in essence the following. We consider the process of gradual deformation of the integrand in an integral which expresses the probability of obtaining at least one hit. The results of this process, for firing with the most advantageous artificial dispersion, yield the possibility of exhibiting a region in which it is necessary to employ this dispersion and of constructing the function of density of shots within this region.

Knowing these principal facts makes it possible to calculate the probability of obtaining at least one hit and the mathematical expectation of the number of hits for firing with the most advantageous artificial dispersion.
Examination of changes in the integrand for the integral which expresses the probability of obtaining at least one hit gives the possibility of obtaining a formula for bounds on the number of shots beginning with which artificial dispersion is useful.

A final application of the concept of density of shots, which is introduced in the beginning of the present article, permits us to solve the problem of the most advantageous artificial dispersion having the form of uniform cannonading of a square (region).

§1. Principal formulas.

Let the target be a parallel piped with edges $\mu$, $\nu$, $\kappa$, parallel to the semi-axes of the ellipsoids of repeating and non-repeating errors. We denote these semi-axes by $a, b, c$, and $\alpha, \beta, \gamma$ respectively. Then, for two groups of errors, the probability of obtaining at least one hit in $n$ shots aimed at the center of distribution of the target is expressed by the formula

$$p_n = 1 - \frac{\rho^3}{\pi^{3/2}abc} \int_{-\infty}^{+\infty} e^{-\rho^2 \left( \frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2} \right)} \left[ 1 - \frac{\rho^3}{\pi^{3/2}abc} \int_{x' - \mu/2}^{x' + \mu/2} d\xi \right]$$

$$\int_{y' - \nu/2}^{y' + \nu/2} \int_{z' - \kappa/2}^{z' + \kappa/2} e^{-\rho^2 \left( \frac{\xi'^2}{a^2} + \frac{\eta'^2}{b^2} + \frac{\zeta'^2}{c^2} \right)} d\xi' d\eta' d\zeta'$$
For the case in which the shots are aimed not at the center of distribution but at some other point, this formula becomes a little more complex. Let the \(i^{th}\) shot be aimed at the point \((x'_i, y'_i, z'_i)\). Then the probability \(P_n\) of obtaining at least one hit is

\[
P_n = 1 - \frac{\rho^3}{\pi^3/2abc} \int_{-\infty}^{+\infty} e^{-\rho^2 \left( \frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2} \right)} d\xi'_i d\eta'_i d\zeta'_i.
\]

\[
\frac{n}{\prod_{i=1}^{n} \left(1 - \frac{\rho^3}{\pi^3/2abc} \int_{-\infty}^{+\infty} e^{-\rho^2 \left( \frac{x'-x'_i+\mu/2}{\alpha^2} + \frac{y'-y'_i+\nu/2}{\beta^2} + \frac{z'-z'_i+\kappa/2}{\gamma^2} \right)} d\xi'_i d\eta'_i d\zeta'_i \right)}.
\]

We assume that the dimensions of the target do not exceed the semi-axes of the ellipsoid of non-repeating errors, i.e., \(\mu < \alpha\), \(\nu < \beta\), \(\kappa < \gamma\). Then one may consider (approximately) that the inner triple integral is equal to the volume of the region of integration \(\mu\), multiplied by the value of the integrand at the center of this region, i.e.,
\[ P_n = 1 - \frac{\rho^3}{\pi^{3/2}abc} \int_{-\infty}^{+\infty} e^{-\rho^2 \left( \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} \right)} \prod_{i=1}^{n} \left[ 1 - \frac{\rho^3 \mu_i \nu_i \kappa}{\pi^{3/2} \alpha_i \beta_i \gamma} \right] \int_{-\infty}^{+\infty} \frac{(x_1 - x_i)^2}{\alpha_i^2} + \frac{(y_1 - y_i)^2}{\beta_i^2} + \frac{(z_1 - z_i)^2}{\gamma_i^2} \right] dx' \, dy' \, dz'. \]

We make the change of variables \( x = \frac{x_1}{\alpha}, \quad y = \frac{y_1}{\beta}, \quad z = \frac{z_1}{\gamma}, \) and find then that

\[ P_n = 1 - \frac{\rho^3 \alpha \beta \gamma}{\pi^{3/2} abc} \int_{-\infty}^{+\infty} e^{-\rho^2 \left[ \left( \frac{\alpha x}{a} \right)^2 + \left( \frac{\beta y}{b} \right)^2 + \left( \frac{\gamma z}{c} \right)^2 \right]} \prod_{i=1}^{n} \left[ 1 - \frac{\rho^3 \mu_i \nu_i \kappa}{\pi^{3/2} \alpha_i \beta_i \gamma_i} \right] dx' \, dy' \, dz'. \]

We introduce the following notation:

\[ \begin{align*}
A &= \frac{\rho^3 \mu \nu \kappa}{\pi^{3/2} \alpha \beta \gamma}, \\
B &= \frac{\rho^3 \alpha \beta \gamma}{\pi^{3/2} abc},
\end{align*} \]

\[ \phi(x, y, z) = B e^{-\rho^2 \left[ \left( \frac{\alpha x}{a} \right)^2 + \left( \frac{\beta y}{b} \right)^2 + \left( \frac{\gamma z}{c} \right)^2 \right]}, \]

\[ \psi(x, y, z) = A e^{-\rho^2 (x^2 + y^2 + z^2)}. \]
With this notation, we have

\[ P_n = 1 - \int_{-\infty}^{+\infty} \phi(x, y, z) \prod_{i=1}^{n} \left[ 1 - \psi(x - x_i, y - y_i, z - z_i) \right] \, dx \, dy \, dz. \]

In the sequel, we shall use the following notation:

\[ \phi = \phi(x, y, z) \]
\[ \psi_1 = \psi(x - x_1, y - y_1, z - z_1) \]
\[ dv = dx \, dy \, dz \]

and instead of the triple integral extended over all of three dimensional space, we shall write one without limits. Thus the formula (2) will be written

\[ P_n = 1 - \int \phi \prod_{i=1}^{n} (1 - \psi_1) \, dv. \]

§2. Conditional distribution of the target after K shots which do not give any hits.

By Bayes's theorem, the conditional density of the probability of finding the target at the point \((x, y, z)\) under the condition that the K first shots have been fired but have not resulted in hits is

\[ \phi_K = \frac{\prod_{i=1}^{K} (1 - \psi_1)}{\int \phi \prod_{i=1}^{K} (1 - \psi_1) \, dv}. \]
The method of the following investigation consists in examining the process of stepwise change of the probability $P_K$ and the conditional probability $\phi_K$ with increase in the number $K$. For $K = 0$, we have

$$P_0 = 0, \quad \phi_0 = \phi,$$

and the further process of changing $P_K$ and $\phi_K$ is governed by the recurrence formulas

$$
\begin{align*}
(P_1) & \quad P_{K+1} = P_K + (1-P_K) \int \phi_K \Psi_{K+1} \, dv, \\
& \quad \phi_{K+1} = \frac{\phi_K(1-\Psi_{K+1})}{\int \phi_K(1-\Psi_{K+1}) \, dv}.
\end{align*}
$$

The recurrence formulas (5) can be made more convenient for the sequel if we introduce the probability

$$Q_K = 1 - P_K = \int \phi \prod_{i=1}^{K} (1-\Psi_i) \, dv$$

of getting nothing but misses on the first $K$ shots, and then instead of the probability $\phi_K$ we consider the function

$$I_K(x, y, z) = \phi(x, y, z) \prod_{i=1}^{K} [1 - \Psi(x-x_i, y-y_i, z-z_i)],$$

from which $\phi_K(x, y, z)$ differs only by the constant factor $\frac{1}{Q_K}$:

$$\phi_K = \frac{1}{Q_K} I_K.$$
For $K = 0$, we have

\[(9) \quad Q_0 = 1, \ I_0 = \phi.\]

For the probability $Q_K$ and the function $I_K$, we have a recursion formula corresponding to (5):

\[
\begin{align*}
Q_{K+1} &= Q_K - \int I_K \Psi_{K+1} dv, \\
I_{K+1} &= I_K (1 - \Psi_{K+1}).
\end{align*}
\]

(10)

We here prove formula (10). The difference $Q_K - Q_{K+1}$ is the probability that the first hit will be made with the $(K+1)^{\text{st}}$ shot. This probability is equal to the product of $Q_K$ and the conditional probability of securing a hit with the $(K+1)^{\text{st}}$ shot under the hypotheses that no hit is secured with the first $K$ shots. In short,

\[
Q_K - Q_{K+1} = Q_K \left(\int \phi_k \Psi_{K+1} dv\right) = \int I_K \Psi_{K+1} dv.
\]

Furthermore, by Bayes's theorem, we have

\[
\phi_{K+1} = \frac{\phi_k (1 - \Psi_{K+1})}{\int \phi_k (1 - \Psi_{K+1}) dv} = \frac{I_K (1 - \Psi_{K+1})}{Q_K (1 - \int \phi_k \Psi_{K+1} dv)} = \frac{I_K (1 - \Psi_{K+1})}{Q_K - Q_{K+1} \int \phi_k \Psi_{K+1} dv}.
\]

\[
\frac{I_K (1 - \Psi_{K+1})}{Q_K - (Q_K - Q_{K+1})}.
\]
that is, \( \phi_{K+1} = \frac{I_K(1 - \psi_{K+1})}{Q_{K+1}} \). Multiplying this result by \( Q_{K+1} \), we obtain (10).

\[ \frac{1}{2} \]

\( \S 3. \) Choice of the points \((x_i, y_i, z_i)\).

It is natural to suppose that in a rational system of firing, after the points \((x_i, y_i, z_i)(i < K)\) have been chosen, and the corresponding probabilities \( P_{K-1} \) and \( Q_{K-1} \) and the functions \( \phi_{K-1} \) and \( I_{K-1} \) have been determined, then the point \((x_K, y_K, z_K)\) must be chosen so as to make the incremental probability (11)

\[ P_K - P_{K-1} - Q_{K-1} - Q_K = \int I_{K-1} \psi_K dv \]

as large as possible.

Strictly speaking, this hypothesis is erroneous. As a matter of fact, the distribution of \((x_i, y_i, z_i)\) making \( P_n \) a maximum will not in general be obtained by choosing the points so as to maximize the integral

\[ \int I_{K-1} \psi_K dv \]

However, as will become evident in the sequel, when \( \frac{\alpha}{\beta}, \frac{\beta'}{\alpha'}, \frac{\delta}{\alpha}, \frac{\epsilon}{\beta}, \frac{\gamma}{\epsilon} \)
tend to zero (i.e., in the case when the target is small in comparison to the ellipsoid of non-repeating errors and this ellipsoid is small in comparison to the ellipsoid of repeating errors), then the most rational density distribution of the points \((x_i, y_i, z_i)\) becomes asymptotically the distribution just defined.

It is plainly seen that the distribution of the points \((x_i, y_i, z_i)(i = 1, 2, \ldots, n)\) yielding a maximum value for \( P_n \) must exactly satisfy the requirement that they make the integral (11) a maximum.
for \( K = n \). Since

\[
I_{n-1} = \frac{I_n}{1 - \psi_n},
\]

the integral (11) for \( K = n \) has the form

\[
\delta_n = \int I_n \frac{\psi_n}{1 - \psi_n} \, dv.
\]

We remark now that neither \( P_n \) nor \( I_n \) depends upon the order in which the points \((x_i, y_i, z_i)\) are numbered. Hence, the distribution of \((x_i, y_i, z_i) (i = 1, 2, \ldots, n)\) giving a maximum value to \( P_n \) must satisfy the requirement that for any \( i = 1, 2, \ldots, r \), a shift in the point \((x_i, y_i, z_i)\) to a new position while we leave all other points stationary, cannot increase the value of the integral

\[
\delta_n^{(i)} = \int I_n \frac{\psi_i}{1 - \psi_i} \, dv.
\]

If the ratios \( \frac{\alpha}{\beta}, \frac{\gamma}{\delta}, \frac{\epsilon}{\phi}, \frac{\psi}{\xi} \) are all small, then it is natural to figure that the function \( I_n = Q_n \phi_n \) changes with sufficient smoothness so that it is almost constant in the neighborhood of \((x_i, y_i, z_i)\) where the factor \( \frac{\psi_i}{1 - \psi_i} \) is markedly different from zero. Under this assumption, it is approximately true that

\[
(13) \quad \delta_n^{(i)} = C I_n(x_i, y_i, z_i),
\]
where \( C = \int \frac{\psi_1}{1 - \psi_1} \, dv \) is a constant. The condition that \( \delta_n^{(1)} \) be a maximum at \((x_1^i, y_1^i, z_1^i)\) leads to this conclusion: all the points \((x_1^i, y_1^i, z_1^i)\) \((i=1, 2, \ldots, n)\) are located in that part of space \( \cap_n \) where the function \( I_n(x, y, z) \) is equal to its maximum value \( 14 \) \( K_n \).

We conclude further from (7), that at a distance from \( \cap_n \) such that all \( \psi_1(x, y, z) \) are nearly zero, we have approximately \( I_n(x, y, z) = \phi(x, y, z) \). Supposing that the function \( I_n(x, y, z) \) varies with sufficient smoothness, we come to the result that on the boundary of \( \cap_n \), it is true that

\[
(14) \quad I_n(x, y, z) = \phi(x, y, z) = K_n.
\]

In consequence, we have finally

\[
(15) \quad I_n(x, y, z) = \begin{cases} K_n & \text{in } \cap_n, \\ \phi(x, y, z) & \text{outside of } \cap_n, \end{cases}
\]

where the region \( \cap_n \) is defined by the inequality \( \phi(x, y, z) > K_n \).

The constant \( K_n \) enjoys a very simple relationship with the probability \( \max P_n \). Indeed, in view of (6) and (7)

\[
(17) \quad \max P_n = 1 - Q_n = 1 - \int I_n \, dv = \int (\phi - I_n) \, dv = \int_{\cap_n} (\phi - K_n) \, dv.
\]

The boundary of the region \( \cap_n \) is given by the equation

\[
\phi(x, y, z) = B e^{-\rho^2 \left[ \left( \frac{\alpha x}{a} \right)^2 + \left( \frac{\beta y}{b} \right)^2 + \left( \frac{\gamma z}{c} \right)^2 \right]} = K_n,
\]
\[
\frac{x^2}{\frac{1}{\rho^2} \left( \frac{a}{\rho} \right)^2 \log \frac{B}{K_n}} + \frac{y^2}{\frac{1}{\rho^2} \left( \frac{b}{\rho} \right)^2 \log \frac{B}{K_n}} + \frac{z^2}{\frac{1}{\rho^2} \left( \frac{c}{\rho} \right)^2 \log \frac{B}{K_n}} = 1,
\]

or

that is, the region \( \cap_n \) is the interior and boundary of the ellipsoid with semi-axes

\[
a \sqrt{\frac{1}{\rho^2} \log \frac{B}{K_n}}, \quad b \sqrt{\frac{1}{\rho^2} \log \frac{B}{K_n}}, \quad c \sqrt{\frac{1}{\rho^2} \log \frac{B}{K_n}}.
\]

From a known formula, we find the volume of this ellipsoid:

\[
V = \frac{4\pi abc}{3\rho^2 a \beta^2 Y} \left( \log \frac{B}{K_n} \right)^{3/2}.
\]

Recalling from (1) the expression for \( B \), we write

\[
V = \frac{4}{3\sqrt{\pi} B} \left( \log \frac{B}{K_n} \right)^{3/2}.
\]

The integral \( \int_{\cap_n} \phi dv \) over an ellipsoid similar to the unit ellipsoid is found by a known formula:

\[
\int_{\cap_n} \phi dv = \sigma(\rho t) - \frac{2\rho t}{\sqrt{\pi}} e^{-\rho^2 t^2},
\]

where \( \sigma(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2} du, \quad t = \frac{1}{\rho} \sqrt{\log \frac{B}{K_n}}. \)

Since \( \sigma(\rho t) = \sigma(t) = \frac{2\rho}{\sqrt{\pi}} \int_0^t e^{-\rho^2 u^2} du \), we obtain from (17) that
\[(18) \quad \max P_n = \left( \frac{1}{\rho} \sqrt{\log \frac{B}{K}} \right) - \frac{2}{\sqrt{\pi}} \sqrt{\log \frac{B}{K}} - \frac{4}{3/\sqrt{\pi}} \left( \log \frac{B}{K} \right)^{3/2} \]

The right of (18) depends only upon \( \frac{B}{K} \), and thus (18) defines a function

\[(19) \quad \max P_n = W_3 \left( \log \frac{B}{K} \right), \]

which can be calculated in tabular form.

For the two-dimensional case, the region \( \cap_n \) is the interior and boundary of an ellipse with semi-axes

\[\frac{a}{\rho} \sqrt{\log \frac{B}{K}} \quad \text{and} \quad \frac{b}{\rho} \sqrt{\log \frac{B}{K}}.\]

Using known formulas for the area of an ellipse and for the integral of the probability function over an ellipse similar to the unit ellipse,

\[\int \phi ds = 1 - e^{-\rho^2 t^2},\]

we obtain the formula

\[(20) \quad \max P_n = 1 - \frac{1}{B/K} (1 - \log \frac{B}{K}) , \]

defining the function

\[(21) \quad \max P_n = W_2 \left( \log \frac{B}{K} \right). \]

Finally, the one-dimensional case leads to the formula

\[(22) \quad \max P_n = \Phi \left( \frac{1}{\rho} \log \frac{B}{K} \right) - \frac{2}{\sqrt{\pi}} \sqrt{\log \frac{B}{K}} , \]

\[\Phi \left( \frac{1}{\rho} \log \frac{B}{K} \right) \]
which defines the function

\[ (23) \quad \max P_n = W_1 \left( \log \frac{B}{K_n} \right) . \]

Thus, knowing \( \max P_n \), we can find the value of the constant \( K_n \).

§4. Density distribution of the points \( (x_1, y_1, z_1) \). If the number of shots, \( n \), is large, there it is natural to suppose that for approximate determination of the probability \( P_n \), it is sufficient to know not the exact co-ordinates of the points \( (x_1, y_1, z_1) \) but the density distribution of these points. We denote by \( N_n(x, y, z) \) the density distribution of the points \( (x_1, y_1, z_1) \) in the neighborhood of the point \( (x, y, z) \). Clearly,

\[ (24) \quad n = \int N_n \, dv . \]

On the basis of the results of the preceding §, we shall suppose that \( N_n(x, y, z) \) differs from zero only within the region \( \mathcal{R}_n \). It remains to define \( N_n(x, y, z) \) at an arbitrary point of the region \( \mathcal{R}_n \). In view of (7) and (15),

\[ I_n(x, y, z) = \phi(x, y, z) \prod_{i=1}^n \left[ 1 - \Psi(x-x_i, y-y_i, z-z_i) \right] = K_n , \]

whence

\[ \prod_{i=1}^n \left[ 1 - \Psi(x-x_i, y-y_i, z-z_i) \right] = \frac{K_n}{\phi(x, y, z)} , \]

or, taking logarithms of both sides.
(25) \[ \sum_{i=1}^{n} \log \left[ 1 - \Psi (x-x_i, y-y_i, z-z_i) \right] = \log \frac{K_n}{\phi(x,y,z)} . \]

Under the assumption that \( \frac{\delta}{a}, \frac{\delta}{b}, \frac{\delta}{c} \) are small, the quantities \( \Psi_i = \Psi (x-x_i, y-y_i, z-z_i) \) are small, and we have approximately \( \log (1 - \Psi_i) = -\Psi_i \). Then from (25), we obtain

(26) \[ \sum_{i=1}^{n} \Psi_i = -\log \frac{K_n}{\phi(x,y,z)} = \log \frac{B}{K_n} - \delta^2 \left[ \left( \frac{\delta x}{a} \right)^2 + \left( \frac{\delta y}{b} \right)^2 + \left( \frac{\delta z}{c} \right)^2 \right] . \]

Assuming that \( N_n(x,y,z) \) varies with sufficient smoothness and is almost constant in the region where \( \Psi_i \) is significantly different from zero, that the points \( (x_i, y_i, z_i) \) are distributed in this region with a density approximately equal to \( N_n(x,y,z) \), sufficiently uniformly, we find approximately

\[ \sum_{i=1}^{n} \Psi_i = N_n(x,y,z) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Psi(x-x', y-y', z-z') \, dx'dy'dz' = \frac{\mu_x \mu_y \mu_z}{\mu_x \mu_y \mu_z} N_n(x,y,z). \]

Combining this result with (26), we find finally in \( \cap_n \),

(27) \[ N_n(x,y,z) = \frac{\mu_x \mu_y \mu_z}{\mu_x \mu_y \mu_z} \left[ \log \frac{B}{K_n} - \delta^2 \left[ \left( \frac{\delta x}{a} \right)^2 + \left( \frac{\delta y}{b} \right)^2 + \left( \frac{\delta z}{c} \right)^2 \right] \right] \]

Outside of \( \cap_n \), as previously stated, we put \( N_n = 0 \).

The equation (27) defines a paraboloid in four-dimensional space having co-ordinates \( (x,y,z,N) \) with vertex displaced from the origin and with axis in the N-direction. In the two-dimensional case (in the two-dimensional case, we have \( B = \frac{\rho^2}{\pi a b} \),
and in the one-dimensional case, we have \( B = \frac{\rho \alpha}{\sqrt{\pi a}} \) we have

\[
N_n(x, y) = \begin{cases} 
\frac{\alpha \beta}{\mu y} \left\{ \log \frac{B}{K_n} - \rho^2 \left[ \frac{(\alpha x)^2}{a} + \frac{(\beta y)^2}{b} \right] \right\} \text{in } \mathcal{N}_n, \\
0 \quad \text{outside of } \mathcal{N}_n,
\end{cases}
\]

where \( \mathcal{N}_n \) is the interior plus boundary of the ellipse

\[
(\frac{\alpha x}{a})^2 + (\frac{\beta y}{b})^2 = \frac{1}{\rho^2} \log \frac{B}{K_n} .
\]

In the one-dimensional case,

\[
N_n(x) = \begin{cases} 
\frac{\alpha}{\mu} \left\{ \log \frac{B}{K_n} - \rho^2 \left( \frac{\alpha x}{a} \right)^2 \right\} \text{ for } |x| \leq \frac{a}{\rho \alpha} \log \frac{B}{K_n}, \\
0 \quad \text{for } |x| > \frac{a}{\rho \alpha} \log \frac{B}{K_n}.
\end{cases}
\]

§5. Relations between the probability of at least one hit and mathematical expectation:

In view of (27),

\[
\frac{\mu \nu \kappa}{\alpha \beta \gamma} \int_{\mathcal{N}_n} N_n(x, y, z) dv = \int_{\mathcal{N}_n} \left\{ \log \frac{B}{K_n} - \delta^2 \left[ \left( \frac{dx}{a} \right)^2 + \left( \frac{\beta y}{b} \right)^2 - \left( \frac{\gamma z}{c} \right)^2 \right] \right\} dv
\]

Since \( \int_{\mathcal{N}_n} N_n dv = n \), we have

\[
n \frac{\mu \nu \kappa}{\alpha \beta \gamma} = \int_{\mathcal{N}_n} \left\{ \log \frac{B}{K_n} - \delta^2 \left[ \left( \frac{\alpha x}{a} \right)^2 + \left( \frac{\beta y}{b} \right)^2 + \left( \frac{\gamma z}{c} \right)^2 \right] \right\} dv,
\]

where \( \mathcal{N}_n \) is the three-dimensional ellipsoid described previously.

We make the change of variables
\[
\frac{\alpha x}{a} = r \sin \phi \cos \lambda, \quad \frac{\beta y}{b} = r \sin \phi \sin \lambda, \quad \frac{\gamma z}{c} = r \cos \phi,
\]
which transforms \( \mathcal{N}_n \) into the sphere with center \((0,0,0)\) and radius \( r \), where \( r = \frac{1}{\rho} \log \sqrt{\frac{B}{K_n}} \). The Jacobian of the transformation is \( |I| = r^2 \frac{a \ b \ c}{\alpha \beta \gamma} \sin \phi \). Hence we have

\[
\int_{\mathcal{N}_n} \left\{ \log \frac{B}{K_n} - \delta^2 \left[ \left( \frac{\alpha x}{a} \right)^2 + \left( \frac{\beta y}{b} \right)^2 + \left( \frac{\gamma z}{c} \right)^2 \right] \right\} \ d\nu =
\]

\[
\frac{a \ b \ c}{\alpha \beta \gamma} \int_0^n \pi \ d\phi \int_0^{2\pi} \ d\lambda \int_0^1 \rho^2 \log \frac{B}{K_n} \ (\log \frac{B}{K_n} - \rho^2 r^2) r^2 \sin \phi \ d\rho \ d\tau =
\]

\[
\frac{8 \pi}{15 \rho^3} \frac{a \ b \ c}{\alpha \beta \gamma} (\log \frac{B}{K_n})^{2/5}.
\]

Putting this expression into equality (31), we obtain

\[
\log \frac{B}{K_n} = \left( \frac{15 \rho^3 \mu \nu \kappa}{8 \pi a \ b \ c} \right)^{2/5}.
\]

Earlier we made the hypothesis that the semi-axes of the ellipsoid of dispersion must be significantly smaller than the corresponding semi-axes of the ellipsoid of distribution of the target. Hence in the expression for the mathematical expectation of number of hits (under the assumption of a small target), we have

\[
M = \frac{\rho^3 \mu \nu \kappa n}{\pi^{3/2} \sqrt{a^2 + \alpha^2 \beta^2 + \gamma^2}}
\]

and we may ignore \( \alpha^2, \beta^2, \) and \( \gamma^2 \) in the radical which appear in the denominator:
(33) \[ M = \frac{\rho^2 \mu v K n}{\pi^{3/2} \ a \ b \ c} \]

and the expression (32) assumes the form

(34) \[ \log \frac{B}{K_n} = \left( \frac{15 \sqrt{\pi}}{6} M \right)^{2/5} \]

For the two-dimensional case, the transformed region has the form of a circle \( S \) of radius \( \frac{1}{\rho} \sqrt{\log \frac{B}{K_n}} \), and the Jacobian of the transformation is \(|I| = r^a b \alpha \beta\); hence we have

\[ \int_S \left\{ \log \frac{B}{K_n} - \rho^2 \left[ \left( \frac{a x}{a} \right)^2 + \left( \frac{b y}{b} \right)^2 \right] \right\} \ ds = \frac{a b}{\alpha \beta} \int_0^{2\pi} d\phi \int_0^1 \left( \log \frac{B}{K_n} - \rho^2 \gamma^2 \right) r^2 \ dr \]

\[ = \frac{\pi a b}{2 \rho^2 \alpha \beta} \left( \log \frac{B}{K_n} \right)^2 , \]

whence finally

(35) \[ \log \frac{B}{K_n} = \left( \frac{2 \rho^2 \frac{a}{b}}{\pi} \right)^{1/2} = \left( 2M \right)^{1/2} . \]

In exactly the same way we find for the one-dimensional case

(36) \[ \log \frac{B}{K_n} = \left( \frac{3 \rho^2 \mu n}{4a} \right)^{2/3} = \left( \frac{3 \sqrt{\pi}}{4} M \right)^{2/3} . \]

Substituting the values just obtained for \( \log \frac{B}{K_n} \) in (18), (20), (22), we obtain

(a) for the three-dimensional case
\[
\max P_n = \Phi\left(\frac{1}{\rho} \left(\frac{15}{8} \sqrt{\pi} M\right)^{1/5}\right) - \frac{10}{3} \sqrt{\pi} \left(\frac{15}{8} \pi M\right)^{1/5} e^{-\left(\frac{15}{8} \sqrt{\pi} M\right)^{1/5}};
\]

(b) for the two-dimensional case,

\[
\max P_n = 1 - \left[1 + (2M)^{1/2}\right] e^{-\left(2M\right)^{1/2}};
\]

(c) for the one-dimensional case,

\[
\max P_n = \Phi\left(\frac{1}{\rho} \left(\frac{3}{4} \sqrt{\pi} M\right)^{1/3}\right) - \frac{2}{\sqrt{\pi}} \left(\frac{3}{4} \pi M\right)^{1/3} e^{-\left(\frac{3}{4} \sqrt{\pi} M\right)^{2/3}}.
\]

For convenience in using these functions, they can be tabulated together with corresponding distributions of shots. Hence the only calculation necessary is to find the mathematical expectation, which will be the argument for these tables.

In this way, we arrive at the conclusion that \(\max P_n\) is an increasing function of the mathematical expectation. From this follows an extremely important conclusion: to compare the effectiveness of two artillery systems, it is sufficient to compare the mathematical expectations. The system giving the larger mathematical expectation will give the larger probability of at least one hit (of course, with corresponding methods of firing).

§6. Broadening the region fired at with increase in the number of shots fired.

In the preceding § we found that in order to obtain the value \(\max P_n\), it is necessary to distribute the points \((x_1, y_1, z_1)\) within the region \(\mathcal{R}_n\) and with the density \(N_n(x, y, z)\). It is easy
to see that the region $\bigcap_n$ increases in size as $n$ increases. Also, the density $N_n(x,y,z)$ increases as $n$ increases. Hence one can produce such a system of firing that after an arbitrary number of shots the probability of at least one hit is equal to $\max P_n$. For this, the shots with numbers between $n$ and $\delta n$ ($\delta n$ small with respect to $n$) must be distributed throughout the region $\bigcap_n$ with uniform density

$$\delta N_n = N_{n+\delta n} - N_n = \frac{a \beta Y}{\mu \nu K} \log \frac{K_n}{K_{n+\delta n}}.$$

From this we extract the practical conclusion: One should fire in such a way that at a given moment one is guaranteed a maximum $P_n$ for the shots fired up to that moment. The firing must start at the center of distribution [of the target] and continue with uniform shelling of a region which increases in size with the number of shots fired.

§7. Conditions under which artificial dispersion is advantageous.

Beginning with the middle of §3, we have been concerned with approximating methods, dealing with the filling up of a region $\bigcap_n$ with points $(x_i, y_i, z_i)$ with a certain density $N_n(x,y,z)$. Investigations of this kind are feasible only when the number of shots, $n$, is large. Furthermore, it was assumed from the very beginning that the shots were subjected to artificial dispersion, and the question led merely to the determination of the region $\bigcap_n$ throughout which the artificial dispersion was to be made. It turned out that it is wise to increase the size of the region $\bigcap_n$ as the number $n$ increases, and furthermore it was found necessary to choose the dimensions of
the region $\mathcal{A}_n$ proportional to the 5th, 4th, and 3rd powers of $n$, for the three-, two-, and one-dimensional cases respectively.

This conclusion concerning the best dimensions for the region $\mathcal{A}_n$ was made, however, under assumptions that these dimensions are large by comparison with the dimensions of the ellipsoid of non-repeating errors. Hence the first stages of increasing the reasonable size of the region of artificial dispersion (when $n$ is not large enough) may not follow the regular law described above. Even more, for values of $n$ which do not exceed a certain $n_0$, artificial dispersion may be actually harmful.

In the present section we adopt the point of view of the beginning of section 3, not assuming that the number $n$ is large.

The first shot must always be aimed at the center of dispersion in order to obtain the maximal value of the number $P_1$, that is, $x_1 = y_1 = z_1 = 0$. We now suppose that the first $n$ shots have been aimed at the origin: $x_i = y_i = z_i = 0$ ($i=1,2,3,\ldots,n$).

Then, by formula (7), we have

$$I_n(x,y,z) = \phi(x,y,z) \left[1 - \psi(x,y,z)\right]^n.$$  \hspace{1cm} (40)

In correspondence with the point of view of the beginning of section 3, we shall suppose that the $(n+1)$st shot must be aimed at the point $(x_{n+1}, y_{n+1}, z_{n+1})$ for which the integral

$$P_{n+1} - P_n = \int I_n(x,y,z) \psi(x-x_{n+1}, y-y_{n+1}, z-z_{n+1})dv$$

has maximal value. Figuring approximately that

$$P_{n+1} - P_n = I_n(x_{n+1}, y_{n+1}, z_{n+1})$$
we arrive at the conclusion that artificial dispersion is useless as long as the function \((40)\) has a maximum at the origin; if for some \(n\), this maximum changes to a minimum, then it is advisable to move the \((n+1)\)st shot away from the origin. (That is, introduce artificial dispersion beginning with the \((n+1)\)st shot.) Such a method is not rigorous, but leads to results which are close to the right ones.

The extrema of the function \(I_n = \phi(1-\psi)^n\) can be obtained by solution of the system of equations

\[
\frac{\partial I_n}{\partial x} = 0, \quad \frac{\partial I_n}{\partial y} = 0, \quad \frac{\partial I_n}{\partial z} = 0,
\]

or, writing the equations in more detail,

\[
\frac{\partial \phi(1-\psi)^n}{\partial x} - n\phi(1-\psi)^{n-1} \frac{\partial \psi}{\partial x} = 0,
\]
\[
\frac{\partial \phi(1-\psi)^n}{\partial y} - n\phi(1-\psi)^{n-1} \frac{\partial \psi}{\partial y} = 0,
\]
\[
\frac{\partial \phi(1-\psi)^n}{\partial z} - n\phi(1-\psi)^{n-1} \frac{\partial \psi}{\partial z} = 0.
\]

Since we always have \(\psi < 1\), the factor \((1-\psi)^{n-1}\) may be cancelled, and we then have

\[
\begin{cases}
\frac{\partial \phi(1-\psi)}{\partial x} - n\phi \frac{\partial \psi}{\partial x} = 0, \\
\frac{\partial \phi(1-\psi)}{\partial y} - n\phi \frac{\partial \psi}{\partial y} = 0, \\
\frac{\partial \phi(1-\psi)}{\partial z} - n\phi \frac{\partial \psi}{\partial z} = 0.
\end{cases}(41)
\]
We calculate the partial derivatives in which we are interested:

\[
\frac{\partial \phi}{\partial x} = \frac{\rho^3 \alpha \beta \gamma}{\pi^{3/2} abc} \frac{\partial}{\partial x} e^{-\rho^2 \left[ \left( \frac{ax}{a} \right)^2 + \left( \frac{by}{b} \right)^2 + \left( \frac{cz}{c} \right)^2 \right]} = -2 \rho^2 \frac{\alpha x}{a^2} \phi;
\]

\[
\frac{\partial \psi}{\partial x} = -2 \rho^2 x \psi
\]
similarly, with corresponding results for

\[
\frac{\partial \phi}{\partial y}, \frac{\partial \psi}{\partial y}, \frac{\partial \phi}{\partial z}, \frac{\partial \psi}{\partial z}.
\]

Substituting these values into (41) and cancelling \(2 \rho^2\), we obtain

\[-\frac{x}{a^2} \phi(1-\psi) + \frac{nx}{a^2} \psi = 0; \]

\[-\frac{y}{b^2} \phi(1-\psi) + \frac{ny}{b^2} \psi = 0; \]

\[-\frac{z}{c^2} \phi(1-\psi) + \frac{nz}{c^2} \psi = 0; \]

or

\[x\phi\left(\frac{ny}{a^2} - \frac{1-\psi}{a^2}\right) = 0, \]

\[y\phi\left(\frac{nx}{b^2} - \frac{1-\psi}{b^2}\right) = 0, \]

\[z\phi\left(\frac{nz}{c^2} - \frac{1-\psi}{c^2}\right) = 0. \]

This system of equations breaks up into a series of systems.

System 1

\[\phi = 0. \]
The solution obtained hereby for the system is

\[ x = \pm \infty, \quad y = \pm \infty, \quad z = \pm \infty, \]

corresponding to a minimum of the function \( \phi_n(x, y, z) \).

**System 2**

\[ x = y = z = 0. \]

**System 3**

\[ x = y = 0; \quad \frac{n \Psi}{\gamma^2} - \frac{1 - \Psi}{c^2} = 0. \]

Using the expression for \( \Psi \) from (1) and putting \( x = y = 0 \) in the third equation of system 3, we have

\[-\frac{1}{c^2} (1 - Ae^{-\rho^2z^2}) + \frac{n}{\gamma^2} Ae^{-\rho^2z^2} = 0,\]

where \( A = \frac{\rho^3 \mu \gamma K}{3^{\frac{3}{2}} \alpha \beta \gamma} \). (\( A \) is clearly the probability of hitting the target when one aims at the center of distribution of the target.)

Then we find

\[ e^{-\rho^2z^2} = \frac{z^2}{A(\gamma^2 + nc^2)}, \]

whence

\[ z = \pm \frac{1}{\rho} \sqrt{\log[A(1 + n \frac{c^2}{\gamma^2})]} \cdot \]

For \( z \) to be a real number, it is necessary that the inequality

\[ A(1 + n \frac{c^2}{\gamma^2}) \geq 1 \]
be satisfied, or that

$$n \geq \left( \frac{\xi}{\varepsilon} \right)^2 \left( \frac{1}{A} - 1 \right).$$

Hence, if we have

$$n < \left( \frac{\xi}{\varepsilon} \right)^2 \left( \frac{1}{A} - 1 \right),$$

system 3 does not yield a real solution.

**System 4**

$$x = 0, \quad -\frac{1}{a^2} (1-\psi) + \frac{ny}{a^2} = 0, \quad z = 0.$$

As in system 3, we find

$$y = \pm \frac{1}{\rho} \sqrt{\log \left[ A(1+n \frac{b^2}{\beta^2}) \right]}$$

and system 4 does not give a real solution for

$$n < \left( \frac{\beta^2}{b^2} \right)^2 \left( \frac{1}{A} - 1 \right).$$

**System 5**

$$-\frac{1}{a^2} (1-\psi) + \frac{ny}{a^2} = 0, \quad y = z = 0.$$

Analogously, we find

$$x = \pm \sqrt{\log \left[ A(1+n \frac{a^2}{\alpha^2}) \right]}$$

and for $$n < \left( \frac{\alpha}{a} \right)^2 (\frac{1}{A} - 1),$$ this system does not yield a real solution.
System 6

\[ x = 0, - \frac{1}{\beta^2} (1 - \psi) + \frac{n \psi}{\beta^2} = 0, - \frac{1}{\gamma^2} (1 - \psi) + \frac{n \psi}{\gamma^2} = 0. \]

For the last two equations to be compatible, we must have

\[ \frac{\beta^2}{\beta^2} = \frac{\gamma^2}{\gamma^2} = \lambda^2, \]

and in this case, the two last equations lead to one:

\[ -\lambda^2 (1 - \psi) + n \psi = 0, \]

whence \[ \psi = \frac{\lambda^2}{n + \lambda^2} \text{ const.} \]; using formula (1), we have

\[ e^{-\rho^2 (y^2 + z^2)} = \frac{\lambda^2}{A(n + \lambda^2)}, \text{ whence } \]

\[ y^2 + z^2 = \frac{1}{\rho^2} \log \left( A(1 + \frac{n}{\lambda^2}) \right). \]

There will be no real solution if \( A(1 + \frac{n}{\lambda^2}) < 1 \), or if

\[ n < \lambda^2 \left( \frac{1}{A} - 1 \right) = \left( \frac{\beta}{b} \right)^2 \left( \frac{1}{A} - 1 \right) = \left( \frac{\gamma}{c} \right)^2 \left( \frac{1}{A} - 1 \right). \]

System 7

\[ - \frac{1}{a^2} (1 - \psi) + \frac{n \psi}{a^2} = 0, y = 0, - \frac{1}{\gamma^2} (1 - \psi) + \frac{n \psi}{\gamma^2} = 0 \]

System 8

\[ - \frac{1}{a^2} (1 - \psi) + \frac{n \psi}{a^2} = 0, - \frac{1}{b^2} (1 - \psi) + \frac{n \psi}{b^2} = 0, z = 0. \]

Systems 7 and 8 are like unto system 6.
System 9

\[- \frac{1}{a^2}(1 - \Psi) + \frac{n}{a^2} \Psi = - \frac{1}{b^2}(1 - \Psi) + \frac{n}{b^2} \Psi = - \frac{1}{c^2}(1 - \Psi) + \frac{n}{c^2} \Psi = 0. \]

These are compatible under the condition that

\[\frac{a^2}{\lambda^2} = \frac{b^2}{\lambda^2} = \frac{c^2}{\lambda^2} = \lambda^2.\]

In this case, the system reduces to

\[-\lambda^2(1 - \Psi) + n \Psi = 0,\]

and as with System 6, we find

\[x^2 + y^2 + z^2 = \frac{1}{\rho^2} \log \left[ A(1 + \frac{n}{\lambda^2}) \right].\]

This equation does not have real solutions for

\[A(1 + \frac{n}{\lambda^2}) < 1; \text{ hence}\]

for \(n < \frac{a^2(1/\lambda^2 - 1)}{\lambda^2}, n < \frac{b^2(1/\lambda^2 - 1)}{\lambda^2}, n < \frac{c^2(1/\lambda^2 - 1)}{\lambda^2}\),

the extrema of the function \(f_n(x,y,z)\) can occur only at the points \(x=0, y=0, z=0\) and \(x=+\infty, y=+\infty, z=+\infty\). It is evident that the function \(f_n(x,y,z)\) has a minimum at infinity.

At the point \((0,0,0)\) we have a maximum of the function, which part can be proved without difficulty. As a matter of fact, \(f_n(0,0,0) > h > 0\). Since \(f_n(x,y,z) \to 0\) as \(x,y,z \to \infty\), there exists a sphere \(V\) of radius \(R\) on the surface of which \(f_n(x,y,z) < h_1 < h\). Thus, in the region \(V\) the function \(f_n\) does not attain its maximum on the boundary. Then there exists a point
inside the region in which \( \phi_n(x,y,z) \) attains its maximum. (See, e.g., Goursat Cours d' Analyse, vol. I., part I, p. 24.) But we have shown that within \( V \), \( (0,0,0) \) is the only extreme point. Hence \( (0,0,0) \) is the point where \( \phi_n(x,y,z) \) assumes its maximum value.

The increase in the probability of at least one hit from the \((n+1)\)st shot is expressed by the formula

\[
I_{n+1} = \int \phi \prod_{i=1}^{n} (1 - \psi_i) \psi_{n+1} \, dv.
\]

Since \( \phi \prod_{i=1}^{n} (1 - \psi_i) \) has a maximum at the origin, \( I_{n+1} \) will have its greatest value when \( x_{n+1} = y_{n+1} = z_{n+1} = 0 \), i.e., the \((n+1)\)st shot must be aimed at the origin. Summarizing the results of this \( \delta \), we may say that artificial dispersion must not be used if the number \( n \) of shots satisfies the inequalities

\[
\begin{align*}
  n &< \left( \frac{\alpha}{\beta} \right)^2 \left( \frac{1}{A} - 1 \right) + 1, \\
  n &< \left( \frac{\beta}{\alpha} \right)^2 \left( \frac{1}{A} - 1 \right) + 1, \\
  n &< \left( \frac{\gamma}{\beta} \right)^2 \left( \frac{1}{A} - 1 \right) + 1.
\end{align*}
\]

Any person can plainly see that these formulas hold also in the two- and one-dimensional cases.

8. Mathematical expectation in firing with the most advantageous artificial dispersion

It is not difficult to compute the mathematical expectation \( M' \) of the number of hits in firing with that artificial dispersion which guarantees the largest probability of obtaining at least one
hit. By definition of the mathematical expectation we have

\[ M' = \sum_{i=1}^{n} \int \phi \psi_i \, dv, \]

where the points \((x_1, y_1, z_1)\) are so chosen that the greatest probability of at least one hit is assured. Rewriting, we have

\[ (42) \quad M' = \int \phi \left( \sum_{i=1}^{n} \psi_i \right) \, dv. \]

From (42) and (26), we obtain

\[ M' = \iiint_{\mathcal{N}_n} \phi(x, y, z) \left\{ \log \frac{B}{K_n} - \rho^2 \left[ \frac{(\alpha x)^2}{a^2} + \frac{(\beta y)^2}{b^2} + \frac{(\gamma z)^2}{c^2} \right] \right\} \, dx \, dy \, dz. \]

The last integral is taken only over \(\mathcal{N}_n\), since outside of \(\mathcal{N}_n\), \(\sum_{i=1}^{n} \psi_i = 0\). Substituting in the last expression the value of \(\phi\) given by (1), we have

\[ (43) \quad M' = \frac{\rho^3 \alpha \beta \gamma}{\pi^{3/2} abc} \iiint_{\mathcal{N}_n} e^{-\rho^2 \left[ \frac{(\alpha x)^2}{a^2} + \frac{(\beta y)^2}{b^2} + \frac{(\gamma z)^2}{c^2} \right]} \left\{ \log \frac{B}{K_n} - \rho^2 \left[ \frac{(\alpha x)^2}{a^2} + \frac{(\beta y)^2}{b^2} + \frac{(\gamma z)^2}{c^2} \right] \right\} \, dx \, dy \, dz. \]

Making the change of variables

\[ \frac{\alpha x}{a} = r \sin \Theta \cos \lambda, \quad \frac{\beta x}{b} = r \sin \Theta \sin \lambda, \quad \frac{\gamma z}{c} = r \cos \Theta, \]

we find

\[ M' = \frac{\rho^3}{\pi^{3/2}} \int_{0}^{2\pi} \, d\lambda \int_{0}^{\pi} \sin \Theta \, d\Theta \int_{0}^{1} \sqrt{\log B/K_n} \]

\[ e^{-\rho^2 r^2 (\log (B/K_n) - \rho^2 r)} \]

after replacing \(\rho r\) by \(t\),
\[
M' = \frac{4}{\sqrt{\pi}} \int_0^\infty \sqrt{\log \frac{B}{K_n}} e^{-t^2} (\log \frac{B}{K_n} - t^2) t^2 dt = \\
4 \log \frac{B}{K_n} \int_0^\infty \sqrt{\log \frac{B}{K_n}} t^2 e^{-t^2} dt - \frac{4}{\sqrt{\pi}} \int_0^\infty \sqrt{\log \frac{B}{K_n}} t^4 e^{-t^2} dt = \\
\frac{4 \log \frac{B}{K_n}}{\sqrt{\pi}} J_1 - \frac{4}{\sqrt{\pi}} J_2 .
\]

We compute the integrals \(J_1\) and \(J_2\) by integration by parts:

\[
J_1 = -\frac{1}{2} \left( \frac{B}{K_n} \right)^{3/2} + \frac{3}{2} J_1,
\]

\[
J_2 = -\frac{1}{2} \left( \frac{B}{K_n} \right)^{3/2} + \frac{3}{2} J_1.
\]

Putting these values in (44), we obtain

\[
M' = \frac{3}{\sqrt{\pi}} \frac{B}{K_n} \sqrt{\log \frac{B}{K_n}} + (\log \frac{B}{K_n} - \frac{3}{2}) \frac{1}{\rho \sqrt{\log \frac{B}{K_n}}} .
\]

For the two-dimensional case, we have analogously:

\[
M' = \frac{p^3}{\pi} \frac{a}{a} \frac{b}{b} \int \int e^{-\rho^2 \left[ \left( \frac{ax}{a} \right)^2 + \left( \frac{by}{b} \right)^2 \right]} \left\{ \log \frac{B}{K_n} - \rho^2 \left[ \left( \frac{ax}{a} \right)^2 + \left( \frac{by}{b} \right)^2 \right] \right\} \, dx \, dy
\]

Making the change of variables \(\frac{ax}{a} = r \cos \theta\), \(\frac{by}{b} = r \sin \theta\), we obtain the following expression:
\[ M' = \frac{\rho^2}{\pi} \int_0^{2\pi} d\theta \int_0^1 \sqrt{\log \frac{B}{K_n}} e^{-\rho^2 r^2} (\log \frac{B}{K_n} - \rho^2 r^2) r dr. \]

Putting \( \rho^2 r^2 = t \), we obtain

\[ M' = \int_0^{\log \frac{B}{K_n}} e^{-t} (\log \frac{B}{K_n} - t) dt = \frac{1}{(\log \frac{B}{K_n})} \log \frac{B}{K_n} + (\log \frac{B}{K_n} - 1) \left[ -e^{-t} \right]_0^{\log \frac{B}{K_n}} \]

whence we obtain as our final result for the two-dimensional case

\[(46) \quad M' = \log \frac{B}{K_n} + \frac{1}{\log \frac{B}{K_n}} - 1.\]

For the one-dimensional case, we obtain

\[ M' = \frac{\rho a}{\sqrt{\pi}} \int_{-\frac{a}{\rho a}}^{+\frac{a}{\rho a}} \frac{1}{\sqrt{\log \frac{B}{K_n}}} e^{-\rho^2 \frac{(ax)^2}{a^2}} \left[ \log \frac{B}{K_n} - \rho^2 \frac{(ax)^2}{a^2} \right] dx, \]

and this integral may be evaluated by elementary computations. As our end result, we obtain

\[(47) \quad M' = \frac{1}{\sqrt{\pi}} \sqrt{\log \frac{B}{K_n}} + \left( \log \frac{B}{K_n} - \frac{1}{2} \right) \Phi \left( \frac{1}{\rho} \sqrt{\log \frac{B}{K_n}} \right).\]

Substituting in (45), (46), and (47) the values obtained in §4 for \( \log \frac{B}{K_n} \), we find the following formulas for \( M' \):

(a) in the three-dimensional case:
\( M' = \frac{3}{\sqrt{\pi}} \left( \frac{15}{8} \sqrt{\frac{\pi}{2}} M \right)^{1/5} \left( \frac{15}{8} \sqrt{\frac{\pi}{2}} M \right)^{1/5} + \left( \frac{15}{8} \sqrt{\frac{\pi}{2}} M \right)^{2/5} - \frac{3}{2} \right) \left( \frac{1}{\rho} \left( \frac{15}{8} \sqrt{\frac{\pi}{2}} M \right)^{1/5} \right) ; \\

(b) in the two-dimensional case,

\( M' = (2M)^{1/2} + e^{-(2M)^{1/2}} - 1 ; \)

(c) in the one-dimensional case,

\( M' = \frac{3}{\sqrt{\pi}} \left( \frac{3}{4} \sqrt{\frac{\pi}{4}} M \right)^{1/3} \left( \frac{3}{4} \sqrt{\frac{\pi}{4}} M \right)^{2/3} + \left( \frac{3}{4} \sqrt{\frac{\pi}{4}} M \right)^{2/3} - \frac{1}{2} \right) \left( \frac{1}{\rho} \left( \frac{3}{4} \sqrt{\frac{\pi}{4}} M \right)^{1/3} \right) . \)

The functions defined by (48), (49), and (50) can be given in the form of tables.

Artificial dispersion which yields the maximum probability of at least one hit leads to a comparatively small decrease in the mathematical expectation. The smaller is the mathematical expectation for firing without artificial dispersion, the less important is the decrease upon introducing artificial dispersion.

9. **Solution for the case of uniform bombardment of a region**

A solution of the problem of finding the best artificial dispersion can be found in the case of uniform bombardment of a region.
We find the solution in the case where the artificial dispersion consists in bombarding uniformly a parallelepiped (rectangle, interval), the edges of which are parallel to the semi-axes of the ellipsoids. Let \( 2\xi, 2\eta, 2\zeta \) be the lengths of the edges of the parallelepiped \( H \) which is to be subjected to uniform bombardment.

We may assert that if the semi-axes of the ellipsoid of non-repeating errors is not large in comparison with the region in which the artificial dispersion is carried out, then we may take

\[
\sum_{i=1}^{n} \pi \left[ 1 - \psi(x-x_i, y-y_i, z-z_i) \right] = \begin{cases} c & \text{for } (x,y,z) \in H \\ 1 & \text{for } (x,y,z) \notin H \end{cases}
\]

Hence we find for the probability of at least one hit

\[
P_n = 1 - \int \phi \prod_{i=1}^{n} (1 - \psi_1) \, dv = 1 - c \int_H \phi \, dv - \int_{H'} \phi \, dv
\]

\((H')\) is the complementary set of \( H \). Recalling that

\[
\int \phi \, dv = \int_H \phi \, dv + \int_{H'} \phi \, dv = 1
\]

we have

\[
P_n = (1 - c) \int_H \phi \, dv
\]

We find the value of \( c \) as similar determinations have been made above. Taking logarithms of (51), we have

\[
\sum_{i=1}^{n} \log \left[ 1 - \psi(x-x_i, y-y_i, z-z_i) \right] = \begin{cases} \log c & \text{for } (x,y,z) \in H \\ 0 & \text{for } (x,y,z) \notin H \end{cases}
\]
or

\[
\sum_{i=1}^{n} \left\{ -\log\left[1 - \psi(x-x_i, y-y_i, z-z_i)\right] \right\} = \begin{cases} 
\log \frac{1}{c} \text{ for } (x,y,z) \in H, \\
0 \text{ for } (x,y,z) \notin H.
\end{cases}
\]

Integrating this equality over the entire space, we have

\[
\int_{\mathbb{R}^3} \sum_{i=1}^{n} \left[ -\log(1 - \psi_i) \right] dv = V \log \frac{1}{c}
\]

when \( V = 8 \xi \eta S \). Since the values of \( \psi_1 \) are small, we replace \( -\log(1 - \psi_i) \) by \( \psi_i \), and find

\[
\sum_{i=1}^{n} \int \psi_i \ dv = 8 \xi \eta S \log \frac{1}{c},
\]

or, substituting for \( \psi_i \) its value as given by (1),

\[
\frac{e^3 \mu \nu K}{\pi^{3/2} \alpha \beta \gamma} \sum_{i=1}^{n} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\rho^2 \left( (x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2 \right)} \ dx \ dy\ dz = 8 \xi \eta S \log \frac{1}{c}.
\]

It is evident that

\[
\frac{e^3}{\pi^{3/2}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\rho^2 \left( (x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2 \right)} \ dx \ dy\ dz = 1,
\]

whence we find

\[
\frac{\mu \nu K}{\alpha \beta \gamma} \sum_{i=1}^{n} = 8 \xi \eta S \log \frac{1}{c};
\]

consequently, we have

\[
\frac{\mu \nu K}{\alpha \beta \gamma} = e^{-\frac{8 \xi \eta S}{\alpha \beta \gamma}}
\]

Substituting this in (52), we obtain
\[ P_n = (1 - e^{-\frac{\mu \nu K}{8a^2 y z}}) \int_0^{\xi} \int_0^{\eta} \int_0^{\zeta} e^{-\int_0^{\gamma} \int_0^{\beta} \int_0^{\alpha} (\frac{ax}{a})^2 + (\frac{by}{b})^2 + (\frac{cz}{c})^2} \, dz, \]

whence it follows that

\[ P_n = (1 - e^{-\frac{\mu \nu K}{8a^2 y z}}) \Phi(\frac{a \xi}{a}) \Phi(\frac{b \eta}{b}) \Phi(\frac{c \zeta}{c}) \]

where \( \Phi(t) = \frac{2\rho}{\sqrt{\pi}} \int_0^t e^{-\rho^2 t^2} \, dt \),

and \( 2\xi, 2\eta, 2\zeta \) are the dimensions of \( H \) in the coordinate system \((x,y,z)\) obtained from the original \((x',y',z')\) by the transformation \( x' = ax, y' = by, z' = cz \). Let \( 2x, 2y, 2z \) be the dimensions of \( H \) in the original system \((x',y',z')\). Thus \( a\xi = x, b\eta = y, c\zeta = z \).

In this way, we obtain

\[ P_n = (1 - e^{-\frac{\mu \nu K}{8a^2 y z}}) \Phi(\frac{x}{b}) \Phi(\frac{y}{b}) \Phi(\frac{z}{c}) \]

(56) \[ P_n = (1 - e^{-\frac{\mu \nu K}{8a^2 y z}}) \Phi(\frac{x}{b}) \Phi(\frac{y}{b}) \Phi(\frac{z}{c}) \]

To find the extrema of \( P_{n1} \) we solve the system of equations

\[ \frac{\partial P_n}{\partial x} = \frac{\partial P_n}{\partial y} = \frac{\partial P_n}{\partial z} = 0. \]

Calculating the partial derivatives, we obtain the following system:

\[ \Phi'(\frac{x}{a}) (1 - e^{-\frac{\mu \nu K}{8a^2 y z}}) - \frac{\mu \nu K}{8a^2 y z} e^{-\frac{\mu \nu K}{8a^2 y z}} \Phi(\frac{x}{a}) = 0, \]

\[ \Phi'(\frac{y}{b}) (1 - e^{-\frac{\mu \nu K}{8a^2 y z}}) - \frac{\mu \nu K}{8a^2 y z} e^{-\frac{\mu \nu K}{8a^2 y z}} \Phi(\frac{y}{b}) = 0, \]

\[ \Phi'(\frac{z}{c}) (1 - e^{-\frac{\mu \nu K}{8a^2 y z}}) - \frac{\mu \nu K}{8a^2 y z} e^{-\frac{\mu \nu K}{8a^2 y z}} \Phi(\frac{z}{c}) = 0. \]
\[
\Phi'(\frac{Y}{b}) (1-e^{\frac{\mu\nu n}{8xyz^2}}) - \frac{\mu\nu n}{8xyz^2} e^{\frac{\mu\nu n}{8xyz^2}} \Phi(\frac{Y}{b}) = 0 ,
\]

\[
\Phi'(\frac{Z}{c}) (1-e^{\frac{\mu\nu n}{8xyz^2}}) - \frac{\mu\nu n}{8xyz^2} e^{\frac{\mu\nu n}{8xyz^2}} \Phi(\frac{Z}{c}) = 0 ,
\]

whence the following:

\[
\frac{\Phi'(\frac{X}{a})}{\Phi'(\frac{X}{a})} = \frac{\Phi'(\frac{Y}{b})}{\Phi'(\frac{Y}{b})} = \frac{\Phi'(\frac{Z}{c})}{\Phi'(\frac{Z}{c})} = 1 - e^{\frac{\mu\nu n}{8xyz^2}} - \frac{\mu\nu n}{8xyz^2} e^{\frac{\mu\nu n}{8xyz^2}}
\]

From the last equations, it follows that

(57) \[ \frac{X}{a} = \frac{Y}{b} = \frac{Z}{c} = t \]

that is, the dimensions of the parallelepiped which we are trying to determine are proportional to the semi–axes of the ellipsoid of repeating errors. In this way, the system leads to a single equation

\[
\Phi'(t) (1-e^{\frac{\mu\nu n}{8abc^2 t^3}}) - \frac{\mu\nu n}{8abc^2 t^3} e^{\frac{\mu\nu n}{8abc^2 t^3}} \Phi'(t) = 0 ,
\]

which defines the function

\[ t = F(\frac{\mu\nu n}{8abc}) = F_3 (M) \]

since by formula (33) \[ \frac{\mu\nu n}{8abc} = \frac{\pi^{3/2}}{\rho_3^3} \] M. Hence we have found the value of t for which \( P_n \) attains its maximum value, that is, we have found the dimensions of the parallelepiped of which the uniform
bombardment guarantees the largest probability of at least one hit.

In a completely analogous fashion we find the following equation for the two-dimensional case:

\[ \Phi'(t) \left( 1 - e^{-\left( \frac{\mu \sqrt{n}}{4 \, abt^2} \right)} - \frac{\mu \sqrt{n}}{4 \, abt^3} \right) e^{-\left( \frac{\mu \sqrt{n}}{4 \, abt^2} \right)} \Phi(t) = 0, \]

defining the function \( t = F_2(M) \).

Finally, for the one-dimensional case we obtain the equation

\[ \Phi'(t) \left( 1 - e^{-\frac{\mu n}{2 \, at}} - \frac{\mu n}{2 \, at^2} e^{-\frac{\mu n}{at}} \right) \Phi(t) = 0, \]

which defines the function \( t = F_1(M) \).

Substituting the values obtained for \( t \) in formula (56), we obtain an expression for the probability of at least one hit for the most advantageous uniform bombardment of a parallelepiped:

\[ P_n = \left( 1 - e^{-\left( \frac{\mu \sqrt{n}}{8 \, abct^3} \right)} \right) \Phi(t)^3 \]

For the two-dimensional case the expression for \( P_n \) assumes the form

\[ P_n = \left( 1 - e^{-\left( \frac{\mu \sqrt{n}}{4 \, abt^2} \right)} \right) \Phi(t)^2, \]

and for the one-dimensional case

\[ P_n = \left( 1 - e^{-\frac{\mu n}{2at}} \right) \Phi(t). \]

In this way, knowing \( M \), it is possible to determine the value of \( t \) and, making appropriate substitutions in the last three formulas, one can find the value of \( P_n \). From these formulas, one can compute tables for \( t \) and \( P_n \) as they vary with \( M \).
It is also simple to compute the mathematical expectation $M''$ for the given method of artificial dispersion, namely, uniform bombardment of the region $H$. It is evident that, analogously to formula (42), we have

$$M'' = \sum_{i=1}^{n} \int \phi \psi_i \, dv = \int \phi \left( \sum_{i=1}^{n} \psi_i \right) \, dv.$$ 

Under the assumption that all the functions $\psi_i$ are small, we obtained the formula (54). From it follows that

$$\sum_{i=1}^{n} \psi_i = - \sum_{i=1}^{n} \log \left[ 1 - \psi_i \right] = -\log \sum_{i=1}^{n} \left[ 1 - \psi_i \right]$$

when, in view of (53) and (55),

$$\sum_{i=1}^{n} \psi_i = \begin{cases} \frac{\mu \nu \kappa}{8 \alpha \beta \gamma \delta \eta \sigma} & \text{for } (x, y, z) \in H \\ 0 & \text{for } (x, y, z) \notin H. \end{cases}$$

Hence, the integration of $\phi \sum_{i=1}^{n} \psi_i$ is extended only over $H$, and $\frac{1}{n} \sum_{i=1}^{n} \psi_i$, as a constant, can be removed from under the integral sign. We shall have

$$M'' = \frac{\mu \nu \kappa}{8 \alpha \beta \gamma \delta \eta \sigma} \left( \frac{\frac{\alpha \xi}{a} + \frac{\gamma \eta}{b} + \frac{\eta \xi}{c}}{\frac{\alpha \xi}{a} + \frac{\gamma \eta}{b} + \frac{\eta \xi}{c}} - e^{-\left( \frac{\alpha \xi}{a} + \frac{\gamma \eta}{b} + \frac{\eta \xi}{c} \right)} \right) \int_{H} \int_{H} \int_{H} e^{-\left( \frac{\alpha \xi}{a} + \frac{\gamma \eta}{b} + \frac{\eta \xi}{c} \right)} \, dz,$$

whence $M'' = \frac{\mu \nu \kappa}{8 \alpha \beta \gamma \delta \eta \sigma} \left( \frac{\frac{\alpha \xi}{a} + \frac{\gamma \eta}{b} + \frac{\eta \xi}{c}}{\frac{\alpha \xi}{a} + \frac{\gamma \eta}{b} + \frac{\eta \xi}{c}} - e^{-\left( \frac{\alpha \xi}{a} + \frac{\gamma \eta}{b} + \frac{\eta \xi}{c} \right)} \right).$

Keeping in mind that

$$t = \frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{\alpha \xi}{a} = \frac{\gamma \eta}{b} = \frac{\eta \xi}{c},$$
we obtain

\[ M'' = \frac{\mu 2kn}{8abc t^3} \left[ \Phi(t) \right]^3. \]

For the two-dimensional case, as in (58), we have

\[ \sum_{i=1}^{n} \psi_i = \begin{cases} \frac{\mu y' n}{4ab \delta n} & \text{for } (x, y) \in H, \\ 0 & \text{for } (x, y) \in H, \end{cases} \]

whence, as in (59), after certain reductions

\[ M'' = \frac{\mu y' n}{4ab t^2} \left[ \Phi(t) \right]^2. \]

In like manner, we find the expression for \( M'' \) in the one-dimensional case to be

\[ M'' = \frac{\mu n}{2} \Phi(t). \]

From the formulas which have been obtained, it is seen that \( M'' \) is expressible as a function of \( M \), the mathematical expectation of hits in the absence of any artificial dispersion.

10. Remark on the applicability of the method.

The method expounded in the present article was based on the following principal hypotheses:

1. The probability of hitting the target with a single shot aimed at the center of the target is not large. For fulfillment of this condition the target's dimensions must be limited.
However, in case the dimensions of the target slightly exceed the natural dispersion, the problem becomes trivial from the point of view of obtaining at least one hit.

2. The dimensions of the region throughout which the artificial dispersion is to be made are significantly larger than the natural dispersion. This condition is often fulfilled in practice, especially in anti-aircraft firing. In those cases where this condition is not fulfilled, artificial dispersion is for the most part disadvantageous.

Application of the method expounded above need not be limited to the case where errors are distributed according to the Gaussian law. When the given conditions are fulfilled, the method is applicable in its entirety to an arbitrary distribution law of errors and to an arbitrary number of dimensions.
APENDIX

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**TABLE II**

\[ H(m-a) \cdot H(m-1, a) - 2H(m-a, a) \]

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**Note:** The table continues with similar entries for each row.
\[ \tau(u) = -\frac{1}{2} \int_{-\infty}^{x} \ln \left(1 - \frac{1}{2} \left| \Phi(x + u) \right| - \sigma(x - u) \right) \, dx \]

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### Table IV

\[ P(N) = \Phi(s_x) - 2s_x \Phi(s_x) \]

\[ s_x(N) = \frac{1}{
\left( \frac{3\sqrt{\pi}}{4} \right)^{1/N^2} \right. \]

\[ d^2(N) = \frac{1}{5} s_x^2 \]

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ПРИЛОЖЕНИЕ

Figure I
graph of the function

\[ H(m, a) = 1 - e^{-\left(1 + \frac{a}{1} + \frac{a^2}{2!} + \cdots + \frac{a^{m-1}}{(m-1)!}\right)} \]

для \( m = 1, 2, \ldots, 10 \)
Figure II
Dependence of $R$ on $a$ for $k = 1/2$, $c = 4/15$
(firing with two aimings)
Figure III
Dependence of \( R \) on \( a \) for \( k = 1/2, \ c = 4/15 \)
(firing with three aimings)