The Benefits of Fractionation in Competitive Resource Allocation

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Abstract

We leverage a new algorithm for numerically solving Colonel Blotto games to gain insight into a version of the game where players have different types of resources. Specifically, the winner of a battlefield is a function of a multi-dimensional allocation vector of each player. Our main focus is on the potential benefits of fractionation, which we define as the degree to which a player can quantize its resources. When players only have one type of resource, we show that the benefits to fractionation are in general, greatest in resource poor environments and against aggregated adversaries. We then extend the model to include random dropout and show that fractionation increases robustness to failure in resource poor environments but not resource rich environments. Finally, we show that when players have different types of resources, the benefits of fractionation are no longer mitigated by an increase in the total force size. Since many real-world resource allocation problems are multi-dimensional, our results illustrate the importance of analyzing multi-resource Blotto games in tandem with the traditional specification.

1 Background

Competitive resource allocation problems are ubiquitous. The canonical example is the case of two military colonels that must allocate a limited number of troops across different battlefields. The colonel that allocates more troops to a battlefield wins the battlefield and the goal of each colonel is to win as many battlefields

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as possible. This now century-old abstraction of a competitive resource scenario is known as the Colonel Blotto Game [1, 2].

Despite its original militaristic framing, the Colonel Blotto game has found wide application in both the social science and engineering communities. For example, social scientists have used Blotto games to understand political campaigning strategies [3], research and development investments [4] and marketing strategies [5]. The engineering community has adopted Blotto games to model computer network attacks [6, 7], communication network formation [8] and infrastructure resilience [9].

Although the framing of the Blotto game is intuitive, simple perturbations to the basic game often pose significant analytical challenges. For example, games where players have different numbers of resources [1] or the battlefields have different values [10] have been solved only relatively recently. One important perturbation that has received scant attention is the case where players have different *types* of resources. This perturbation of the Blotto game is as ubiquitous as the standard formalism. For example, military commanders must allocate different resources such as people and equipment, firms must decide how to allocate different types of employees to different tasks, advertisers must allocate different product placement mechanisms (commercials and endorsements, for example) over time and space, and political campaigners must determine how to allocate funds, volunteers and managers. Despite its wide prevalence, there has yet to be a systematic study of such a resource allocation scenario.

To address this gap, we analyze what has been called a multi-resource Blotto game [11]. In the game, the players allocate *platforms* across a set of *battlefields*. Each platform has a set of *capabilities*. Generally, a platform may have more than one unit of each capability. Each unit of a capability carried by a platform is called a *resource*. The winner of a battlefield is a function of the joint allocation of resources and capabilities to a battlefield and the players' goal is to win as many battlefields as possible.

Consider, for example, the scenario where two aerial combat colonels must allocate their resources among 3 different battlefields. One colonel has a fleet of 10 large aircraft, each with 5 missiles per aircraft. Another colonel has 50 small autonomous aerial vehicles (drones), each with one missile per drone. The winner of the battlefield is the colonel that allocates more total missiles to the battlefield. In this case, the platforms are the aircraft, the capabilities are the missiles and the number of resources are 5 per aircraft and 1 per drone.

However, the multi-resource Blotto game is more general than simply capturing aggregation of resources on platforms. For a more complex example, consider two colonels that now must allocate their resources between two theaters. Specifically, each colonel must allocate 10 battleships and 15 airplanes. Each battleship has 5 missiles and is equipped with radar. Each airplane has 2 missiles and is likewise equipped with radar, but in addition carries a camera to conduct surveillance. In this case the theaters are the battlefields. Battleships and airplanes are the platforms. The capabilities are missiles, radar and camera. The resources for each battleship are five missiles and one radar while the resources for each airplane are two missiles, one radar and one camera. The winner of the theater is some function of the joint allocation of missiles, radars, and cameras. For example, the function might be "the player that allocates more missiles to a theater, provided they allocate at least two radars and one camera, is the winner."

Applications of the multi-resource Blotto games are not confined to the military domain. For example, consider two political parties that are lobbying for and against a particular bill in several counties. Each party seeks to have their side of the bill supported in as many counties as possible. To sway voters, the party allocates orators, pamphlet distributors and journalists among the counties. However, each "platform" (orator, pamphlet distributors and journalists) affects voters in each county differently and the winner of each county is some function of the joint allocation of orators, pamphlets and journalists.

In this work, we use the general multi-resource Blotto game to determine the potential benefits of having resources spread across many platforms versus having resources aggregated on fewer platforms. Specifically, we examine the equilibrium payoffs as a function of the platforms' degree of fractionation. We "overload" the term fractionation to take on different meanings depending on the context. In the case where each force only has one capability, we define *fractionation* as the number of resources per platform (the number of missiles per aircraft, in the example above). In the case of multiple capabilities, we define the degree of fractionation as the number of number of unique capabilities per platform. We refer to this second type of fractionation as *fractionation of heterogenous resources* (FHR). As shorthand, we refer to the force that is more fractionated (fewer resources and capabilities per platform) as the fractionated force and the force that is less fractionated as the aggregated force.

We vary the key parameters — the number of battlefields and total number of resources — to fully characterize the potential benefit of having a fractionated force. To limit the scope, we examine how the most fractionated force performs against varying types of aggregated forces. Specifically, we first consider the case where there is only one capability and determine how a force with one resource per platform (fractionated) performs against a force with several resources per platform (aggregated). We augment the model to include the case where platforms are imperfect and subject to error. We then analyze the potential benefits of FHR in scenarios with multiple capabilities. That is, we examine whether it is better to have several different capabilities aggregated onto one platform or several platforms each with a unique capability.

Of course, we do not cover the complete parameter space of the very general multi-resource Blotto game. For example, we do not consider the case where forces have mixed degrees of fractionation. In reference to the aerial example, this means we don't consider the case where forces have large aircraft *and* drones. Furthermore, we do not consider the case where both forces' platforms have multiple capabilities but a different number of resources per platform. Again, in reference to the aerial example, this means we do not consider the case of a force that has platforms with 3 missiles and 3 cameras per platform against a force that has 10 missiles and 10 cameras per platform. While these are interesting extensions, our goal is to specifically disentangle the benefits of fractionation and therefore, we simplify the experiments in a way that best highlights the impacts of fractionation and eliminates other confounding factors.

The enabler of this work is the recently developed algorithm for solving multi-resource Blotto games [11]. The key to the algorithm is to transform the zero sum game into a flow problem on a layered graph. The authors show that with this representation, the game can be solved with $O(N^{2c}K)$ constraints where N is the number of platforms, c is the number of different platform types and K is the number of battlefields. This allows us to solve for equilibrium payoffs in games where players have up to 50 platforms and there are up to 9 battlefields. Thus our numerical results cover enough of the parameter space to draw robust conclusions regarding the benefits of fractionation.

In addition to the vast literature on Blotto games and its variants [1, 2, 12, 13, 14, 15, 16] our work is related to the work on multi-activity contests [17, 18]. In multi-activity contests, players contribute costly resources in order to win a contest. The winner of the contest is a stochastic function of the players' joint contributions. While this is similar to the multi-resource Blotto game, it has two main differences. First, and most crucially, multi-activity contests often only have one battlefield and therefore players do not have to allocate resources across different spaces. Secondly, multi-activity contests usually assume players' contributions are continuous, which precludes an analysis of fractionation since continuous pure strategy spaces are infinitely divisible for both players.

2 Model

In the model, there are two players: a *fractionated* force denoted by the subscript "f," and an *aggregated* force which we denote the subscript "a." In general, players must allocate *platforms* that have a number of *resources* of various *capabilities.*

Let $\mathcal{C} = \{c_1, c_2 ... c_M\}$ be the set of possible capabilities in the game. Let $\mathcal{P}_i = \{p_{i1}, p_{i2} ... p_{iK_i}\}$ be the set of platforms available to player i. Note that the number of platforms K_i depends on the player and players are not, in general, symmetric. Each platform p_{ij} is an m dimensional vector of integers where the k'th element of p_{ij} corresponds to the number of resources of capability k on the jth platform of player i. For example, if $p_{ij} = \{2, 1, 1\}$, then the j'th platform of player i has two resources of c_1, and 1 resource each of c_2 and c_3. This would be the case if the platform was the airplane as described in section 1. In some cases, it may be convenient to represent player i's platforms as a tuple of a vector and a matrix, $(\mathbf{v_i}, \mathbf{P_i})$, where each row of $\mathbf{P_i}$ represents the resources and capabilities of a platform and v_i represents the number of platforms. For example the values $\mathbf{v_i} \begin{bmatrix} 15 \\ 10 \end{bmatrix}$ and $\mathbf{P_i} = \begin{bmatrix} 2 & 1 & 1 \\ 5 & 1 & 0 \end{bmatrix}$, represent the battleship and airplane scenario described in section 1. A specific quantity of interest in later sections if $\mathbf{v_i}^T \mathbf{P_i}$ which gives a vector that denotes the total number of resources of each capability for player i. We refer to the tuple of $(\mathbf{v_i}, \mathbf{P_i})$ as player i's *force*.

There are T battlefields. A player's pure strategy is a mapping $S_i : \mathcal{P}_i \rightarrow \mathbb{N}^{K_i}_{n<T}$ that assigns each of the K_i platforms to one of T battlefields. A mixed strategy σ_i is a distribution over all possible S_i and the set of mixed strategies, ΔS_i is the set of all possible distributions over S_i. For a given strategy S_i, let $A_t^{S_i}$ be an M dimensional vector such that the j'th element in $A_t^{S_i}$ denotes how many resources of capability j player i allocated to battlefield t.

Given the joint allocations, player i's payoff is given by

$$W_i(S_i, S_{-i}) = \sum_{t=1}^{T} U_{it}(A_t^{S_i}, A_t^{S_{-i}}) \tag{1}$$

where U_t is a function for battlefield t that maps the joint allocation of resources to a payoff to the battlefield. For example, in the simple Blotto game, U_t gives a payoff of 1 if i allocates more resources that $-i$ to battlefield t, 0 if the resource allocation are the same and -1 otherwise.. Since we are restricted to zero-sum games, $U_{-it} = -U_{it}$. A Nash equilibrium is then a strategy profile where both players are maximizing the expectation of W, where the expectation is taken over the distributions of the mixed strategy.

3 Results

In this section, we illustrate the benefits of fractionation. First, we examine the simplest case where players only have to allocate one capability. Then, we illustrate how the potential benefits change when platforms are subject to random error. Finally we examine the case with multiple capabilities and heterogeneous battlefields and characterize the benefits of fractionation of capabilities.

3.1 Environment with a Single Capability

We begin by characterizing the benefits of fractionation when there is only one capability. Importantly, we want to ensure that the results examine only the benefits of fractionation and not the potential benefits from an asymmetric number of resources. To achieve this, we only consider cases when both players have the same number of resources of the capability but the resources may be aggregated differently across platforms.

As a simple example, consider the scenario in Figure 1. The fractionated force has 4 platforms, each with

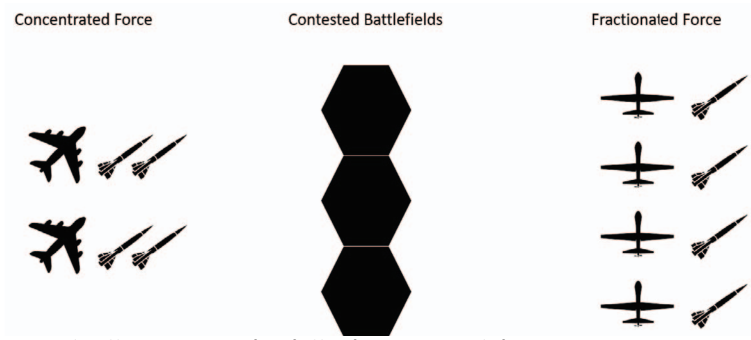

Figure 1: A simple illustration of a fully fractionated force against an aggregated force.

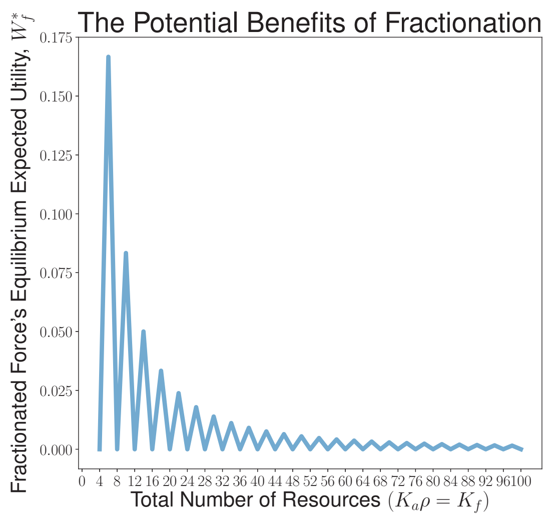

Figure 2: The potential benefits of fractionation as a function of force size. In the plot , $\rho = 2$ and $T = 4$.

one resource of a single capability type, while the aggregated force has 2 platforms, each with 2 resources of the same single capability type. In an aerial context this might mean that the fractionated force has 4 airplanes, each with one missile, while the aggregated force might have 2 airplanes, each with 2 missiles.

In all of the experiments in this section, we assume that the fractionated force has one resource per platform. The parameter ρ sets the number of resources per platform for the aggregated force. Recalling that K_i is the number of platforms belonging to player i, it must be that $K_f = K_a\rho$ in order to ensure that the aggregated and fractionated forces have the same number of resources. Of course, since the game is zero sum, the fractionated force's benefits to being fractionated is the negative of the aggregated force's cost to being aggregated so it is not necessary to consider the case where $\rho < 1$. In this model, we restrict our analysis to integer values of ρ.

In our analysis, we will first describe the empirical insights gleaned from the computational experiments and then we will discuss the mechanisms that lead to such results. The reason for this approach is to first fully understand which patterns generalize to all parameter sets and then, with a firm grasp on the trends, determine why such trends exist.

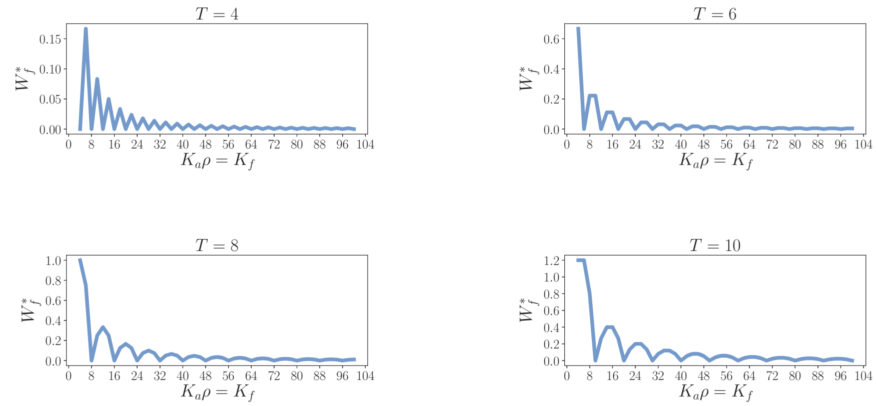

Figure 3: The fractionated force's expected utility for various number of battlefields as a function of force size. In the plot, $\rho = 2$

3.1.1 Descriptive Results

As an initial exposition, figure 2 shows the fractionated force's equilibrium expected utility when the aggregated force has 2 resources per platform as a function of the total number of resources.

First, we notice that the fractionated force's expected utility is never less than 0. However, this is not surprising because the fractionated force could always "mimic" a more aggregated force by coupling together two platforms and earn an expected utility of 0. In other words, if both forces had two resources per platform, both the fractionated and aggregated forces' expected utility would be 0 since the game would be a symmetric zero-sum game. Therefore, when one force is more fractionated than the other, at the very worst, it can always be allocated in sets to earn an expected utility of 0. Therefore, the fractionated force should never earns an expected utility less than 0.

There are other and arguably more surprising features that emerge. Specifically, the fractionated force's expected utility is a) decaying and b) periodic. This behavior is not unique to the case where $T = 4$. Figure 3 holds ρ constant at 2 and shows that decaying periodicity appears when $T = 6, 8$ and 10. Examining the case when $\rho = 2$ and T is odd demonstrates a similar pattern but the relationship is more complex as the periodicity is less obvious as shown in figure 4. Finally, the dampening periodicity is also present when $\rho = 3$ and T is both even and odd, as shown in figure 5.

By examining the relative heights of plots for different values of T, it is evident that the potential benefits to fractionation increase as the number of battlefields increase. This is further illustrated in figure 6, which plots the fractionated force's expected utility for various force sizes, with a rolling average across force sizes, to remove the periodic behavior.

Another feature of the plots is that the fractionated force's expected utility is *not* normalized by T. It is reasonable to assume that the benefits to fractionation would increase as T increases simply because the

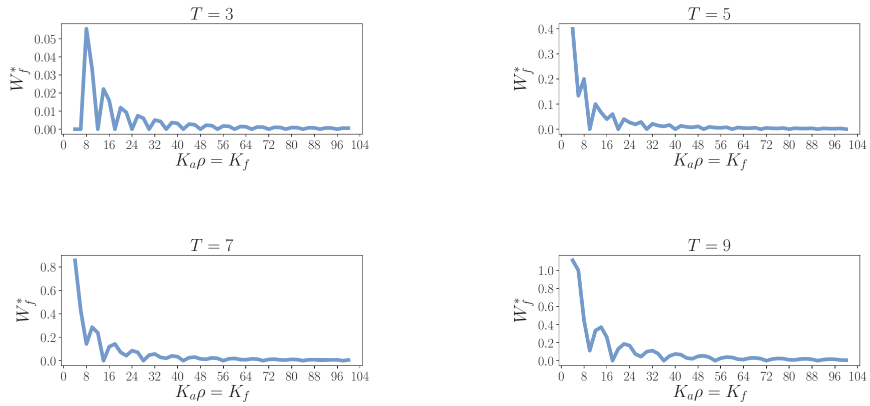

Figure 4: The fractionated force's expected utility for various number of battlefields as a function of force size. In the plot, $\rho = 2$

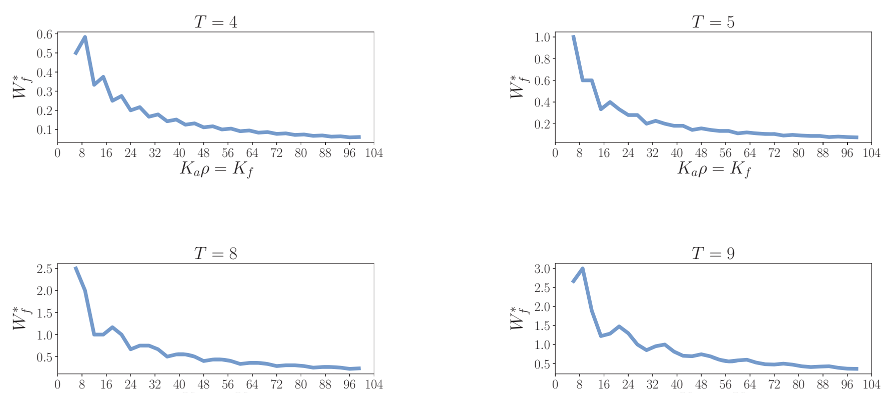

Figure 5: The fractionated force's expected utility for various number of battlefields as a function of force size. In the plot, $\rho = 3$. The plot includes examples with both an even number of battlefields and an odd number of battlefields.

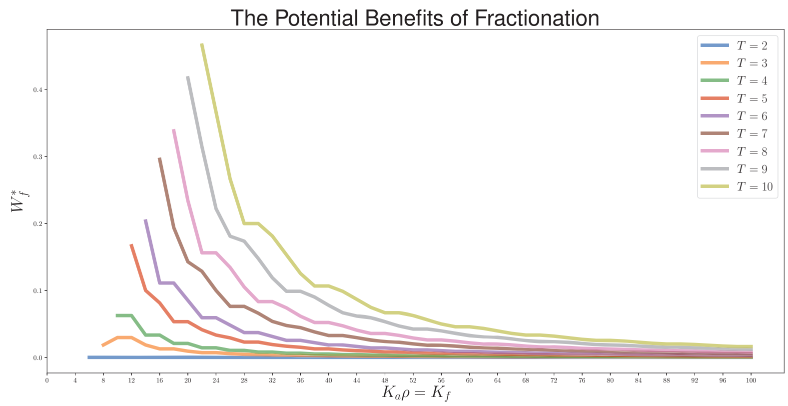

Figure 6: The fractionated force's expected utility for various number of battlefields as a function of force size. In the plot, $\rho = 3$. The plot includes examples with both an even number of battlefields and an odd number of battlefields.

total possible utility increases. In other words, with $T = 3$, the best the fractionated force could do is earn 3 units of utility whereas if $T = 9$, the most fractionation could earn is 9 units of utility. However, we are interested in comparing how well the fractionated force does *relative* to an aggregated force, in which case both players would earn an expected utility of 0. Therefore, we plot the unnormalized expected utilities. For comparison, figure 7 plots the fractionated force's expected utility normalized by T. The plot shows that in some cases increasing T has no effect on the fractionated force's normalized expected utility and in the remainder of the cases, increasing T does improve normalized expected utility. In summary, the fractionated force's unnormalized utility is strictly increasing in T whereas the normalized utility is only weakly increasing in T.

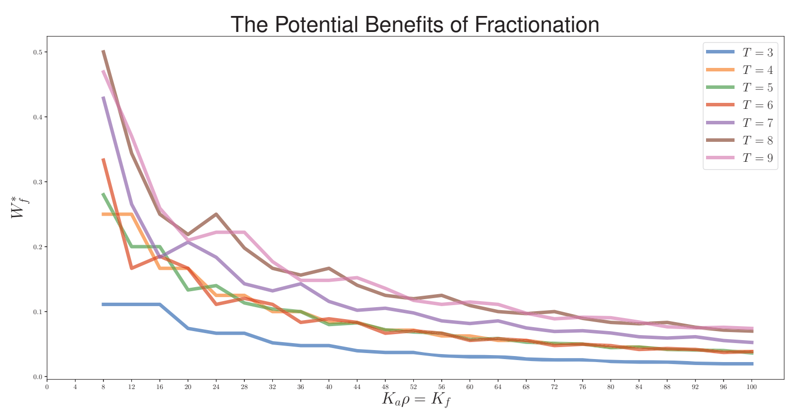

Figure 7: The fractionated force's normalized expected utility for various number of battlefields as a function of force size. In the plot, $\rho = 4$.

Finally, we examine the benefits of fractionation as a function of ρ, the relative degree of fractionation. As figure 8 shows, the benefits of fraction increase in ρ for all values of T.

In summary, the empirical results yield four robust patterns. First, the benefits of fractionation decrease

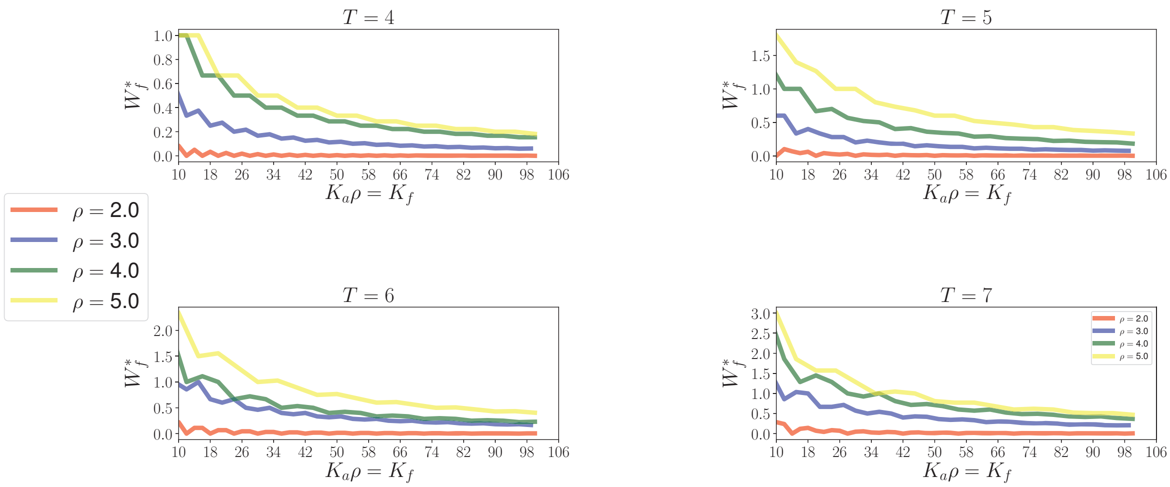

Figure 8: The fractionated force's expected utility for various number of battlefields and various values of ρ. It shows the benefits of fractionation increase with ρ for each value of T.

in the total number of resources. This is shown in figures 3, 4 and 5. The benefits to fractionation at least weakly increase in the number of battlefields as shown in figure 6 and how aggregated the opposing force is, as shown in figure 8. Finally, the benefits to fractionation are periodic in the size of the forces.

3.1.2 Mechanism driving Decaying Utility, varying T and ρ

To better understand these insights, it is imperative to determine the mechanisms behind such behavior. It turns out, the mechanism driving the decaying utility is the same as the mechanism behind the benefits of fractionation as a function of T and ρ. The mechanism driving the periodicity is different and will be discussed later.

The key to understanding the mechanism behind decaying utility is that the benefits of fractionation only depend on the probability that for a given strategy for the fractionated force, the aggregated force's allocation of resources to a battlefield fall within the number of resources it has per platform, ρ. For example, if the aggregated force has 2 resources per platform, for a fixed (mixed) strategy for the fractionated force, the aggregated force is only affected by its inability to fractionate if its allocation is within two resources of the fractionated force's allocation. That is, if the aggregated force loses a battlefield by 3 or more resources, its inability to fractionate did not impact the outcome of that battlefield. This is because the aggregated force could have allocated another platform (of two resources) to the battlefield and it still would have lost the battlefield. Similarly, if the aggregated force wins a battlefield by more than 3 resources, its inability to fractionate is not the source of the excessive allocation because the aggregated force could have allocated one platform away from the battlefield and still won the battlefield. Therefore, any impact of the aggregated forces ability to fractionate on its expected utility is when its allocation is "close to" that of the fractionated force.

With this intuition in hand, it becomes clear that the benefits of fractionation decrease in the size of the force because as forces grow, the probability that the aggregated force's allocation is within ρ of the fractionated force's decreases. Or in other words, the aggregated force's ability to fine-tune its allocation is only relevant when it should expect its allocation to be "close to" the fractionated force's. Otherwise, the aggregated force's utility is not greatly impacted by its inability to fractionate but simply a classic ex-post misallocation.

Considering the case where $\rho = 2$, figure 9 examines this systematically by computing the fractionated force's average equilibrium probability mass on a 5-resource window.[1] As the figure shows, the average probability in any $5 - resource$ window decreases as the fractionated force's total number of resources grow. This implies that for any allocation of the aggregated force, the probability that it's allocation is within 2 of the fractionated force's decreases and therefore the effects of fractionation decrease. This can also be observed by examining the distribution of fractionated force's resources for a given battlefield. For example, figure 10 shows the fractionated force's equilibrium allocation cumulative density function (CDF) for a given battlefield for two values of K_f. Clearly, the CDF is smoother and covers a wider range when K_f is higher, implying that the probability that the aggregated force's allocation is within two of the fractionated force's is less.

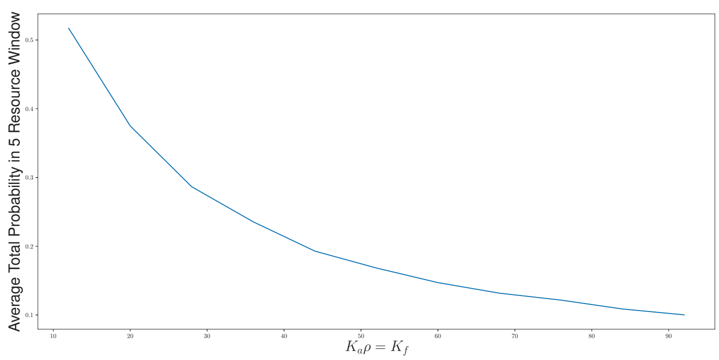

Figure 9: Average probability that for a given allocation of the aggregated force and fixed mixed strategy of the fractionated force, the aggregated force's allocation is within 2 of the fractionated force's allocation. In the figure, $\rho = 2$ and $T = 4$.

The same logic holds when ρ and T increase. Specifically, when ρ increases, the probability that the aggregated force's allocation is within ρ of the fractionated force's allocation increases and thus the benefit

[1]Specifically, consider the case where the fractionated force has 50 total resources. Then from the equilibrium strategy we calculate $\frac{1}{23}\sum_{i=2}^{25}\sum_{j=-2}^{2} P(A_1 = i + j)$. That number, which is approximately 1.8, is plotted on the y-axis of figure 9 at an x-axis value of 50. Repeat the calculation for all values of total resources to trace out the curve. We exclude from the average any values over $K_f/2$ as the fractionated force always places 0 probability on such allocations. This is why the upper limit on the sum in the example is 25 and not 50.

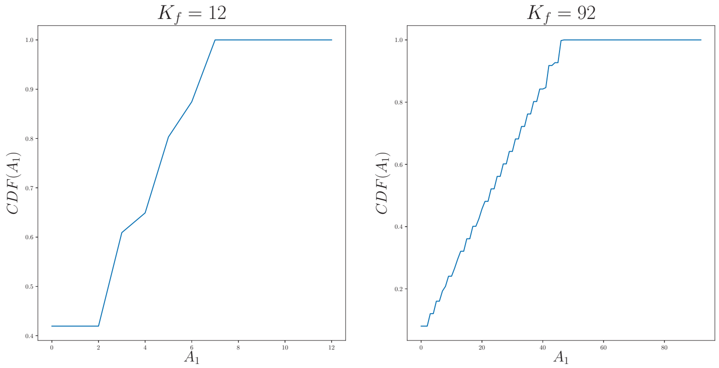

Figure 10: Cumulative Distribution of the equilibrium total number of resources the fractionated force allocates to battlefield 1. In the figure $\rho = 2$ and $T = 4$.

of fractionation increase. In other words, the aggregated force's ability to "fine tune" its allocations subject to the fractionated force's strategy decreases and thus the fractionated force's expected utility increases. Finally, when the number of battlefields increase but the total number of resources stays constant, both players must allocate their resources over more battlefields and thus, allocate fewer resources per battlefield. This reduced range of allocations implies that the probability that the aggregated force's allocation is within ρ of the fractionated force's increases and thus there is a greater benefit to having a fractionated force.

3.1.3 Mechanism Driving Periodic Utility

Understanding the mechanisms giving rise to periodicity is slightly more convoluted though not completely opaque. First, figures 2 — 5 show that the period length of the fractionated force's expected utility is equal to $\frac{\rho T}{2}$ when T is even and ρT when T is odd. This is equivalent to saying that when the fractionated force's equilibrium expected utility is plotted against the number of the aggregated force's platforms, the period length is either T or $T/2$ depending on the parity of T. Crucially, the minima occur at integer multiples of ρT

To provide some intuition of this result, consider the case where there are 5 battlefields and $\rho = 3$. In all cases, both players play a mixed strategy. Figure 11 shows the equilibrium *marginal* probability of the aggregated force's equilibrium allocation when it has 15 platforms (or 45 total resources). Like in the traditional Blotto game, the aggregated force's equilibrium strategy has the property of uniform marginal distributions for all battlefields [19, 12]. So, for example, the marginal probability that the aggregated force allocates 3 platforms to battlefield 1 is the same as the probability that it allocates 0 platforms to battlefield 1 which is the same as the probability it allocates 2 platforms to battlefield 2.

However, when the aggregated force's resources are not a multiple of ρT (or $\frac{\rho T}{2}$ when T is even), equilibrium marginal allocation probabilities are not exactly uniform as shown in figure 12. Thus fractionation

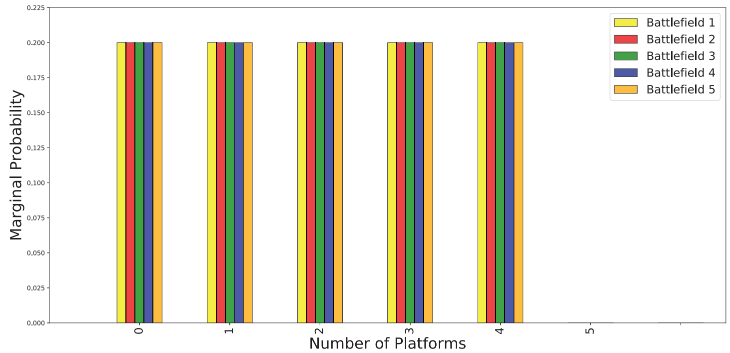

Figure 11: The aggregated force's equilibrium marginal distribution for the number of platforms allocated to each battlefield. In this case, there are 5 battlefields with $\rho = 3$ and $K_a = 10$

causes the aggregated player to insert asymmetries into its equilibrium strategy and thus, the fractionated force can exploit those asymmetries to gain an advantage.

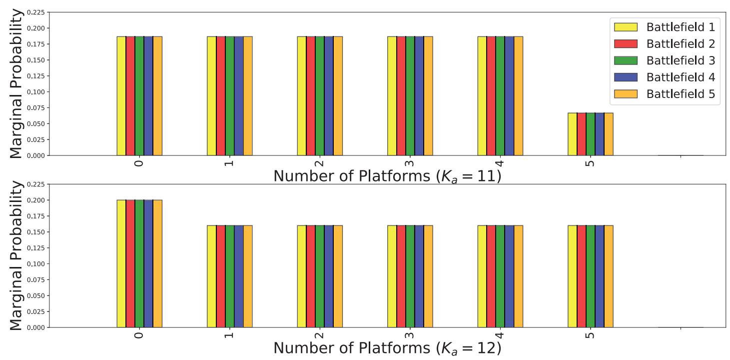

Figure 12: The aggregated force's equilibrium marginal distribution for the number of platforms allocated to each battlefield. In this case, there are 5 battlefields with $\rho = 3$ and $K_a = 11$ (top)or $K_a = 12$ (bottom). The aggregated force's equilibrium strategy strays from uniform marginals.

To understand the driver of the aggregated force's non-uniform marginal distribution, recall that from the simple Blotto game with symmetric forces, the equilibrium strategy is for players to have uniform marginal distributions on the interval $[0, \lfloor \frac{2x}{T} \rfloor]$ where x is the number of troops and T is the number of battlefields. Since the fractionated force only has one resource per platform, it can randomize uniformly on that interval. However, this is not always possible for the aggregated force. For example if $\rho = 4$, $T = 5$ and $\rho K_a = K_f = 24$, then $2\frac{K_f}{5} = 9.6$ and the aggregated player *cannot* place positive probability on allocating 9 resources to a target because its allocations must be in multiples of 4. This inability to randomize on the same interval as the fractionated force causes the aggregated force to introduce an asymmetry into its strategy, which is then exploited by the fractionated force.

3.2 Fractionation and Random Platform Failure

We now consider the case when each of the aggregated and fractionated force's platforms are subject to random and independent failure at the platform level. Drop outs may represent a number of phenomena in a system of interest. In a warfare context, an assigned unit may be unable to reach the assigned battlefield by some necessary time due to transportation breakdowns, navigation error, or interference. In computer network applications, drop out may reflect equipment failures or outages. Political or marketing campaign resources may be reallocated due to opportunity cost. For example, a commercial may be canceled at the last minute because the firm needs to funds a new project.

To formally represent independent failure, let q_i be the probability that player i's platform fails and thus is not counted toward the player's allocation toward the battlefield nor any other battlefield. If the fractionated and aggregated forces allocate n_f and n_a platforms to battlefield t, then the fractionated force's expected utility accrued at battlefield t is given by:

$$U_{f,t}(n_f, n_a) = P\left\{n_f - B(n_f, q_f) > n_a - B(n_a, q_a)\right\} - P\left\{n_f - B(n_f, q_f) < n_a - B(n_a, q_a)\right\} \tag{2}$$

where $B(x, y)$ is a binomial random variable with x trials and success probability y. In other words, the contribution to the fractionated force's expected utility from a specific battlefield *given a joint allocation* is the probability that the fractionated player allocates more than the aggregated player to the battlefield minus the probability that the aggregated player allocates more than the fractionated player. Of course, when considering mixed strategies the expectation is over the aggregated and fractionated player's strategy as well as the distribution that governs the random failure process.

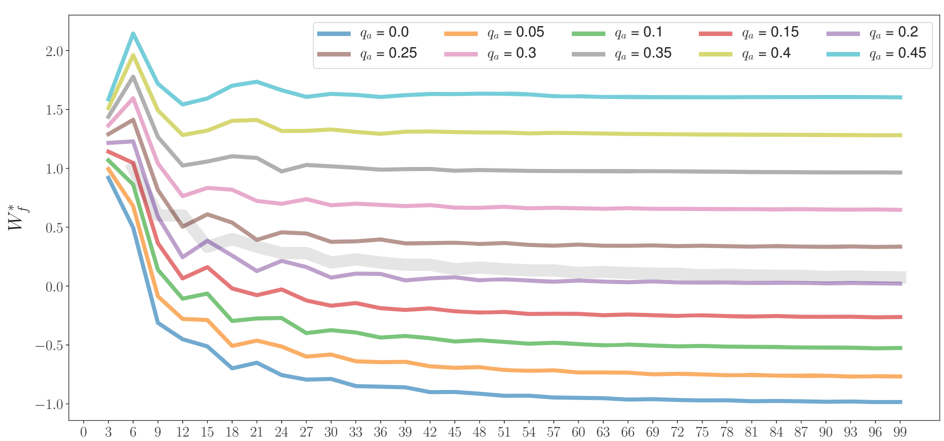

Figure 13: The fractionated force's equilibrium expected utility when forces are subject to random error. In the figure, $T = 5$, $\rho = 3$, $q_f = .2$.

Figure 13 plots the fractionated force's expected utility when it's failure probability is .2, the aggregated force has 3 resources per platform, and there are 5 battlefields. Each line represents a different platform failure probability for the aggregated force. The thick gray line is included for comparison and represents the fractionated player's expected utility when neither platform is subject to failure.

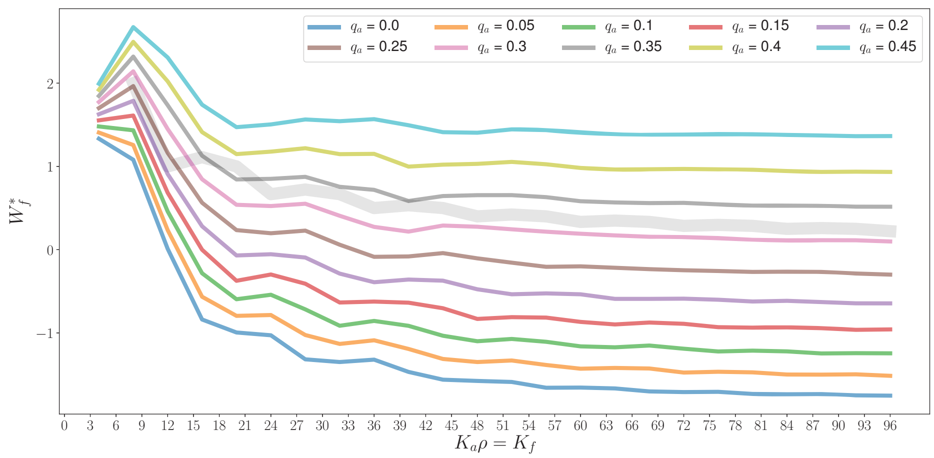

Figure 14: The fractionated force's equilibrium expected utility when forces are subject to random error. In the figure, $T = 6$, $\rho = 4$, $q_f = .3$. The expected utility when $q_a = .3$ is slowly converging to 0.

As the figure shows, when the total number of resources is small, The fractionated force may have an advantage over the aggregated force, even when it's probability of failure is higher than the aggregated force's. This is represented by the bottom four lines (blue, orange green and red), which show positive utility for fractionation when the total number of resources is small (less than 20). However, as the force size (i.e. number of resources) gets larger, the benefits to being fractionated disappear. In other words, only when $q_a = q_f$ are the expected force sizes of the aggregated and fractionated forces the same. If $q_a < q_f$ then the expected force size for the aggregate force is higher than the expected force size for the fractionated force. As the results show, for large enough force sizes, the aggregated force's advantage due to having a larger expected force size outweighs the disadvantage it faces from having an aggregated force. This result generalizes to other parameterizations as shown in figure 14.

Finally, figure 15 shows that when the joint error rate increases, the fractionated force's expected utility still converges to 0 but does so much slower than for lower error rates. This is because increasing the error rate decreases each player's *expected* number of resources allocated for a fixed force size and effectively increases the scarcity of resources that do not fail.

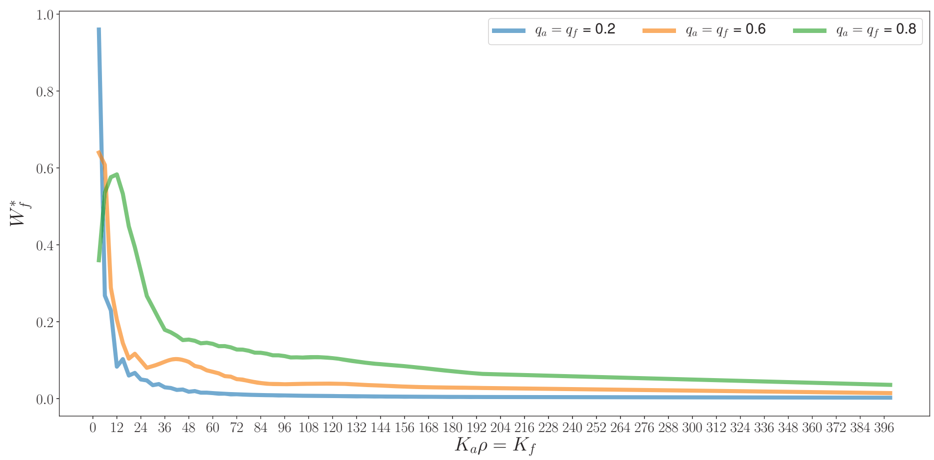

Figure 15: The fractionated force's equilibrium expected utility when forces are subject to random error. In the figure, $T = 3$, $\rho = 3$. For $K_f < 198$, we compute the equilibrium value of the fractionated force's expected utility for all values of K_f such that $\rho K_a = K_f$. We then compute the fractionated force's expected utility at $K_a\rho = K_f = 399$ to demonstrate convergence. In other words, from 198 to 399 on the x-axis is a linear interpolation.

3.3 Fractionation of Capabilities

In this section, we investigate the potential benefits of fractionation of capabilities when the battlefields are heterogeneous. That is, the function that determines the winner of a battlefield is different for each battlefield, and may not respond equally to all available capabilities. We define a force that is fractionated in capabilities as having only one capability per platform, and consider a game with only two different capabilities, c_1 and c_2. We assume that the aggregated force has K_a platforms, each with one resource of c_1 and one resource of c_2. The fractionated force [2] has $K_f = 2K_a$ platforms where $\frac{K_f}{2}$ platforms have one resource of c_1 and the other $\frac{K_f}{2}$ platform has one resource of c_2. The aggregated and fractionated forces have the same total number of resources with the only difference being that the aggregated player's resources are coupled on platforms where each of the fractionated force's resources are isolated on platforms.

Our setup ensures the results capture the benefits of fractionation of capabilities and not the fractionation of resources, as was covered in the last two chapters. The difference is that fractionation of capabilities results in platforms that are specialized , where our aggregated force is composed of platforms that carry both of the desired capabilities. In other words, we explicitly don't consider the case where one player has 30 platforms, each with one resource of two distinct capabilities and another player has 15 platforms, each with two resources of two distinct capabilities. In such a case, both players have the same number of capabilities per platform (both forces are similarly composed of universal platforms, rather than specialized ones). These

[2] We abuse terminology in this section and refer to the force with fewer capabilities per platform as the fractionated force, whereas in the last section, fractionation was used to describe forces with fewer resources per platform.

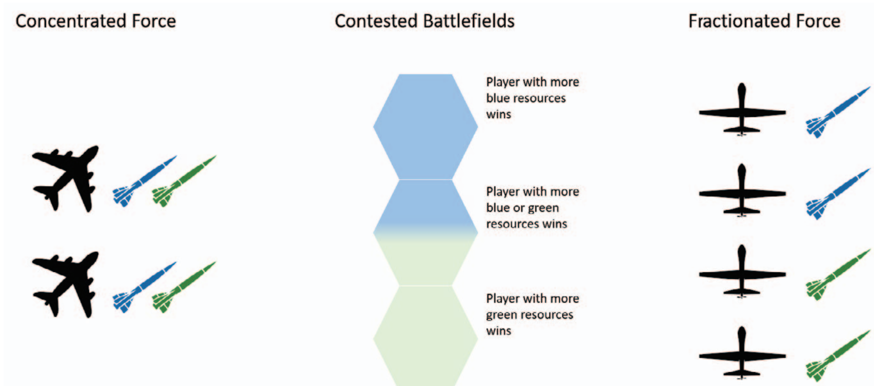

Figure 16: In this pictorial representation, each player has 2 resources of each capability indicated by the blue and green colored missiles. However, the player on the left has 2 platforms with one missile of each color, while the player on the right has 4 platforms, each with a single missile. The player that allocates more of blue missiles to the top (blue) battlefield wins the battlefield. The player that allocates more of the green missiles to the bottom (green) battlefield wins the battlefield. Finally, the player that allocates more total missiles (blue or green missiles) wins the center battlefield.

other cases are important, but are not part of our analysis.

Like in the traditional Blotto game, players win battlefields by allocating more resources to the battlefield. However, unlike the single resource Blotto game, the function that determines the winner of a battlefield depends on the battlefield. We consider three simple types of battlefields and associated functions that determine the winner in each case. For the first type of battlefield, the player that allocates more of c_1 wins the battlefield. For the second type of battlefield, the player that allocates more of c_2 wins the battlefield. For the third type of battlefield, the player that allocates more of both resources (the sum of c_1 resources and c_2 resources) wins the battlefield. Figure 16 sketches an example of the scenario using aircraft. This heterogeneity in battlefields is designed to elicit inefficiencies in the aggregated force. If all battlefields were the same, there would be no opportunity for a fractionated force in our scenario to outperform the aggregated force.

To determine the effect of the number of battlefields, we compute the fractionated force's expected utility when there are $3, 6$, and 9 battlefields. In each case, one third of the battlefields have a winner decided by the joint allocation of c_1 only, one third of the battlefields have a winner decided by the joint allocation of c_2 only and the last third of the battlefields have a winner determined by the sum total of c_1 and c_2 resources allocated to it.

Figure 17 shows that unlike in the case of a single capability, the benefits of fractionation remain roughly unaffected by the force size. As the plot demonstrates, the fractionated force's expected utility is roughly $\frac{T}{3}$ for all force sizes.

This result suggests that it might be a Nash equilibrium for the aggregated player to allocate its platforms only to battlefields where both of its resources are used and for the fractionated player to allocate its platforms

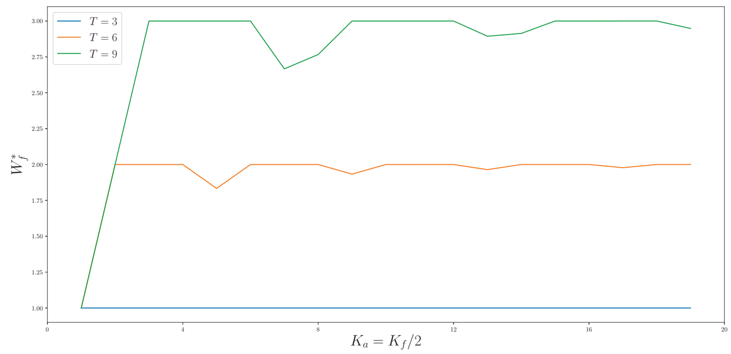

Figure 17: The fractionated force's expected utility with heterogeneous platforms. The plot shows that as the force size grows, the benefits of being fractionated are relatively constant.

to battlefields that require only one capability. In fact, the following proposition proves such an insight for the case when $T = 3$.

Proposition 1. *Consider the scenario described above and $T = 3$. Then a Nash equilibrium strategy is for the fractionated player to allocate all of the $K_f/2$ platforms with c_1 to battlefield 1, all of the $K_f/2$ platforms with c_2 to battlefield 2 and for the aggregated player to allocate all K_a platforms to battlefield 3 and the fractionated player earns an equilibrium expected utility of 1.*

Proof. Obviously, the fractionated force would never allocate platforms with c_1 to battlefield 2 and vice-versa. The only other potential change for the fractionated force is to allocate all of its platforms to battlefield 3 and thus allocate the same number of resources as the aggregated force at all battlefields and earn a payoff of 0, which is less than the proposed equilibrium payoff of 1. The only strategy for the aggregated force that can change the battlefield winners is if it allocate all of its platforms to battlefield 1 (the case of battlefield 2 is symmetric). In this case, the aggregated force would tie on battlefields 1 and 3 and still lose on battlefield 2, which earns a payoff of -1, which is the same expected utility from simply allocating all of its resources to battlefield 3. Therefore, neither the aggregated or fractionated player have an incentive to change their strategy and thus the proposed strategy is a Nash equilibrium □

In short, proposition 1 says that it is an equilibrium strategy for the aggregated force to "specialize" at battlefields where both of its resources are counted toward its allocation and the fractionated force to specialize on battlefields that only require 1 capability. Phrased another way, platforms with many capabilities are allocated to battlefields that reward both capabilities, whereas fractionated platforms (those with only one capability) are allocated to battlefields that reward only one capability or the other. Of course, a key assumption we made is that the value of the battlefields are the same and changing the relative importance of the battlefields may impact these results. This remains a topic of future work.

This intuition extends to higher values of T. However, the numerical properties are not as straightforward since the fractionated player cannot always evenly split its resources between battlefields. For example, if the fractionated force has an odd number of platforms with c_1 and there are two battlefields that require c_1, it cannot split its platforms evenly between the two battlefields. Nevertheless, as figure 17 shows, the fractionated force's inability to perfectly split its platforms between battlefields has only a modest impact on its equilibrium expected utility.

4 Future Work

We present an initial investigation into the multi-resource Blotto game. We focus on examining the benefits of fractionation and how they vary as a function of the number of battlefields, total force size, and the level of aggregation in the opposing force. We show that when there is only one type of resource, the benefits of fractionation decrease with increasing force size, increase with increasing target quantity, and increase with increase in the opponent's level of aggregation. These results are driven by the aggregated force's inability to adopt an ideal mixed-strategy (equal marginal distribution across battlefield allocations) as a result of having multiple resources per platform. There is also a curious periodicity due to the aggregate force having to restrict the range of the uniform marginal distribution of its allocations.

Random platform failure can prolong this degradation by reducing the *effective* size of each force, but ultimately leads to the same relationship. That is, as long as the expected force sizes are the same, the benefits of fractionation decrease in resource-rich environments.

Finally, we found that fractionation of capabilities, in the presence of heterogeneous battlefields, removed this dependence on force size, and allowed the fractionated force to persistently exploit the inability of the aggregated force to efficiently allocate resources to battlefields that required only one of the two capabilities its platforms carried.

Although our numerical experiments cover a significant portion of the parameter space, there are still several questions that remain. Most glaringly is the possibility of an analytical solution to the multi-resource Blotto game. As the results showed, in some cases the uniform marginal distribution result of the standard Blotto game extended to the multi-resource version. However, due to a player's inability to fractionate, the uniform marginal distribution result does not hold for all values of the parameters. Therefore, a key question is characterizing the marginal distributions in the multi-resource Blotto game and explaining the deviation from uniform marginal distributions.

There is also a plethora of further computational experiments that can illuminate insight in the multi-resource Blotto game. This includes more complex functions that determine the winner of the battlefield, different configurations of heterogeneity and correlated platform failures. Furthermore, a full analysis that considers how the benefits of fractionation are affected by force size asymmetry is pertinent to both economic and military resource allocation problems. The richness of our results, despite the model's simple formulation, hint at the potential value of pursuing such additional investigations in the multi-resource Blotto game.

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